AN I.L.B. LIE SUBGROUP OF $O(\mathcal{S})$.

By

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1. Introduction.

In the earlier papers [1] and [2], concerning with the transformation group of the Gaussian White Noise, the author investigated subgroups of $O(\mathcal{S})$, the rotation group of $\mathcal{S}$, where $\mathcal{S}$ is the nuclear space of all rapidly decreasing functions on the real line. In [2], he found a method of constructing one-parameter subgroups of $O(\mathcal{S})$ and defined a Lie group structure in a subgroup $\mathfrak{G}$ of $O(\mathcal{S})$ with associated Lie algebra and exponential map. However he did not introduce a true differentiable structure into the subgroup $\mathfrak{G}$.

In this work, modifying the notion of an I.L.H. Lie group which is defined by H. Omori [3], we introduce a true differentiable structure (an I.L.B. Lie group structure) into the subgroup $\mathfrak{G}$ and estimate the associated bracket product and the exponential map explicitly in view of this structure.

In Section 2, we define an I.L.B. Lie group. In Section 3, we show the results of this paper and in Section 4, we give the proofs of them.

The author wishes to show his hearty thanks to Professor H. Omori for his many valuable suggestions on the theory of I.L.H. Lie group.

2. Definition of an I.L.B. Lie group.

Concerning with the study of the diffeomorphism group of a compact manifold, H. Omori [3] defined an I.L.H. (inverse limit Hilbertian) Lie group. Without any difficulty, the notion of an I.L.H. Lie group can be modified to the notion of an I.L.B. (inverse limit Banach) Lie group. In fact we define an I.L.B. Lie group as follows.

A sequence of Banach space $\{E^k\}_{k=0}^{+\infty}$ is called an I.L.B.-system if

every $\mathbf{E}^k$ is a separable Banach space and imbedded continuous linearly as a dense subspace of $\mathbf{E}^{k-1}$ for every k . A typical example of such an I.L.B.-system is derived from a complete separable countably normed space.

A topological group $\mathfrak{G}$ is called an I.L.B. Lie group modeled on an I.L.B.-system $\{\mathbf{E}^k\}$ if $\mathfrak{G}$ satisfies the following conditions:

(ILB. 1) Let $\mathbf{E}$ be the intersection of all $\mathbf{E}^k$ with the inverse limit topology. Then there exists an open neighborhood $\tilde{U}$ of the identity e of $\mathfrak{G}$ and a homeomorphism φ of the intersection of $\mathbf{E}$ and an open neighborhood U of 0 of $\mathbf{E}^0$ onto $\tilde{U}$, where the topology of $U \cap \mathbf{E}$ is the relative topology in $\mathbf{E}$.

(ILB. 2) There exists an open neighborhood V of 0 of $\mathbf{E}_0$ such that $\varphi(V \cap \mathbf{E}) = \varphi(V \cap \mathbf{E})^{-1}$, $\varphi(V \cap \mathbf{E})^2 \subset \varphi(U \cap \mathbf{E})$.

(ILB. 3) The map $\psi(u, v) = \varphi^{-1}(\varphi(u)\varphi(v))$ can be extended to a C^p -map of $(V \cap \mathbf{E}^k) \times (V \cap \mathbf{E}^{k+p})$ into $U \cap \mathbf{E}^k$ for every nonnegative p and k .

(ILB. 4) The map $\psi_u(v) = \psi(u, v)$ can be extended to a C^∞ -map of $V \cap \mathbf{E}^k$ into $U \cap \mathbf{E}^k$ for every fixed u in $V \cap \mathbf{E}^k$ and k .

(ILE. 5) The map $\zeta(w, u, v) = (d\psi_u)_v w$, can be extended to a C^p -map of $\mathbf{E}^{k+p} \times (V \cap \mathbf{E}^k) \times (V \cap \mathbf{E}^{k+p})$ into $\mathbf{E}^k$ for every k .

(ILB. 6) The map $\theta: V \cap \mathbf{E} \rightarrow V \cap \mathbf{E}$ defined by $\theta(u) = \varphi^{-1}(\varphi(u)^{-1})$ can be extended to a continuous map of $V \cap \mathbf{E}^k$ onto itself for every k .

(ILB. 7) For every g in $\mathfrak{G}$, there exists an open neighborhood W of 0 of $\mathbf{E}^0$ such that $g^{-1}\varphi(W \cap \mathbf{E})g \subset \varphi(U \cap \mathbf{E})$ and the map $\mu_g(u) = \varphi^{-1}(g^{-1}\varphi(u)g)$ can be extended to a C^∞ -map of $W \cap \mathbf{E}^k$ into $\mathbf{E}^k$ for every k .

3. An I.L.B. Lie group $\mathfrak{G}$.

A linear homeomorphism g of $\mathcal{S}$ onto itself is called a rotation of $\mathcal{S}$ if

$$\int_{-\infty}^{+\infty} |g\xi(x)|^2 dx = \int_{-\infty}^{+\infty} |\xi(x)|^2 dx,$$

for every ξ in $\mathcal{S}$ and $O(\mathcal{S})$ is the group of all rotations of $\mathcal{S}$. We define an I.L.B. Lie subgroup of $O(\mathcal{S})$ as follows.

For every u in $\mathcal{S}$, define a transformation $\hat{g}(u)$ on $\mathcal{S}$ by

$$\hat{g}(u): \xi(x) \rightarrow \xi\left(\int_0^x \exp u(y) dy\right) \exp \frac{1}{2}u(x).$$

Then the collection

$$\mathfrak{G} = \{\hat{g}(u); u \in \mathcal{S}\}$$

is shown to be a subgroup of $O(\mathcal{S})$ (H. Sato [2]). Since the correspondence between u and $\hat{g}(u)$ is bijective, we introduce the topology of $\mathcal{S}$ into $\mathfrak{G}$ through the map g . Then $\mathfrak{G}$ becomes a topological group (H. Sato [2]). Considering $\mathcal{S}$ is a countably normed space with an increasing sequence of norms

$$|\xi|_k = \sup_{0 \leq \nu \leq k} \sup_{-\infty < x < +\infty} (1+x^2)^k |\xi^{(\nu)}(x)|,$$

$k=0, 1, 2, \dots$, we define an I.L.B.-system $\{\mathcal{S}^k; k=0, 1, 2, \dots\}$ where $\mathcal{S}^k$ is the $|\cdot|_k$ -completion of $\mathcal{S}$.

THEOREM 1. The topological group $\mathfrak{G}$ is an I.L.B. Lie group modeled on the I.L.B.-system $\{\mathcal{S}^k\}$.

THEOREM 2. The corresponding bracket product on $\mathcal{S}$, which is the tangent space of the identity of $\mathfrak{G}$, is given by

$$[u, v] = F_u v' - u' F_v,$$

for every u and v in $\mathcal{S}$, where

$$F_u(x) = \int_0^x u(y) dy, \quad -\infty < x < +\infty.$$

The explicit formula of the exponential map is the same as in [2]. To begin with, we define a function $h(t, x; u)$ as follows. For every u in $\mathcal{S}$, define

$$A = \{x \in \mathbf{R}^1; F_u(x) \neq 0\}.$$

Then A is an at most countable union of open intervals

$$A = \bigcup_n (a_n, b_n).$$

Define a function η_n on each interval (a_n, b_n) by

$$\eta_n(x) = \int_{c_n}^x \frac{dy}{F_u(y)}, \quad a_n < x < b_n,$$

where c_n is an arbitrary fixed point in (a_n, b_n) and define

$$h(t, x; u) = \begin{cases} x, & x \notin A, \quad -\infty < t < +\infty, \\ \eta_n^{-1}(\eta_n(x) + t), & a_n < x < b_n, \quad -\infty < t < +\infty. \end{cases}$$

THEOREM 3. The exponential map of the tangent space $\mathcal{S}$ into $\mathfrak{G}$ is given by

$$\exp tu = \hat{g}\left(\int_0^t u(h(r, x; u)) dr\right), \quad -\infty < t < +\infty,$$

for every u in $\mathcal{S}$.

4. Proof of theorems.

To prove Theorem 1, we check the conditions (ILB. 1)–(ILB. 7) in Section 2.

(ILB. 1) The intersection of all $\mathcal{S}^k$ with the inverse limit topology is the nuclear space of $\mathcal{S}$. Let $\tilde{U}$ be $\mathfrak{G}$, U be $\mathcal{S}^0$ and φ be $\hat{g}$. Then by the definition of the topology of $\mathfrak{G}$, $\hat{g}$ is a homeomorphism of $U \cap \mathcal{S} = \mathcal{S}^0 \cap \mathcal{S} = \mathcal{S}$ onto $\tilde{U} = \mathfrak{G}$.

(ILB. 2) Let V be $\mathcal{S}^0$. Then we have

$$\begin{aligned} \hat{g}(V \cap \mathcal{S}) &= \hat{g}(\mathcal{S}^0 \cap \mathcal{S}) = \hat{g}(\mathcal{S}) = \mathfrak{G} \\ &= \mathfrak{G}^{-1} = \hat{g}(V \cap \mathcal{S})^{-1}, \\ \hat{g}(V \cap \mathcal{S}^0)^2 &= \hat{g}(\mathcal{S})^2 = \mathfrak{G}^2 = \mathfrak{G} = \hat{g}(\mathcal{S}^0 \cap \mathcal{S}). \end{aligned}$$

(ILB. 3) For every u and v in $\mathcal{S}$, we define

$$\begin{aligned} \psi(u, v) &= \hat{g}^{-1}(\hat{g}(u)\hat{g}(v)) \\ &= u + v \circ f_u \end{aligned}$$

where

$$f_u(x) = \int_0^x \exp u(y) dy, \quad -\infty < x < +\infty.$$

Then it is shown in [2] that ψ is a continuous map from $\mathcal{S} \times \mathcal{S}$ into $\mathcal{S}$. Since it is obvious that the map $(u, v) \rightarrow u$ can be extended to a C^∞ -map of $\mathcal{S}^k \times \mathcal{S}^{k+p}$ to $\mathcal{S}^k$, we have only to prove that the map

$$\gamma(u, v) = \phi(u, v) - u = v \circ f_u$$

can be extended to a C^p -map of $\mathcal{S}^k \times \mathcal{S}^{k+p}$ to $\mathcal{S}^k$. We prove it for $p=0$ and positive p separately.

PROPOSITION 1. For every non-negative integer k , the map γ can be extended to a continuous map of $\mathcal{S}^k \times \mathcal{S}^k$ to $\mathcal{S}^k$.

To prove Proposition 1, we utilise the estimations in [2]. For every ξ in $\mathcal{S}$ and every non-negative integers n and p , define

$$\|\xi\|_{np} = \sup_{-\infty < x < +\infty} |x^p \xi^{(n)}(x)|.$$

Then the norm $\|\xi\|_k$ is equivalent to the norm defined by

$$\sup_{0 \leq n \leq k} \sup_{0 \leq p \leq 2k} \|\xi\|_{np}, \quad k=0, 1, 2, \dots$$

In accordance with the estimations in the proofs of Proposition 7, Lemma 9 and Lemma 11 in [2], we can easily prove the proposition.

PROPOSITION 2 For every non-negative integer k and every positive integer p , the map γ can be extended to a C^p -map of $\mathcal{S}^k \times \mathcal{S}^{k+p}$ to $\mathcal{S}^k$.

To prove Proposition 2, we use the following lemmas.

LEMMA 1. For every u, u_1 in $\mathcal{S}^0$, v in $\mathcal{S}^p$ and x in $\mathbf{R}^1$,

$$\gamma(u + tu_1, v)(x) = v\left(\int_0^{xu(y)+tu_1(y)} e^{dy}\right), \quad t \in \mathbf{R}^1,$$

is p -times continuously differentiable in t and we have

$$\begin{aligned} & \frac{d^q}{dt^q} \gamma(u + tu_1, v)(x) \Big|_{t=0} \\ &= \sum_{\nu=1}^q \sum_{\substack{0 \leq i_1, i_2, \dots, i_q \leq q \\ \sum_{j=1}^p j i_j = q}} c_{i_1 i_2 \dots i_p}^{(\nu)} (df_u(u_1))^{i_1} (d^2 f_u(u_1))^{i_2} \\ & \dots (d^q f_u(u_1))^{i_q} \gamma(u, v^{(\nu)})(x), \quad q=0, 1, \dots, p, \end{aligned}$$

where $c_{i_1 i_2 \dots i_p}^{(\nu)}$'s are real coefficients defined independently of the choice of u, u_1 and v and

$$= d^\nu f_u(u_1) \int_0^x e^{u(y)} u_1(y)^\nu dy, \quad x \in \mathbf{R}^1, \quad \nu=0,1,2,\dots,p$$

PROOF. This lemma is proved easily by mathematical induction with respect to p .

PROOF OF PROPOSITION 2. For every u, u_1 in $\mathcal{S}^k$ and v, v_1 in $\mathcal{S}^{k+p}$, by Leibnitz' formula and Lemma 1 we have

$$\begin{aligned} & d^q \gamma((u, v), (u_1, v)) \\ &= \frac{d^q}{u t^q} \gamma(d + t u_1, v + t v_1) \Big|_{t=0} \\ &= \frac{d^q}{d t^q} \gamma(u + t u_1, v) \Big|_{t=0} + q \frac{d^{q-1}}{d t^{q-1}} \gamma(u + t u_1, v_1) \Big|_{t=0} \\ &= \sum_{\nu=1}^q \left(\sum_{\substack{0 \leq i_1, i_2, \dots, i_q \leq q \\ \sum_{j=1}^p j i_j = q}} c_{i_1 i_2 \dots i_q}^{(\nu)} (d f_u(u_1))^{i_1} (d^2 f_u(u_1))^{i_2} \right. \\ &\quad \left. \dots (d^q f_u(u_1))^{i_q} \gamma(u, v^{(\nu)}) \right. \\ &\quad \left. + q \sum_{\mu=1}^{q-1} \left(\sum_{\substack{0 \leq i_1, i_2, \dots, i_{q-1} \leq q-1 \\ \sum_{j=1}^{q-1} j i_j = q-1}} c_{i_1 i_2 \dots i_{q-1}}^{(\mu)} (d f_u(u_1))^{i_1} (d^2 f_u(u_1))^{i_2} \right. \right. \\ &\quad \left. \left. \dots (d^{q-1} f_u(u_1))^{i_{q-1}} \gamma(u, v_1^{(\mu)}) \right) \right) \quad q=1, 2, 3, \dots, p. \end{aligned}$$

We shall prove that $d^q \gamma$, $q=1, 2, \dots, p$ are derivatives of γ .

It is obvious that $d^q \gamma$ is q -linear in $(u_1, v_1) \in \mathcal{S}^k \times \mathcal{S}^{k+p}$.

In order to show the continuity of $d^q \gamma$, it is enough to show that of

$$\Gamma(u, u_1, v) = d f_u(u_1)^{i_1} d^2 f_u(u_1)^{i_2} \dots d^p f_u(u_1)^{i_p} \gamma(u, v^{(\mu)})$$

and $\Gamma(u, u_1, v_1)$ where $\sum_{j=1}^p j i_j \leq p$ and $1 \leq \nu \leq p$. Using the fact that

$$\begin{aligned} & \frac{d^k}{d x^k} \Gamma(u, u_1, v) \\ &= \sum_{j=0}^k \left(\sum_{\substack{0 \leq j_1, j_2, \dots, j_p \leq p \\ \sum_{l=1}^p l j_l \leq p}} Q_{j_1 \dots j_p}^{j_1 \dots j_p} [u_1, u'_1, \dots, u_1^{(k-1)}, e^u, \right. \end{aligned}$$

$$u', \dots, u^{(k-1)}] df_u(u_1)^{j_1} d^2 f_u(u_1)^{j_2} \dots d^b f_u(u_1)^{j_p} \gamma(u, v^{(\nu+j)})$$

where $Q_j^{j_1 j_2 \dots j_p}$ are polynomials of $(u_1, u'_2, \dots, u_1^{(k-1)}, e^u, u', u'' \dots, u^{(k-1)})$ defined independently of the choice of (u, u_1, v) and the estimation

$$\begin{aligned} & |d^l f_u(u_1)^{i_l}| \\ &= \left| \left(\int_0^x e^u(y) u^1(y)^l dy \right)^{i_l} \right| \\ &\leq |x|^{i_l} e^{|u|_0} |u_1|_0 \end{aligned}$$

we can prove the continuity of $d^q \gamma$ in $((u, v), (u_1, v_1)) \in \mathcal{S}^k \times \mathcal{S}^{k+p} \times (\mathcal{S}^k \times \mathcal{S}^{k+p})$, $q=1, 2, 3, \dots, p$. Furthermore, we can prove the continuity of $d^q \gamma$ in (u, v) uniformly over $\mathbf{B}^q$ where $\mathbf{B} = \{(u_1, v_1); |u_1|_k + |v_1|_{k+p} = 1\}$.

It remains to prove that $d^q \gamma$ is the true q -th derivative of γ . We prove it by mathematical induction with respect to q ; $0 \leq q \leq p$. In the case of $q=0$, $d^0 \gamma = \gamma$ is continuous by Proposition 1. Assume that it is true for $q=0, 1, 2, \dots, r$, ($r \leq p-1$). Then, considering the fact that $d^{r+1} \gamma$ is an $(r+1)$ -linear map of $(\mathcal{S}^k \times \mathcal{S}^{k+p})^{r+1}$ into $\mathcal{S}^k$ for every fixed (u, v) and Taylor's formula, we have

$$\begin{aligned} & \sup_{|u_1|_k + |v_1|_{k+p} = 1} |d^r \gamma((u, v) + (u_0, v_0), (u_1, v_1)) \\ & - d^r \gamma((u, v), (u_1, v_1)) - d^{r+1}((u, v); (u_0, v_0), (u_1, v_1), \\ & \dots, (u_1, v_1))|_k \\ &= \sup_{|u_1|_k + |v_1|_{k+p} = 1} \left| \int_0^1 d^{r+1} \gamma((u, v) + s(u_0, v_0); (u_0, v_0), (u_1, v_1), \right. \\ & \quad \left. \dots, (u_1, v_1)) ds - d^{r+1} \gamma((u, v); (u_0, v_0), (u_1, v_1), \dots, (u_1, v_1)) \right|_k \\ & \leq (|u_0|_k + |v_0|_{k+p}) \sup_{|u_1|_k + |v_1|_{k+p} = 1} |d^{r+1} \gamma((u, v) + s(u_0, v_0); \\ & \quad \frac{(u_0, v_0)}{|u_0|_k + |v_0|_{k+p}}, (u_1, v_1), \dots, (u_1, v_1))|_k = o(|u_0|_k + |v_0|_{k+p}). \end{aligned}$$

Thus we have proved Proposition 2 and consequently (ILB. 3).

(ILB. 4) By a simple calculation we have

$$(d\psi_u)_v(v_1) = v_1 \circ f_u = \gamma(u, v_1)$$

and since it is independent of v , $\psi_u(v) = \phi(u, v)$ is a C^∞ -map of $\mathcal{S}^k$ into $\mathcal{S}^k$.

(ILB. 5) By the above calculation we have

$$\zeta(w, u, v) = (d\psi_u)_v w = \gamma(u, w)$$

which is a C^p -map of $\mathcal{S}^{k+p} \times \mathcal{S}^k \times \mathcal{S}^{k+p}$ into $\mathcal{S}^k$ for every k .

(ILB. 6) Reformulating the proof of Proposition 7 in [2], we can prove

(ILB. 6) without any difficulty.

(ILB. 7) Let $\mathcal{W}$ be $\mathcal{S}^0$. Then we have

$$g^{-1}\hat{g}(\mathcal{W} \cap \mathcal{S})g = g^{-1}\mathfrak{G}g = \mathfrak{G} = \hat{g}(\mathcal{S}^0 \cap \mathcal{S}).$$

Since g is in $\mathfrak{G}$, there exists u_0 in $\mathcal{S} = \mathcal{S}^\infty$ such that $g = g(u_0)$. By (ILB. 3) and (ILB. 4), the map

$$\mu_g(u) = \phi(u_0^{-1}, \phi(u, u_0)) = \phi_{u_0}^{-1}(\phi(u, u_0))$$

is extended to a C^∞ -map of $\mathcal{S}^k$ into $\mathcal{S}^k$ for every k .

Thus we have proved Theorem 1.

PROOF OF THEOREM 2. We estimate the Lie bracket in the tangent space $\mathcal{S}$ of the identity e of $\mathfrak{G}$. Let L_g be the left translation of $\mathfrak{G}$, that is,

$$L_g g' = gg', \quad g, g' \in \mathfrak{G}.$$

Then we have a vector field $\tilde{v}_g$ over $\mathfrak{G}$;

$$\begin{aligned} \tilde{v} = \tilde{v}_g &= d(L_g)_0(v) = d(g^{-1}(g(u)g(0)))(v) \\ &= d(\psi_u)_0(v) = \zeta(v, u, 0) \\ &= v \circ f_u = v \circ f_{\hat{g}^{-1}(g)}, \end{aligned}$$

where $u = g^{-1}(g)$. Furthermore the differential $d(\tilde{v})_0(u)$ is given by

$$d(\tilde{v})_0(u) = \frac{d}{dt} v \circ f_{tu} \big|_{t=0}$$

$$\begin{aligned}
&= \frac{d}{dt} v \left(\int_0^x \exp tu(y) dy \right) \Big|_{t=0} \\
&= v'(x) \int_0^x u(y) dy \\
&= F_u v'.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
[u, v] &= d(\tilde{v})_0(u) - d(\tilde{u})_0(v) \\
&= F_u v' - u' F_v, \quad u, v \in \mathcal{S}
\end{aligned}$$

(see [3] p. 76).

PROOF OF THEOREM 3. The exponential map is defined as the integral curve of the vector field $\tilde{u}$ with initial condition e . For every u in $\mathcal{S}$, put

$$\exp tu = \hat{g} \left(\int_0^t u(h(r, \cdot; u)) dr \right), \quad -\infty < t < +\infty.$$

Then we have

$$\begin{aligned}
d(\exp)_0 u &= \frac{d}{dt} \hat{g}^{-1}(\exp tu) \Big|_{t=0} \\
&= u(h(0, \cdot; u)) = u(\cdot)
\end{aligned}$$

([2], Lemma 3) and this proves the theorem ([3] p. 76).

References

- [1] H. SATO: *A family of one-parameter subgroups of $O(\mathcal{S}_r)$ arising from the variable change of the WhiteNoise*. Publ. R.I.M.S., Kyoto Univ., Vol. 5 (1969) pp. 165–191.
- [2] H. SATO: *One-parameter subgroups and a Lie subgroup of an infinite dimensional rotation group*. J. Math. Kyoto Univ., Vol. 11–2 (1971) pp. 253–300.
- [3] H. OMORI: *Theory of infinite dimensional Lie group*. Lecture note at Tokyo Metropolitan Univ., (1972). (in Japanese)
- [4] S. LANG: *Differential Manifolds*. Addison-Wesley, (1972).
- [5] J. DIEUDONNE: *Foundations of Modern Analysis*. Academic Press, N.Y., (1960).