INITIAL-BOUNDARY VALUE PROBLEM FOR A NONLINEAR HEAT EQUATION

By

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§0. Introduction

Recently nonlinear parabolic equations and nonlinear hyperbolic equations have been studied by many authors. (see references) Main tools in these papers are the theory of nonlinear semigroups, monotone operators, and various compact arguments.

In this paper we employ a compact argument to prove the existence, the uniqueness, and the continuity with respect to the data of the strict solution for the initial boundary problem of a nonlinear heat equation of the type;

$$(0.1) \quad \begin{cases} \frac{\partial}{\partial t} u - \Delta u + F'(u) = f(x, t) \\ u|_{\partial\Omega} = 0 \quad \text{and } u(x, t)|_{t=0} = u_0 \end{cases}$$

where $F'(u) = \frac{\partial}{\partial s} F(x, s)|_{s=u(x,t)}$.

Our method in this paper was already applied by Lions and Straus [7] to nonlinear evolution equations and by J. Sather to nonlinear wave equations and so on. Especially our argument is parallel to J. Sather [9]. We note that our solution is in the large with respect to the time variable.

§1. Definitions and notations

Let Ω be any domain (open) in the n -dimensional Euclidean space R^n and $\partial\Omega$ its boundary. $I = [0, T)$ for some positive number T is a in-

terval in the real line R^1 and we denote by $\mathcal{Q}_T$ the strip $\mathcal{Q} \times I$. All of the functions considered are real valued so that the function spaces are real vector spaces. We summe up the function spaces which will be used constantly. For p with $1 \leq p < \infty$, we denote by $L^p \equiv L^p(\mathcal{Q})$ the space consisting of all of the measurable functions on $\mathcal{Q}$ that satisfy

$$\|u\|_p \equiv \left(\int_{\mathcal{Q}} |u|^p dx \right)^{\frac{1}{p}} < +\infty$$

If $p = \infty$, $\|u\|_p$ is to be understood as the essential sup. of $|u|$ in $\mathcal{Q}$. As the special case, L^2 becomes Hilbert space with innerproduct;

$$(u, v) = \int_{\mathcal{Q}} uv dx \quad \text{for } u, v \in L^2.$$

$H_1 \equiv H_1(\mathcal{Q})$ is the Hilbert space of functions u which not only belong to L^2 but have derivative $u_{x_i} (i=1, 2, \dots, n)$ in L^2 , that is

$$\|u\|_1 = (\|u\|^2 + \|u_x\|^2)^{\frac{1}{2}} < +\infty$$

where $\|u_x\| = \left(\sum_{i=1}^n \|u_{x_i}\|^2 \right)^{\frac{1}{2}}$. The innerproduct of $H_1(\mathcal{Q})$ is denoted by $(u, v)_1$. The most important Hilbert space $\hat{H}_1$ is the completion in H_1 of the space of all infinitely differentiable functions with compact support in $\mathcal{Q}$. If H is a Hilbert space, we denote by $L^p(I, H)$ the space of all p -Bochner integrable H -valued functions defined on I , say, (see [14])

$$\|u\|_{L^p(I, H)} \equiv \left[\int_0^T \|u\|_H^p dx \right]^{\frac{1}{p}} < +\infty$$

If $p = \infty$, we replace the righthand by $\text{ess. sup}_{t \in I} \|u(t)\|_H$. We note that $[L^p(I, H)]^*$; the dual space of $L^p(I, H)$, becomes $L^q(I, H)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and a bounded set in $L^p(I, H)$ is weakly compact since $L^p(I, H)$ is separable if H is separable.

DEF. 1.1

$u \in L^1(I, H)$ is said to be weakly differentiable and have weak derivative $v \in L^1(I, H)$ if

$$(1.1) \quad \int_0^T (u(t), \varphi'(t))_{H^{\alpha t}} dt = - \int_0^T (v, \varphi(t))_{H^{\alpha t}} dt$$

for every $\varphi \in C_0^1(I^0, H)$, where $(\cdot, \cdot)_H$ is the innerproduct in H , $I^0 = (0, T)$, $C_0^1(I^0, H)$ is the set of continuously strong (ordinary) differentiable H -valued functions defined on I^0 , and φ' is strong derivative of φ .

It is known that if $u \in L^1(I, H)$ has a weak derivative $v \in L^1(I, H)$, then it holds that

$$(1.2) \quad u(t) - u(\tau) = \int_{\tau}^t v(s) ds \quad \text{in } H \text{ for } t \text{ and } \tau \text{ (a. e.)}$$

Thus $u(t)$ has a strong derivative $v(t)$ for almost every $t \in I$ and we denote also by $u'(t)$ the weak derivative $v(t)$. We may assume by (1.2) that u belongs to $C(I, H)$, the set of continuous operators from I to H .

We put several assumptions on $f(x)$, $F(x, s)$ and initial value u_0 , they are;

ASSUMPTION I

$$(1.3) \quad 0 \leq F(x, s) \leq \text{const. } |s|^{\rho+2}, \quad \left| -\frac{\partial}{\partial s} F(x, s) \right| \leq \text{const. } |s|^{\rho+1}$$

$$\text{and } \left| \frac{\partial^2}{\partial s^2} F(x, s) \right| \leq \text{const. } |s|^{\rho}$$

and

$$(1.4) \quad (F'(u), u) \geq 0$$

where ρ is a positive number such that

$$(1.5) \quad \begin{aligned} 0 < \rho < \frac{2}{n-2} & \quad \text{if } n > 2 \\ \text{or } 0 < \rho < \infty & \quad \text{if } n = 2 \end{aligned}$$

ASSUMPTION II

$$(1.6) \quad f \in L^2(I, H_0) \quad \text{and } f' \in L^2(I, H_0)$$

and

$$(1.7) \quad u_0 \in \dot{H}_1 \cap H_2.$$

If we assume the monotonicity of $F'(u)$ i.e

$$(F'(u_1) - F'(u_2), u_1 - u_2) \geq 0,$$

then we can easily show the uniqueness and the continuous dependence (in a sense) of the weak solution on the data. (see Lions and Strauss [7]) But we don't want to put such a strong assumption though we cannot stop setting the weaker assumption (1.4). The necessity of (1.5) follows from well known Sobolev's inequality. We state it as a *lemma* without proof.

LEMMA 1.1 (Sobolev)

If $u \in \dot{H}_1 \cap L^p (p \geq 2)$, then

$$(1.8) \quad \|u\|_q^{\alpha+\beta} \leq \text{const.} \cdot \|u\|_p^\alpha \|u\|_1^\beta$$

for $p \leq q \leq \infty$ when $\alpha = n\left(\frac{1}{q} - \frac{1}{2}\right) + 1 \geq 0$ and $\beta = n\left(\frac{1}{p} - \frac{1}{q}\right)$ but that $p < \infty$ and $\alpha = 0$ where $\|u\|_1^2 = \|u\|_1^2 - \|u\|^2$. From this we can easily show using Hölder's inequality that

COR. 1.1

If ρ satisfies (1.5), then

$$(1.9) \quad \|u\|^{\rho+1} \leq \text{const.} \cdot \|u\|_1 \quad \text{for } u \in \dot{H}_1$$

DEF. 1.2

A $\dot{H}_1$ -valued function u said to be a weak solution of the problem (0.1)

if

$$(1.10) \quad u \in L^\infty(I, \dot{H}_1) \quad \text{and it holds that}$$

$$(1.11) \quad \int_0^T \{-(u(t), \varphi'(t)) + ((u(t), (t)))_{(1)} + (F'(u), \varphi)\} dt \\ = (u_0, \varphi(0)) + \int_0^T (f(t), \varphi(t)) dt \quad \text{for all } \varphi \in \mathcal{F}$$

where we put

$$\mathcal{F} \equiv \{\varphi \mid \varphi \in L_1(I, H_1), \varphi' \in L^1(I, H_0)\}$$

and $\varphi(x) \equiv 0$ in a neighborhood of $t = T$

and

$$((u, v))_{(1)} \equiv \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} u, \frac{\partial}{\partial x_i} v \right) \quad \text{for } u, v \in H_1$$

DEF. 1.3

A $\dot{H}_1$ -valued function u is said to be a strict solution for the problem (0.1) if

$$(1.12) \quad u(t) \in \dot{H}_1 \cap H_2 \text{ and } u'(t) \in H_0 \quad \text{for all } t \in I$$

and it holds that

$$(1.13) \quad u'(t) - \Delta u(t) + E'(u(t)) = f(t) \text{ in } H_0 \quad \text{for all } t \in I.$$

and

$$(1.14) \quad u(0) = u_0$$

§2. Existence theorem

In this section, we shall prove the existence of a strict solution. Firstly we state

THEOREM 2.1

Under ASSUMPTION I and II, there exists a $\dot{H}_1$ -valued function $u(t)$ which satisfies

$$(2.1) \quad u \in L^\infty(I, \dot{H}_1), u' \in L^\infty(I, H_0)$$

and

$$(2.2) \quad \int_0^T \{ (u'(t), \varphi(t)) + ((u(t), \varphi(t)))_{(1)} + (F'(u) \varphi(t)) \} dt = \int_0^T (f(t), \varphi(t)) dt$$

for all $\varphi \in L^1(I, H_1)$

moreover we have

$$(2.3) \quad u(t) = u_0 + \int_0^t u'(\tau) d\tau \text{ in } H_0 \quad \text{for (a.e.) } t \in I.$$

REMARK

We note that the term $\int_0^T (f(t), \varphi(t)) dt$, in (1.11) or (2.2) is meaningful since $\|f(t)\|$ is bounded under Assumption II. The function u in Theorem 2.1 clearly becomes a weak solution. This solution is furthermore easily modified using a weakly convergent theorem and a standard theory of elliptic partial differential equations to become a strict solution (see J. Sather [9]). Thus we have

THEOREM 2.2 *Let Ω be C^2 -class.*

Under the same condition as in Theorem 2.1, there exists a strict solution of (0.1) which satisfies $u \in L^\infty(I, \dot{H}_1)$ and $u' \in L^\infty(I, H_0)$.

We devote the remainder of this section to the proof of theorem 2.1. At first we choose a linearly independent set $\{\phi_k\}_{k=1}^\infty \subset \dot{H}_1 \cap H_2$ whose linear combinations hulls the space $\dot{H}_1 \cap H_2$. This is possible since $\dot{H}_1 \cap H_2$ is separable. We determine u_{m_0} and $\lambda_k^m(0)$ so that

$$(2.4) \quad u_{m_0} \equiv \sum_{k=1}^m \lambda_k^m(0) \phi_k \longrightarrow u_0 \quad \text{in } \dot{H}_1 \cap H_2 \text{ when } m \rightarrow \infty$$

and define next $\lambda_k^m(t)$ ($k=0, 1, 2, \dots, m$) as the solution of the following system of the ordinary differential equations;

$$(2.5) \quad \left\{ \begin{array}{l} \left(\sum_{k=0}^m \dot{\lambda}_k^m(t) \phi_k, \phi_k \right) + \left(\left(\sum_{k=1}^m \lambda_k^m(t) \phi_k, \phi_k \right) \right)_{(1)} \\ \quad + (F'(\sum_{k=1}^m \lambda_k^m(t) \phi_k), \phi_k) = (f(t), \phi_k) \\ k=1, 2, \dots, m \\ \text{and} \\ \lambda_k^m(t)|_{t=0} = \lambda_k^m(0) \end{array} \right.$$

By the fundamental theorem of ordinary differential equation, (2.5) has a unique solution on a certain interval $[0, T_m]$. If we set,

$$(2.6) \quad u_m(t) = \sum_{k=1}^m \lambda_k^m(t) \phi_k \in \dot{H}_1 \cap H_2$$

and

$$(2.7) \quad u'_m(t) = \sum \dot{\lambda}_k^m(t) \phi_k \in \dot{H}_1 \cap H_2,$$

then (2.5) becomes

$$(2.5)' \quad \begin{cases} (u'_m(t), \phi_k) + ((u_m(t), \phi_k))_{(1)} + (F'(u_m), \phi_k) \\ = (f(t), \phi_k) & (k=1, 2, \dots, m) \\ \text{and} \\ u_m(0) = u_{m_0} \end{cases}$$

It is our work remained to show that $\lambda_k^m(t)$ is defined in the large interval $[0, T)$ and $u_m(t)$ (or its subsequence) converges in various topologies. For this purpose, we estimate the various norms of $\{u_m(t)\}$.

LEMMA 2.1

$u_m(t)$ is defined on $[0, T)$ and it holds that

$$(2.8) \quad \begin{aligned} & \|u_m(t)\|^2 + \int_0^t \|u_m(s)\|_{(1)}^2 ds \\ & \leq \text{const.} \left(\|u_0\|, \int_0^T \|f\|^2 dt \right) e^t \\ & \text{for } 0 \leq t \leq T \end{aligned}$$

and

$$(2.9) \quad \begin{aligned} & \int_0^t \|u'_m(s)\|^2 ds + \|u_m(t)\|_{(1)}^2 + |F(u_m(t))|_1 \\ & \leq \text{const.} \left(\|u_0\|_1, \int_0^T \|f\|^2 dt \right) < +\infty \end{aligned}$$

where we denote by $\text{const.}(Q)$ various constants which depend on Q and are bounded if Q is bounded.

PROOF

We multiply the both sides of (2.5)' by $\lambda_k^m(t)$ and add all of the equations to obtain;

$$(u'_m, u_m) + ((u_m, u_m))_{(1)} + (F'(u_m), u_m) = (f, u_m),$$

and

$$\frac{1}{2} \frac{d}{dt} \|u_m\|^2 + \|u_m\|_{(1)}^2 + (F'(u_m), u_m) = (f, u_m)$$

By integration,

$$\begin{aligned} & \frac{1}{2} \{ \|u_m(t)\|^2 - \|u_m(0)\|^2 \} + \int_0^t \|u_m\|_{(1)}^2 ds \\ & + \int_0^t (F'(u_m), u_m) ds = \int_0^t (f, u_m) ds \leq 2 \int_0^t (f u_m) ds \leq 2 \int_0^t \|f\|^2 ds \\ & + \frac{1}{2} \int_0^t \|u_m\|^2 ds \end{aligned}$$

Since $(F'(u_m), u_m) \geq 0$ by hypothesis, we have

$$\begin{aligned} & \|u_m(t)\|^2 + 2 \int_0^t \|u_m\|_{(1)}^2 ds \\ & \leq \|u_m(0)\|^2 + 4 \int_0^T \|f\|^2 ds + \int_0^t \|u_m\|^2 ds \end{aligned}$$

By Gronwall's lemma,

$$\begin{aligned} (2.10) \quad & \|u_m(t)\|^2 + 2 \int_0^t \|u_m\|_{(1)}^2 ds \\ & \leq \left(\|u_m(0)\|^2 + 4 \int_0^T \|f\|^2 ds \right) e^t \\ & \text{for } 0 \leq t \leq T_m \end{aligned}$$

Since the righthand of (2.10) is bounded on $[0, T)$ and also $\lambda_k^m(t)$, the domain $\lambda_k^m(t)$ is defined is extended to $0 \leq t \leq T$ from the well-known theory of ordinarily differential equations. It completes the proof of (2.8). To say (2.9), we multiply the both sides of (2.5)' by $\dot{\lambda}_k^m(t)$ and summe up with respect to k to obtain

$$\|u'_m\|^2 + ((u_m, u''_m))_{(1)} + (F'(u_m), u''_m) = (f, u'_m)$$

Since $F'(u_m)u'_m = \frac{\partial}{\partial t} F(x, u_m)$ and $F(x, u_m) \geq 0$ by hypothesis, we have

$$\|u'_m\|^2 + \frac{1}{2} \frac{d}{dt} \|u_m\|_{(1)}^2 + \frac{d}{dt} |F(x, u_m)|_1 = (f, u'_m)$$

Integrating the both sides and using Schwartz's inequality,

$$\begin{aligned} & \int_0^t \|u'_m\|^2 ds + \frac{1}{2} \|u_m\|_{(1)}^2 + |F(x, u_m)|_1 \\ & \leq \|u_m(0)\|_{(1)}^2 + |F(x, u_m(0))|_1 \\ & + \frac{1}{2} \int_0^t \|f\|^2 ds + \frac{1}{2} \int_0^t \|u'_m\|^2 ds \end{aligned}$$

Hence

$$\begin{aligned} (2.11) \quad & \int_0^t \|u'_m\|^2 ds + \|u_m\|_{(1)}^2 + 2|F(x, u_m)|_1 \\ & \leq 2\|u_m(0)\|_{(1)}^2 + 2|F(x, u_m(0))|_1 + \int_0^t \|f\|^2 ds \end{aligned}$$

$\|u_m(0)\|_{(1)}^2 \leq \text{const.} (\|u_0\|_1)$ by definition and $|F(x, u_m(0))|_1 \leq \text{const.}$
 $|u_m(0)^{p+2}|_1 \leq \text{const.} \|u_m(0)\|_1^2$ by Sobolev's inequality. Thus we have (2.9).
 q.e.d.

Furthermore we can show

LEMMA 22.

$$\begin{aligned} (2.12) \quad & \|u'_m\|^2 + \int_0^t \|u'_m\|_{(1)}^2 ds \\ & \leq \text{const.} \left(\|u(0)\|_1, \|u(0)\|_2, \int_0^T \|f\|^2 ds, \int_0^T \|f'\|^2 ds \right) \\ & \text{for } t \in I \end{aligned}$$

PROOF

$f' \in L^2(I, H_0)$ implies that $(f(t), \psi_k)$ is continuously differentiable for almost every t , and $\lambda_k^m(t)$ has twice derivative (a.e.). Differentiating the both sides of (2.5)',

$$\begin{aligned} & (u_m''(t), \psi_k) + ((u'_m(t), \psi_k))_{(1)} + (F''(u_m(t))u'_m(t), \psi_k) \\ & = (f', \psi_k) \quad (\text{a.e. in } I) \end{aligned}$$

where $F''(u) = \frac{\partial^2}{\partial s^2} F(x, s)|_{s=u(x,t)}$.

By the same way as we used to get (2.9),

$$-\frac{1}{2} \frac{d}{dt} \|u'_m\|^2 + \|u'_m\|_{(1)}^2 + (F''(u_m)u'_m, u'_m) = (f', u'_m)$$

Integrating with respect to t again, we have

$$(2.13) \quad \left\{ \frac{1}{2} \|u'_m(t)\|^2 - \|u'_m(0)\|^2 \right\} + \int_0^t \|u'_m\|_{(1)}^2 ds \\ \leq \int_0^t |(F''(u_m)u'_m, u'_m)| ds + \int_0^t |(f', u'_m)| ds$$

Schwarz's inequality and hypothesis on F'' imply

$$|(F''(u_m)u'_m, u'_m)| \leq \text{const.} \|u_m^p u'_m\| \|u'_m\|$$

We can show (see J. Sather [9]) using Sobolev's inequality and Hölder's inequality that

$$(2.14) \quad \|u_m^p u'_m\| \leq \text{const.} (\|u_m(t)\|_1) \|u'_m\|_{(1)}$$

and also by LEMMA 2.1 which assures the boundedness of $\|u_m(t)\|_1$.

$$\leq \text{const.} (\|u_0\|_1, \int_0^T \|f\|^2 dt) \|u'_m\|_{(1)}$$

Thus we have

$$(2.15) \quad |(F''(u_m)u'_m, u'_m)| \leq \varepsilon \|u'_m\|_{(1)}^2 + C(\varepsilon) \|u'_m\|^2$$

Where ε is any positive number and $C(\varepsilon)$ is a constant which depends on $\|u_0\|_1, \int_0^T \|f\|^2 dt$ and ε .

(2.13) and (2.15) with $\varepsilon < \frac{1}{2}$ imply that

$$\|u'_m(t)\|^2 + \int_0^t \|u'_m\|_{(1)}^2 ds \leq \|u'_m(0)\|^2 \\ + \int_0^t \|f'\|^2 ds + (2C(\varepsilon) + 1) \int_0^t \|u'_m\|^2 ds,$$

and by *Grouwall's* Lemma

$$\|u'_m(t)\|^2 + \int_0^t \|u'_m\|_{(1)}^2 ds \leq C_m e^{C_0 t}$$

where $C_m = \|u'_m(0)\|^2 + \int_0^T \|f'\|^2 ds$ and $C_0 = 2C(\varepsilon) + 1$. Since $u_m(t)$ is the solution of (2.5)' and $u_m(0) \in \dot{H}_1 \cap H_2$, it is easy to prove $\|u'_m(0)\| \leq \text{const.}$ ($\|u_0\|_1, \int_0^T \|f\|^2 dt$). Thus the proof of Lemma is complete. q.e.d.

We next show that we can extract a subsequence which converges in various topologies.

LEMMA 2.3

There exists a subsequence of $\{u_m\}$, which we denote by $\{u_m\}$ again, such that

$$(2.16) \quad u_m(t) \longrightarrow u(t) \text{ in the weak topology of } L^\infty(I, \dot{H}_1)$$

$$(2.17) \quad u_m(t) \longrightarrow u(t) \text{ in the weak topology of } H_0 \text{ for a.e } t \in I$$

$$(2.18) \quad u'_m(t) \longrightarrow u'(t) \text{ in the weak topology of } L^\infty(I, H_0)$$

and

$$(2.19) \quad F'(u_m) \longrightarrow F'(u) \text{ in the weak topology of } L^\infty(I, H_0)$$

PROOF

By Lemmas 2.1 and 2.2, the sequences $\{u_m(t)\}$, $\{u'_m(t)\}$ and $\{D_j u_m(t)\}$ ($j=1, \dots, n$) are bounded as subsets of $L^2(\mathcal{Q}_T)$. Hence, if K is an arbitrarily but fixed compact set in $\mathcal{Q}$, we can choose a subsequence $\{u_m\}$ by Rellich's theorem which converges in $L^2(K \times I)$, and also we may assume, using the diagonal method, that $\{u_m\}$ converges in $L^2(K \times I)$ for any compact set $K \subset \mathcal{Q}$. Thus we have

$$(2.20) \quad u_m(x, t) \longrightarrow U(x, t) \quad \text{a.e in } \mathcal{Q} \times I$$

$$\text{and in } L_2^{loc}(\mathcal{Q} \times I)$$

for a measurable function $U(x, t)$. Precisely we must take a subsequence of $\{u_m(x, t)\}$ for (2.20), but denote by $\{u_m\}$ again the subsequence. We do so in what follows.

Since $\{u'_m(t)\}$ is bounded in $L^\infty(I, H_0)$, $u'_m(t)$ converges weakly to a point in $L^\infty(I, H_0)$, which we denote by $\bar{u}(t)$. Hence, from

$$(u_m, v) = (u_m(0), v) + \int_0^t (u'_m(s), v) ds \quad \text{for all } v \in H_0,$$

we have by taking the limit,

$$\lim_{m \rightarrow \infty} (u_m, v) = (u_0, v) + \int_0^t (\tilde{u}(s), v) ds$$

This shows that

$$u_m(t) \longrightarrow u(t) \quad \text{weakly in } H_0 \text{ for each } t \in I$$

where we put

$$u(t) = u_0 + \int_0^t \tilde{u}(s) ds \quad (\text{in } H_0)$$

By the uniqueness of weak derivative, we have

$$(2.21) \quad u' = \tilde{u} \quad \text{in } L^1(I, H_0)$$

We show next (2.16) and (2.19).

Since Sobolev's inequality and Lemma 2.1 imply

$$\|F'(u_m)\| \leq \text{const.} |u_m|_{\frac{1}{2}+\frac{\rho}{p}} \leq \text{const.} \|u_m\|_1^{1+\rho} < \text{const.} < +\infty$$

$F'(u_m)$ converges weakly to a point $\tilde{F}' \in L^\infty(I, H_0)$ and also in the sense of distribution on $\mathcal{Q} \times I$. On the other hand, from that $u_m \longrightarrow U$ in $L^2_{loc}(\mathcal{Q}_T)$ we can show using Sobolev's inequality that

$$F'(u_m) \longrightarrow F'(U) \quad \text{in distribution.}$$

Hence we have

$$\tilde{F}' = F'(U) \text{ in } H_0 \quad \text{for almost all } t \in I.$$

From the boundedness of u_m in $L^\infty(I, \dot{H}_1)$, we have also

$$u_m \longrightarrow u_* \quad \text{weakly in } L^\infty(I, \dot{H}_1)$$

for some $u_* \in L(I, H_1)$. Finally it is seen easily that

$$u(x, t) = u_*(x, t) = U(x, t) \quad \text{a.e in } \mathcal{Q}_T$$

q.e.d.

We have now been ready to prove Theorem 2.1.

PROOF OF THEOREM 2.1

If we denote by $\chi_k(t)$ various characteristic functions dedfined on I , the set of step functions of the type $\Phi(t) = \sum_{k=1}^N C_k \chi_k(t) \psi_k$ is dense in $L^1(I, \dot{H}_1)$.

If we take $m > N$, we have by (2.5)' that

$$\begin{aligned} & (u'_m(t), \Phi(t)) + ((u_m(t), \Phi(t))_{(1)} + (F'(u_m(t)), \Phi(t))) \\ &= (f(t), \Phi(t)) \end{aligned}$$

Integrating and taking the limit, we have in view of the above Lemma

$$\begin{aligned} & \int_0^T \{(u'(t), \Phi(t)) + ((u(t), \Phi(t))_{(1)} + (F'(u(t)), \Phi(t)))\} dt \\ &= \int_0^T (f(t), \Phi(t)) dt. \end{aligned}$$

This shows (2.2). And (2.3) is already proved in the proof of Lemma 2.3.
q.e.d.

§3. Uniqueness and continuity of the solution with respect to the data.

In this section we prove the uniqueness and the continuous dependence of the solution on the data.

The uniqueness is proved for weak solution, but it remains valid for strict solution since a strict solution becomes, of course, a weak solution.

THEOREM 3.1 (uniqueness)

Let u and v be the weak solutions of (0.1) with initial values u_0 and v_0 respectively, and with inhomogeneous terms f and g respectively. We assume that

$$u, v \in L^\infty(I, \dot{H}_1) \text{ and } u', v' \in L^\infty(I, H_0)$$

If we set $w = u - v$, $w_0 = u_0 - v_0$, and

$$h = f - g, \quad \text{it holds then that}$$

$$(3.1) \quad \|w(t)\|^2 \leq \left(\|w_0\|^2 + \int_0^T \|h(s)\|^2 ds \right) e^{K_c t}$$

and

$$\|u(t)\|^2 \leq \text{const.} \left(\|u_0\|^2 + \int_0^T \|f\|^2 dt \right)$$

for almost all $t \in I$

where K_c is a constant depending on $C = \text{ess. sup}_{t \in I} (\|u(t)\|_1, \|v(t)\|_1)$

PROOF

Under the assumptions, we can easily prove $u(0)=u_0$ and $v(0)=v_0$ with aid of a function $\theta=\theta(t) \in C^1(I)$ such that

$$(3.2) \quad \theta(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \tau \quad \text{and} \quad 0 \leq \theta(t) \leq 1 \\ 0 & \text{if } \tau + \varepsilon \leq t \end{cases}$$

for $\tau, \varepsilon > 0$ with $\tau + \varepsilon < T$. This is done by putting $\theta\varphi$ with $\varphi \in H_1$ in place of $\varphi \in \mathcal{F}$ in (1.11) and taking the limit as $\varepsilon \rightarrow 0$ and $\tau \rightarrow 0$. We have thus

$$(3.3) \quad \begin{aligned} & \int_0^T \{ -(w(t), \varphi(t)) + ((w(t), \varphi(t)))_{(1)} \\ & + F'(u) - F'(v), (t) \} dt = (w(0), \varphi(0)) \\ & + \int_0^T (h(t), \varphi(t)) dt \quad \text{for all } \varphi \in \mathcal{F} \end{aligned}$$

The mean value theorem implies that

$$|F'(u) - F'(v)| \leq F''(\xi) |u - v| \quad \text{for } (x, t) \in \mathcal{Q}_T$$

where $\xi(x, t)$ is a vector between $u(x, t)$ and $v(x, t)$. By hypothesis, we have

$$|F''(\xi)| \leq \text{const.} \quad |\xi|^\rho \leq \text{const.} \quad (|u| + |v|)^\rho$$

and by Sobolev's inequality and Minkowskii's inequality

$$(3.4) \quad |F''(\xi)|^{\frac{1}{p}} \leq \text{const.} \quad \{\|u(t)\|_1 + \|v(t)\|_1\} (a.e)$$

if p satisfies that

$$(3.5) \quad \frac{1}{\rho} \leq p \leq \frac{2n}{\rho(n-2)} \quad (n \neq 2) \text{ or } p < +\infty (n=2)$$

Consequently, we have

$$(3.4)' \quad \operatorname{ess. sup}_{t \in I} |F''(\xi)|_p^{\frac{1}{p}} \leq Kc < +\infty$$

if (3.5) is satisfied. Since $\theta w \in \mathcal{F}$ where θ is the same function as (3.2), we have by (3.3) that

$$\begin{aligned} & \int_0^T \{-(w, (\theta w)') + ((w, \theta w))_{(1)} + (F'(u) - F'(v), \theta w)\} dt \\ &= w(0)^2 + \int_0^T \theta(h, w) dt \end{aligned}$$

Since $w(t)$ has a strong derivative $w'(t)$ for almost all $t \in I$, it follows that

$$(w, (\theta w')) = (w, \theta w)' - (w', \theta w) \quad (a.e)$$

and also connecting this with the above equality,

$$\begin{aligned} (3.6) \quad & \int_0^T \theta(w', w) dt + \int_0^T \theta \|w\|_{(1)}^2 dt \\ &= \int_0^T \theta(h, w) dt - \int_0^T \theta(F'(u) - F'(v), w) dt \end{aligned}$$

If ε tends to 0, it is easily seen that

$$\begin{aligned} & \int_0^T \theta(w', w) dt \longrightarrow \frac{1}{2} (\|w(\tau)\|^2 - w(0)^2) \\ & \int_0^T \theta(h, w) dt \longrightarrow \int_0^\tau \theta(h, w) dt \\ & \int_0^T \theta(F'(u) - F'(v), w) dt \longrightarrow \int_0^\tau \theta(F'(u) - F'(v), w) dt \end{aligned}$$

and

$$\int_0^T \theta \|w\|_{(1)}^2 dt \longrightarrow \int_0^\tau \|w\|_{(1)}^2 dt$$

And consequently from (3.5)

$$(3.7) \quad \frac{1}{2} \|w(\tau)\|^2 + \int_0^\tau \|w\|_{(1)}^2 dt = \frac{1}{2} \|w_0\|^2 \\ + \int_0^\tau (h, w) dt - \int_0^\tau (F'(u) - F'(v), w) dt$$

From Hölder's inequality and Sobolev's inequality, we can show (see J. Sather [9])

$$(3.8) \quad |(F'(u) - F'(v), w)| \leq \text{const.} \|w\|^{1-\beta} \|w\|_{(1)}^\beta |F''(\xi)|_p \|w\|$$

with $0 < \beta < 1$

for p with $\frac{2}{\rho} \leq p \leq \frac{2n}{\rho(n-2)}$ if $n=2$ or $0 < p < +\infty$ if $n=2$.

Thus if we denote by Kc various constants which depend on C , we obtain from (3.4)' and (3.8) that

$$|(F'(u) - F'(v), w)| \leq Kc \|w\|^{2-\beta} \|w\|_{(1)}^\beta \quad \text{with } 0 < \beta < 1$$

and by Young's inequality,

$$\leq Kc \{ (2-\beta)^{\frac{-2}{2-\beta}} \|w\|^2 + \beta \cdot \epsilon_\beta^2 \|w\|_{(1)}^2 \}$$

for any $\epsilon > 0$. In other words, for arbitrarily $\delta > 0$, there exists $C(\delta) > 0$ such that

$$(3.9) \quad |F'(u) - F'(v), w| \leq Kc \{ C(\delta) w^2 + \delta \|w\|_{(1)}^2 \}$$

This, (3.7) and the inequality

$$|(h, w)| \leq \|h\| \|w\| \leq \frac{1}{2} (\|h\|^2 + \|w\|^2)$$

imply that

$$\frac{1}{2} \|w(\tau)\|^2 + \int_0^\tau \|w\|_{(1)}^2 dt \leq \frac{1}{2} \|w_0\|^2$$

$$\begin{aligned}
& + \left(\frac{1}{2} + KcC(\delta) \right) \int_0^\tau \|w\|^2 dt + \frac{1}{2} \int_0^\tau \|h\|^2 dt \\
& + Kc\delta \int_0^\tau \|w\|^2 dt
\end{aligned}$$

Fixing $\delta > 0$ so that $Kc\delta \leq \frac{1}{2}$ and changing the notations, we have

$$\|w(t)\|^2 \leq \|w_0\|^2 + \int_0^t \|h(s)\|^2 ds + Kc \int_0^t \|w(s)\|^2 ds$$

and by Gronwall's lemma

$$\|w(t)\|^2 \leq \left[\|w_0\|^2 + \int_0^t \|h\|^2 ds \right] e^{Kc \cdot t(a.e)} \quad \text{in } I.$$

q.e.d.

We note that in the proof of Theorem 3.1 the assumption; $f', g' \in L^2(I, H_0)$ is not used, so that the uniqueness is valid under the assumption $f, g \in L^2(I, H_0)$, but if we assume that a weak solution u satisfies $u \in L^\infty(I, \dot{H}_1)$ and $u' \in L^\infty(I, H_0)$, moreover inhomogeneous term f satisfies

$$f \in L^2(I, H_0) \text{ and } f' \in L^2(I, H_0),$$

we conclude then from the above uniqueness theorem that u is nothing but the strict solution the existence of which is already assured by Theorem 2.1 and Theorem 2.2. Thus the solution is the limit of $\{u_m\}$ constructed in §2 and satisfies

$$\text{ess. sup}_{t \in I} \|u\|_1 \leq K(\|u\|_{L^\infty(I, H)}, \|f\|_{L^2(I, H_0)})$$

by virtue of Lemma 2.1. We denote, here and in what follows, by $K(\cdot)$ various constants which depends on $(\cdot)$ continuously. Connecting this consideration with the estimate of the above Theorem, we can obtain

$$\begin{aligned}
& \|w(t)\|^2 \leq K(\|u_0\|_1, \|v_0\|_1, \\
& \|f\|_{L^2(I, H_0)}, \|g\|_{L^2(I, H_0)}) (\|w_0\|^2 + \int_0^t \|h\|^2 dt).
\end{aligned}$$

This shows that the mapping $(u_0, f) \rightarrow u(x)$ is continuous from $\dot{H}_1 \times$

$L^2(I, H_0)$ to $L^\infty(I, H_0)$.

But we can obtain better results than this concerned with the continuous dependence of the solution on the data, that is;

THEOREM 3.2

Let's put the same assumptions as Theorem 3.1. It holds then that

$$\begin{aligned}
 (3.10) \quad & \|u(t) - v(t)\|^2 + \int_0^t \|u - v\|_{(1)}^2 dt \\
 & \leq K(\|u_0\|_1, \|v_0\|_1, \|f\|_{L^2(I, H_0)}, \\
 & \|g\|_{L^2(I, H_0)}) \left(\|u_0 - v_0\|^2 + \int_0^t \|f - g\|^2 dt \right) \quad \text{for } t \in I
 \end{aligned}$$

and

$$\begin{aligned}
 (3.11) \quad & \int_0^t \|u' - v'\|^2 dt + \|u - v\|_{(1)}^2 \\
 & \leq K(\|u_0\|_1, \|v_0\|_1, \|f\|_{L^2(I, H_0)}, \\
 & \|g\|_{L^2(I, H_0)}) \left\{ \left[\int_0^1 \|f - g\|^2 dx + \|u_0 - v_0\|^2 \right]^\alpha \right. \\
 & \left. + \int_0^t \|f - g\|^2 dt \right\} \quad \text{for } t \in I
 \end{aligned}$$

for some $\alpha > 0$.

REMARK

This theorem implies that the mapping; $(u_0, f) \rightarrow (u, u')$ is continuous from $H_1 \times L^2(I, H_0)$ to $L^\infty(I, H_1) \times L^2(I, H_0)$

PROOF OF THEOREM

As is mentioned just before, we can construct sequences $\{u_m\}$ and $\{v_m\}$ which satisfy the properties of lemma 2.3. Furthermore it is easily seen that

$$u_m \rightarrow u \quad \text{weakly in } L^2(I, H_1)$$

and

$$u'_m \rightarrow u' \quad \text{weakly in } L^2(I, H_0)$$

Repeating the similar argument to one in the proof of lemma 2.1, we have

$$\begin{aligned} & \|u_m(t) - v_m(t)\|^2 + \int_0^t \|u_m - v_m\|_{(1)}^2 ds \\ & \leq \text{const.} \left\{ \|u_m(0) - v_m(0)\|^2 + \int_0^t \|f - g\|^2 ds \right. \\ & \quad \left. + \int_0^t (F'(u_m) - F'(v_m), u_m - v_m) ds \right\} \end{aligned}$$

Since

$$\begin{aligned} & |(F'(u_m) - F'(v_m), u_m - v_m)| \\ & \leq K(\cdot) \{C(\delta) \|u_m - v_m\|^2 + \delta \|u_m - v_m\|_{(1)}^2\} \end{aligned}$$

from 3.9, we have, fixing δ sufficiently small, that

$$\begin{aligned} & \|u_m(t) - v_m(t)\|^2 + \int_0^t \|u_m - v_m\|_{(1)}^2 ds \\ & \leq K(\cdot) \left\{ (\|u_m(0) - v_m(0)\|^2 + \int_0^t \|f - g\|^2 ds) \right. \\ & \quad \left. + \int_0^t \|u_m - v_m\|^2 ds \right\} \end{aligned}$$

and also by Gronwall's lemma

$$\begin{aligned} (3.12) \quad & \|u_m(t) - v_m(t)\|^2 + \int_0^t \|u_m - v_m\|_{(1)}^2 ds \\ & \leq K(\cdot) \left\{ (\|u_m(0) - v_m(0)\|^2 + \int_0^t \|f - g\|^2 ds) \right\} e^{K(\cdot)t} \end{aligned}$$

where $K(\cdot) = K(\|u_0\|_1, \|v_0\|_1, \|f\|_{L^2(I, H_0)}, \|g\|_{L^2(I, H_0)})$ by (3.9) and

Lemma 2.1.

$u_m(t) \rightarrow u(t)$ and $v_m(t) \rightarrow v(t)$ in various weak topologies imply that;

$$\|u(t) - v(t)\|^2 + \int_0^t \|u - v\|_{(1)}^2 ds$$

$$\begin{aligned} &\leq \underline{\lim} K(\cdot) \left\{ \|u_m(0) - v_m(0)\|^2 + \int_0^t \|f - g\|^2 ds \right\} e^{K(\cdot)t} \\ &= K(\cdot) \left\{ \|u(0) - v(0)\|^2 + \int_0^t \|f - g\|^2 ds \right\} e^{K(\cdot)t} \end{aligned}$$

This proves (3.10). (3.11) is also shown similarly along the line of the proof of Lemma 2.1, though we need not use Gronwall's lemma. The only point that we must note is the estimate of the term;

$$A \equiv \int_0^T (F'(u_m) - F'(v_m), u'_m - v'_m) dt$$

We estimate this as follows

$$\begin{aligned} |A| &\leq K(\cdot) \int_0^T (C(\delta) \|u_m - v_m\|^{2\alpha} + \|u'_m - v'_m\|^2) dt \\ &\leq K(\cdot) \int_0^T C(\delta) K(\cdot) [\|u_m(0) - v_m(0)\|^2 \\ &\quad + \int_0^t \|f - g\|^2 ds]^\alpha dt + K(\cdot) \int_0^T \|u'_m - v'_m\|^2 dt \\ &\leq K(\cdot) C(\delta) [\|u_m(0) - v_m(0)\|^2 \\ &\quad + \int_0^T \|f - g\|^2 ds]^\alpha + K(\cdot) \delta \int_0^T \|u'_m - v'_m\|^2 dt \end{aligned}$$

for some $\alpha > 0$ and any $\delta > 0$. This is done using Sobolev's inequality, Hölder's inequality, Young's inequality, Lemma 2.1, and (3.12)

q.e.d.

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