RELATIONS IN CATEGORIES WITH PULLBACKS

By

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Introduction

Many authors considered relations for abelian categories (S. MacLane [7], D. Puppe [9], P.J. Hilton [5]) and for exact categories (M.S. Calenko [2], H.-B. Brinkmann [1]). In the present paper we will try to define relations in categories with pullbacks by pairs of morphisms and study their properties. The main results are Theorem A, B and C below. This paper is self-contained.

We now recall the definition of an *I*-category (a category with involution), introduced by D. Puppe [9]. In the definition of categories we do not require that the morphisms from one object to another form a set. A category $\mathcal{A}$ is called an *I*-category if it satisfies the following axioms:

- (Ia) The class $\mathcal{A}(A, B)$ of morphisms from A to B is partially ordered by $\subset$ for any objects A, B in $\mathcal{A}$.
- (Ib) If $\alpha, \alpha' \in \mathcal{A}(A, B)$, $\beta \in \mathcal{A}(B, C)$ and $\alpha \subset \alpha'$, then $\alpha\beta \subset \alpha'\beta$.
- (Ic) For any objects A, B in $\mathcal{A}$, there is a mapping

$$\#: \mathcal{A}(A, B) \longrightarrow \mathcal{A}(B, A),$$

called an *involution* such that

$$(c.1) \quad (\alpha\beta)^\# = \beta^\# \alpha^\#$$

$$(c.2) \quad \alpha^{\#\#} = \alpha \quad \text{and}$$

$$(c.3) \quad \alpha_1 \subset \alpha_2 \quad \text{implies} \quad \alpha_1^\# \subset \alpha_2^\#$$

(for any morphisms $\alpha, \alpha_1, \alpha_2 \in \mathcal{A}(A, B)$, $\beta \in \mathcal{A}(B, C)$).

Let $\mathcal{C}$ be a category and let $\mathcal{A}$ be an *I*-category. A functor $G: \mathcal{C} \rightarrow \mathcal{A}$ is called a *relational functor* if and only if it satisfies the following

conditions:

(RF. 1) For each morphism $f \in \mathcal{C}$, $G(f)^*G(f) \subset \Delta$,
where Δ is an identity in $\mathcal{A}$.

(RF. 2) If $\begin{array}{ccc} & x & \\ y \downarrow & \square & \downarrow f \\ & g & \end{array}$ is a pullback in $\mathcal{C}$, then $G(f)G(g)^* = G(x)^*G(y)$.

THEOREM A. *Let $\mathcal{C}$ be a category with pullbacks and let $\mathcal{E}$ be a retractive subcategory (cf. §1) of $\mathcal{C}$. Then we can construct an I -category $Rel(\mathcal{C}; \mathcal{E})$, called the category of relations in $\mathcal{C}$ relative to $\mathcal{E}$, and a relational embedding $\Gamma: \mathcal{C} \rightarrow Rel(\mathcal{C}; \mathcal{E})$, called the graphic embedding of $\mathcal{C}$ relative to $\mathcal{E}$.*

An I -functor of I -categories is a functor between I -categories which preserves partial orders and commutes with involutions.

THEOREM B. *Let $\mathcal{C}$ be a category with pullbacks, $\mathcal{E}$ a retractive subcategory of $\mathcal{C}$ and $G: \mathcal{C} \rightarrow \mathcal{A}$ a relational functor.*

Then the following statements (a) and (b) are equivalent:

- (a) *For each morphism $f \in \mathcal{E}$, $G(f)^*G(f) = \Delta$.*
- (b) *There is a (unique) I -functor $\tilde{G}: Rel(\mathcal{C}; \mathcal{E}) \rightarrow \mathcal{A}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Gamma} & Rel(\mathcal{C}; \mathcal{E}) \\ G \searrow & & \swarrow \tilde{G} \\ & \mathcal{A} & \end{array}$$

is commutative.

For a relational functor $G: \mathcal{C} \rightarrow \mathcal{A}$, consider the following axioms:

(REL. 1) If $\Delta \subset \alpha G(f)$ in $\mathcal{A}$ for some $\alpha \in \mathcal{A}$, then $G(f)^*G(f) = \Delta$.

(REL. 2) (Universality) For each relational functor $H: \mathcal{C} \rightarrow \mathcal{B}$ such that $G(f)^*G(f) = \Delta$ in $\mathcal{A}$ implies $H(f)^*H(f) = \Delta$ in $\mathcal{B}$, there exists a unique I -functor $\tilde{H}: \mathcal{A} \rightarrow \mathcal{B}$ which makes the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{A} \\ H \searrow & & \swarrow \tilde{H} \\ & \mathcal{B} & \end{array}$$

commutative.

Theorem B and Proposition 4.3 show that the graphic embedding $\Gamma: \mathcal{C} \rightarrow \text{Rel}(\mathcal{C}; \mathcal{E})$ satisfies the above axioms. Conversely, the axioms (REL. 1) and (REL. 2) characterize graphic embeddings as follows.

THEOREM C. *Let $\mathcal{C}$ be a category with pullbacks. If a relational embedding $G: \mathcal{C} \rightarrow \mathcal{A}$ satisfies the axioms (REL. 1) and (REL. 2), then there exist a unique retractive subcategory $\mathcal{E}$ of $\mathcal{C}$ and a unique isomorphism $\mathcal{A} \approx \text{Rel}(\mathcal{C}; \mathcal{E})$ of I-categories such that the diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Gamma} & \text{Rel}(\mathcal{C}; \mathcal{E}) \\ & \searrow G & \downarrow \\ & & \mathcal{A} \end{array}$$

is commutative.

§1 and §2 are preliminary sections. In §1, the definition of retractive subcategories is described precisely. For spans in a category, namely, pairs of morphisms with common domains, their compositions by pullbacks and a partial order on them concerning a retractive subcategory are investigated in §2.

In §3, we consider the construction of relations and we prove the first half of Theorem A. Here relations in the category relative to the retractive subcategory are constructed by classifying the spans with the smallest equivalence relation induced by the above partial order. The construction of relations can be dualized with no difficulty.

In §4, the elementary properties of relations, as in the classical case, will be proved (cf. Formula 4.6) and consequently, we will obtain the latter half of the proof of Theorem A. In §5, the induced morphism theorem (cf. Theorem 5.1) will be shown, as a simple application of relations. And we give a necessary and sufficient condition for proper relations to be morphisms (cf. Theorem 5.2). §6 is devoted to the proofs of Theorem B and Theorem C. In §7, we will define the notions of $\mathcal{E}$ -subobjects and $\mathcal{E}$ -decompositions. Here it is shown that the relations in a category relative to a retractive subcategory are subobjects relative to the retractive subcategory if the category has products and we give sufficient conditions for the class of relations among two objects to be a set.

In §8, the examples of relations are given and we discuss the com-

parison between the classical relations and this theory.

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§ 1. Retractive Subcategories

Throughout the rest of the paper, we will assume that $\mathcal{C}$ is a category with pullbacks. The class of epimorphisms (respectively, monomorphisms and isomorphisms) of $\mathcal{C}$ is denoted by $Epi(\mathcal{C})$ (respectively, $Mon(\mathcal{C})$ and $Iso(\mathcal{C})$). A subcategory $\mathcal{E}$ of $\mathcal{C}$ is called retractive if and only if it satisfies the following three axioms:

$$(E.1) \quad Iso(\mathcal{C}) \subset \mathcal{E} \subset Epi(\mathcal{C}).$$

$$(E.2) \quad \text{If } fg \in \mathcal{E}, \text{ then } g \in \mathcal{E}.$$

$$(E.3) \quad (\text{Pullback Axiom}) \quad \text{If } \begin{array}{ccc} & \xrightarrow{\hat{x}} & \\ \hat{f} \downarrow & & \downarrow f \\ & \xrightarrow{x} & \end{array} \text{ is a pullback in } \mathcal{C} \text{ and } f \in \mathcal{E}, \\ \text{then } \hat{f} \in \mathcal{E}.$$

As easily seen, the class of retractions in $\mathcal{C}$, denoted by $Ret(\mathcal{C})$, is a retractive subcategory of $\mathcal{C}$. For each retractive subcategory $\mathcal{E}$ of $\mathcal{C}$, identities in $\mathcal{C}$ are epimorphisms in $\mathcal{E}$ by (E.1) and therefore the class $Ret(\mathcal{C})$ of retractions in $\mathcal{C}$ is included in $\mathcal{E}$ by (E.2). So we have proved the following

PROPOSITION 1.1. *$Ret(\mathcal{C})$ is the smallest retractive subcategory in any category $\mathcal{C}$.*

PROPOSITION 1.2. *Let $\mathcal{S}$ be a subcategory of $\mathcal{C}$ satisfying the axioms (E.1) and (E.2). Define $\mathcal{S}^*$ to be the class of all epimorphisms $f \in \mathcal{S}$ such that any pullback $\begin{array}{ccc} & \xrightarrow{\hat{x}} & \\ \hat{f} \downarrow & & \downarrow f \\ & \xrightarrow{x} & \end{array}$ implies $\hat{f} \in \mathcal{S}$. Then $\mathcal{S}^*$ is a retractive subcategory of $\mathcal{C}$.*

PROOF. The proofs of (E.1) and (E.2) for $\mathcal{S}^*$ are clear from the pullback properties and omitted. For the proof of (E.3), let $f \in \mathcal{S}^*$ and let $\hat{x}f = \hat{f}x$ be a pullback. Then it is obvious that $\hat{f} \in \mathcal{S}$. We must show that $\hat{f} \in \mathcal{S}^*$. Assume that $\hat{y}f = gy$ is a pullback, then $(\hat{y}\hat{x})f = g(yx)$ is also a pullback and therefore $g \in \mathcal{S}$ since $f \in \mathcal{S}^*$. Hence we have $\hat{f} \in \mathcal{S}^*$.

More precisely, $\mathcal{S}^*$ obtained in the above proposition is the largest

retractive subcategory contained in $\mathcal{S}$ and we have

COROLLARY 1.3. *$Epi^*(\mathcal{C})$ is the largest retractive subcategory in any category $\mathcal{C}$.*

PROOF. It easily follows since the axioms (E.1) and (E.2) are always valid for $Epi(\mathcal{C})$.

§2. Spans in Categories

Let $\mathcal{C}$ be a (fixed) category with pullbacks and $\mathcal{E}$ a (fixed) retractive subcategory of $\mathcal{C}$ throughout the rest of the paper.

Let A, B be two objects in $\mathcal{C}$. A *span* α from A to B is an ordered pair (f, g) of morphisms in $\mathcal{C}$ such that

and a span (sf, sg) is denoted by $s^*\alpha$ (or $s^*(f, g)$). Remark that we identify $\alpha = (f, g)$ with $i^*\alpha = (if, ig)$ for any $i \in Iso(\mathcal{C})$. The class of spans from A to B is denoted by $Spn(A, B)$.

For $\alpha = (a_1, a_2) \in Spn(A, B)$ and $\beta = (b_1, b_2) \in Spn(B, C)$, the composition $\alpha\beta \in Spn(A, C)$ is defined by the pair (xa_1, yb_2) , where in the following diagram

the square (1) is a pullback.

A span $(1_A, 1_A)$ from A to A is denoted by Δ_A , or simply by Δ . It is clear that $\alpha\Delta_B = \alpha$ and $\Delta_A\alpha = \alpha$ for any $\alpha \in Spn(A, B)$.

PROPOSITION 2.1. (*Associativity*) For $\alpha \in \text{Spn}(A, B)$, $\beta \in \text{Spn}(B, C)$ and $\gamma \in \text{Spn}(C, D)$, we have $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

PROOF. Let $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ and $\gamma = (c_1, c_2)$. Consider the following diagram

in which the commutative squares (1), (2) and (3) are pullbacks. As the commutative square obtained by juxtaposing (2) and (3) is also a pullback, we have $(\alpha\beta)\gamma = (x_2x_1a_1, y_2y_1c_2)$. Similarly, we have $\alpha(\beta\gamma) = (x_2x_1a_1, y_2y_1c_2)$. This completes the proof.

For $\alpha, \alpha' \in \text{Spn}(A, B)$, we define $\alpha < \alpha'$ if and only if there exist $e \in \mathcal{E}$ and $s \in \mathcal{C}$ such that $e^*\alpha = s^*\alpha'$. Since identities are epimorphisms in $\mathcal{E}$, it is obvious that $\alpha < \alpha$ for any $\alpha \in \text{Spn}(A, B)$.

PROPOSITION 2.2. Let α, α' and $\alpha'' \in \text{Spn}(A, B)$. If $\alpha < \alpha'$ and $\alpha' < \alpha''$, then $\alpha < \alpha''$.

PROOF. Let $e^*\alpha = s^*\alpha'$ and $d^*\alpha' = t^*\alpha''$ for $e, d \in \mathcal{E}$ and $s, t \in \mathcal{C}$. Now construct a pullback

of s and d , then $(xe)^*\alpha = (xs)^*\alpha' = (yd)^*\alpha' = (yt)^*\alpha''$. From the pullback axiom, $x \in \mathcal{E}$ and so $xe \in \mathcal{E}$. Hence we have $\alpha < \alpha''$, which completes the proof.

LEMMA 2.3. *Let $\alpha \in \text{Spn}(A, B)$, $\beta \in \text{Spn}(B, C)$ and let s be a morphism in $\mathcal{C}$. If $(s^*\alpha)\beta$ is defined, then there exists a morphism $\hat{s} \in \mathcal{C}$ such that $(s^*\alpha)\beta = \hat{s}^*(\alpha\beta)$. If $s \in \mathcal{E}$ in particular, then $\hat{s} \in \mathcal{E}$.*

PROOF. Let $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$. Observing the following diagram

where the squares (1) and (2) are pullbacks, one can easily verify that $(s^*\alpha)\beta = \hat{s}^*(\alpha\beta)$. If $s \in \mathcal{E}$, then $\hat{s} \in \mathcal{E}$ by virtue of (E.3), and the proof of this lemma is complete.

COROLLARY 2.4. *Let $\alpha, \alpha' \in \text{Spn}(A, B)$ and $\beta \in \text{Spn}(B, C)$. If $\alpha < \alpha'$, then $\alpha\beta < \alpha'\beta$.*

PROOF. For the proof, let $e^*\alpha = s^*\alpha'$ for $e \in \mathcal{E}$ and $s \in \mathcal{C}$. By Lemma 2.3, there exist $\hat{e} \in \mathcal{E}$ and $\hat{s} \in \mathcal{C}$ such that $(e^*\alpha)\beta = \hat{e}^*(\alpha\beta)$ and $(s^*\alpha')\beta = \hat{s}^*(\alpha'\beta)$. Then we have $\hat{e}^*(\alpha\beta) = \hat{s}^*(\alpha'\beta)$ and hence $\alpha\beta < \alpha'\beta$.

For $\alpha = (a_1, a_2) \in \text{Spn}(A, B)$, we define $\alpha^* \in \text{Spn}(B, A)$ by (a_2, a_1) . A simple inspection of the definitions shows the following

PROPOSITION 2.5.

$$(c'.1) \quad (\alpha\beta)^* = \beta^*\alpha^*.$$

$$(c'.2) \quad \alpha^{**} = \alpha.$$

$$(c'.3) \quad \alpha_1 < \alpha_2 \Rightarrow \alpha_1^* < \alpha_2^*.$$

Combining Corollary 2.4. and Proposition 2.5, we have

PROPOSITION 2.6. *If $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, then $\alpha_1\beta_1 < \alpha_2\beta_2$.*

PROOF. $\alpha_1\beta_1 < \alpha_2\beta_1 = (\beta_1^*\alpha_2^*)^* < (\beta_2^*\alpha_2^*)^* = \alpha_2\beta_2$.

§3. Relations in Categories

Let $\alpha, \alpha' \in \text{Spn}(A, B)$. We define $\alpha \sim \alpha'$ if and only if $\alpha < \alpha'$ and $\alpha' < \alpha$. By Proposition 2.2, the relation " $\sim$ " is an equivalence relation on $\text{Spn}(A, B)$. For example, if $e^*\alpha = e'^*\alpha'$ for $e, e' \in \mathcal{E}$, then $\alpha \sim \alpha'$. The equivalence class of $\alpha = (a_1, a_2) \in \text{Spn}(A, B)$ is called a *relation from A to B relative to $\mathcal{E}$* and denoted by $[\alpha] = [a_1, a_2]$. The class of relations from A to B in $\mathcal{E}$ is denoted by $\text{Rel}_{\mathcal{E}}(A, B)$ or $\text{Rel}(A, B)$.

For two relations $\alpha = [\alpha] \in \text{Rel}(A, B)$ and $\beta = [\beta] \in \text{Rel}(B, C)$, we define the composition $\alpha\beta \in \text{Rel}(A, C)$ by $[\alpha\beta]$. By Corollary 2.6, the composition is well-defined. Moreover, the associativity $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ of relations can be easily seen from Proposition 2.1.

The equivalence class of $\mathbf{1}_A = (1_A, 1_A)$ is denoted by $\mathbf{1}_A$ or simply $\mathbf{1}$. Now by the notation above, $\alpha\mathbf{1}_B = \alpha$ and $\mathbf{1}_A\alpha = \alpha$ hold for any $\alpha \in \text{Rel}(A, B)$. Thus one obtains a category $\text{Rel}(\mathcal{E}; \mathcal{E})$ with the same objects as $\mathcal{E}$ and with the relations as its morphisms. The category $\text{Rel}(\mathcal{E}; \mathcal{E})$ is called *the category of relations in $\mathcal{E}$ relative to $\mathcal{E}$* . Next we introduce a partial order into $\text{Rel}(A, B)$, namely, for $\alpha = [\alpha]$, $\alpha' = [\alpha'] \in \text{Rel}(A, B)$, define $\alpha < \alpha'$ if and only if $\alpha < \alpha'$. This definition is independent of the choice of representatives in view of Proposition 2.2.

PROPOSITION 3.1. *If $\alpha < \alpha'$, then $\alpha\beta < \alpha'\beta$.*

PROOF. It is immediate from Corollary 2.4.

For $\alpha = [\alpha] \in \text{Rel}(A, B)$, define $\alpha^* = [\alpha^*] \in \text{Rel}(B, A)$, then from Proposition 2.5 we have immediately

PROPOSITION 3.2.

$$(c.1) \quad (\alpha\beta)^* = \beta^*\alpha^*.$$

$$(c.2) \quad \alpha^{**} = \alpha.$$

$$(c.3) \quad \alpha_1 < \alpha_2 \Rightarrow \alpha_1^* < \alpha_2^*.$$

Thus we have proved the following

THEOREM 3.3. *The category $Rel(\mathcal{C}; \mathcal{E})$ of relations in $\mathcal{C}$ relative to $\mathcal{E}$ is an I -category.*

It is not difficult to verify

PROPOSITION 3.4.

$$(a) \quad \mathbf{A}_A^* = \mathbf{A}_A.$$

$$(b) \quad \alpha_1 \subset \alpha_2 \text{ and } \beta_1 \subset \beta_2 \Rightarrow \alpha_1 \beta_1 \subset \alpha_2 \beta_2.$$

For $f \in \mathcal{C}(A, B)$, define $\mathbf{I}(f) = [1_A, f] \in Rel(A, B)$, and $\mathbf{I}(f)$ is called *the graph* of a morphism f . Since $\mathbf{I}(1_A) = [1_A, 1_A] = \mathbf{A}_A$ and

$$\mathbf{I}(f)\mathbf{I}(g) = [1_A, f][1_B, g] = [1_A, fg] = \mathbf{I}(fg)$$

for $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, we obtain a (covariant) functor $\mathbf{I}: \mathcal{C} \rightarrow Rel(\mathcal{C}; \mathcal{E})$, called *the graphic embedding of $\mathcal{C}$ relative to $\mathcal{E}$* .

To prove Theorem A, it suffices to show that the graphic embedding $\mathbf{I}$ is actually an embedding and a relational functor, which will be proved in Proposition 4.2.

As a consequence of Theorem A, we have the following

COROLLARY 3.4. *Every category with pullbacks can be embedded into an I -category.*

PROOF. It is clear since $Ret(\mathcal{C})$, $Epi^*(\mathcal{C})$ and $Ext^*(\mathcal{C})$ are retractive subcategories of any category $\mathcal{C}$ with pullbacks. (Where, $Ext(\mathcal{C})$ denotes the class of *extremal epimorphisms* [4] in $\mathcal{C}$.)

§4. The Elementary Properties of Relations and Proof of THEOREM A.

First we will prove that every relation has a canonical factorization as the inverse of a morphism followed by a morphism.

LEMMA 4.1. *For any $\alpha = [f, g] \in Rel(\mathcal{C}; \mathcal{E})$, $\alpha = [f, g] = \mathbf{I}(f)^* \mathbf{I}(g)$.*

PROOF. Noticing that the square (1) is a pullback in a diagram

we have $[f, 1][1, g] = [f, g]$ and so

$$\mathbf{I}(f) * \mathbf{I}(g) = [1, f] * [1, g] = [f, 1][1, g] = [f, g],$$

which completes the proof.

The completion of the proof of Theorem A is essentially due to the following

PROPOSITION 4.2.

- (a) If $\begin{array}{ccc} & \xrightarrow{x} & \\ y \downarrow & & \downarrow f \\ & \xrightarrow{g} & \end{array}$ is a pullback, then $\mathbf{I}(f)\mathbf{I}(g)^* = \mathbf{I}(x)^*\mathbf{I}(y)$.
- (b) For every morphism $f \in \mathcal{C}$, $\mathbf{I}(f)^*\mathbf{I}(f) \subset \mathbf{A}$.
- (c) If $\mathbf{I}(f) \subset \mathbf{I}(g)$ for $f, g \in \mathcal{C}$, then $f = g$.
- (d) If $\mathbf{I}(f) = \mathbf{I}(g)$ for $f, g \in \mathcal{C}$, then $f = g$.

PROOF. (a) Since the square (1) is a pullback in the diagram

from the assumption, we have

$$\mathbf{I}(f)\mathbf{I}(g)^*=[1, f][1, g]^*=[1, f][g, 1]=[x, y]=\mathbf{I}(x)^*\mathbf{I}(y)$$

by the definition and Lemma 4.1.

(b) Since $1^*(f, f)=(f, f)=f^*(1, 1)=f^*\mathbf{A}$ and $1 \in \mathcal{E}$, we have $[f, f] \subset \mathbf{A}$ and hence $\mathbf{I}(f)^*\mathbf{I}(f) \subset \mathbf{A}$ by Lemma 4.1.

(c) If $\mathbf{I}(f) \subset \mathbf{I}(g)$, then we have $e^*(1, f)=s^*(1, g)$ for some $e \in \mathcal{E}$ and $s \in \mathcal{E}$. From this, we have $ef=eg$ and so $f=g$ since $e \in \text{Epi}(\mathcal{E})$. The statement (d) is trivial from (c). This completes the proof.

PROOF OF THEOREM A.

An I -category $\text{Rel}(\mathcal{C}; \mathcal{E})$ and a functor $\mathbf{I}: \mathcal{C} \rightarrow \text{Rel}(\mathcal{C}; \mathcal{E})$ have been constructed in §3. Proposition 4.2 (b) and (a) mean the conditions (RF. 1) and (RF. 2) for $\mathbf{I}$, respectively. Proposition 4.2 (d) implies that $\mathbf{I}$ is an embedding. Hence the proof of Theorem A is completed.

The retractive subcategory $\mathcal{E}$ may be characterized in $\text{Rel}(\mathcal{C}; \mathcal{E})$ as follows.

PROPOSITION 4.3. *The following three statements are equivalent:*

- (a) $f \in \mathcal{E}$.
- (b) $\mathbf{I}(f)^*\mathbf{I}(f) = \mathbf{A}$.
- (c) $\mathbf{A} \subset \alpha\mathbf{I}(f)$ for some $\alpha \in \text{Rel}(\mathcal{C}; \mathcal{E})$.

PROOF. (a) $\Rightarrow$ (b) Assume that $f \in \mathcal{E}$. Then we have that $(f, f) = f^*(1, 1) \sim (1, 1) = \mathbf{A}$. Thus $\mathbf{I}(f)^*\mathbf{I}(f) = [f, f] = \mathbf{A}$ by Lemma 4.1.

(b) $\Rightarrow$ (c) It is trivial.

(c) $\Rightarrow$ (a) Assume (c) and let $\alpha = [x, y] = \mathbf{I}(x)^*\mathbf{I}(y)$. Since

$$\alpha\mathbf{I}(f) = \mathbf{I}(x)^*\mathbf{I}(y)\mathbf{I}(f) = \mathbf{I}(x)^*\mathbf{I}(yf) = [x, yf],$$

there exist $e \in \mathcal{E}$ and $s \in \mathcal{E}$ such that $e^*(1, 1) = s^*(x, yf)$. Then we have $syf = e \in \mathcal{E}$ and hence $f \in \mathcal{E}$ by (E. 2). This completes the proof.

COROLLARY 4.4. $\mathbf{I}(f)$ is a retraction in $\text{Rel}(\mathcal{C}; \mathcal{E})$ if and only if $f \in \mathcal{E}$.

REMARK. Consequently, for two retractive subcategories $\mathcal{E} \neq \mathcal{E}'$ in $\mathcal{C}$, $Rel(\mathcal{C}; \mathcal{E}) \neq Rel(\mathcal{C}; \mathcal{E}')$.

Next we describe the general properties of relational functors.

LEMMA 4.5. Let $G: \mathcal{C} \rightarrow \mathcal{A}$ be a relational functor. Then,

- (a) $\Delta \subset G(f)G(f)^*$ for every morphism $f \in \mathcal{C}$, and
- (b) if $f \in Mon(\mathcal{C})$, then $G(f)G(f)^* = \Delta$.

PROOF. (a) Let $xf = yf$ be a pullback in $\mathcal{C}$. Then we have $G(f)G(f)^* = G(x)^*G(y)$ by (RF. 2). Since $1f = 1f$, there exists a unique $s \in \mathcal{C}$ such that $sx = sy = 1$. Therefore we have

$$\Delta = G(sx)^*G(sy) = G(x)^*G(s)^*G(s)G(y) \subset G(x)^*G(y) = G(f)G(f)^*$$

by (RF. 1), which proves (a).

(b) Assume that $f \in Mon(\mathcal{C})$. Then $1f = 1f$ is a pullback and we have $G(f)G(f)^* = G(1)^*G(1) = \Delta$ by (RF. 2). This completes the proof.

Henceforth, for each morphism $f \in \mathcal{C}$, we will often identify the graph $\Gamma(f)$ with f for the sake of simplicity, because Γ is a natural embedding. We summarize the fundamental formulae of relations in the following

FORMULA 4.6.

- (a) $[f, g] = f^*g$.
- (b) $f \in \mathcal{E} \Leftrightarrow f^*f = \Delta \Leftrightarrow \Delta \subset \alpha f$ for some $\alpha \in Rel(\mathcal{C}; \mathcal{E})$.
- (c) $f \in Mon(\mathcal{C}) \Leftrightarrow ff^* = \Delta$.
- (d) For every pullback $y \begin{array}{c} \xrightarrow{x} \\ \downarrow \quad \downarrow f \\ \xrightarrow{g} \end{array} x$, $fg^* = x^*y$.
- (e) For every morphism $f \in \mathcal{C}$, $f^*f \subset \Delta$ and $\Delta \subset ff^*$.
- (f) For every morphism $f \in \mathcal{C}$, $ff^*f = f$ and $f^*ff^* = f^*$.
- (g) For two morphisms $f, g \in \mathcal{C}$, $f \subset g \Rightarrow f = g$.

PROOF. The assertions (a), (b), (d), (e) and (g) have been already seen and (f) is obtained from (e) without difficulty. To prove (c), it suffices to show that $ff^* = \mathbf{A}$ implies $f \in \text{Mon}(\mathcal{C})$, in view of Lemma 4.5 (b). Assume that $\Gamma(f)\Gamma(f)^* = \mathbf{A}$ and let $xf = yf$ in $\mathcal{C}$. Then we have

$$\Gamma(x) = \Gamma(xf)\Gamma(f)^* = \Gamma(yf)\Gamma(f)^* = \Gamma(y),$$

and therefore $x = y$ since Γ is an embedding. Hence f is a monomorphism in $\mathcal{C}$, which proves (c). This completes the proof.

§5. The Properties of Relations in Special Cases

First we will recall the definition of regular epimorphisms. Given two morphisms $x, y: A \rightarrow B$, we say that $f: B \rightarrow K$ is a *coequalizer* for x and y if $xf = yf$ and if, for any morphism g such that $xg = yg$, there exists a unique morphism t such that $g = ft$. Then we write $f \in \text{Coequ}(x, y)$. A *regular epimorphism* is a coequalizer for some pair of morphisms and the class of regular epimorphisms in a category $\mathcal{C}$ is denoted by $\text{Reg}(\mathcal{C})$. Note that the class of regular epimorphisms in a general category is not always a subcategory.

THEOREM 5.1. *The following statements (a) and (b) are equivalent:*

- (a) $f \in \text{Reg}(\mathcal{C})$.
- (b) $ff^* \subset gg^*$ always implies that there exists a unique morphism $t \in \mathcal{C}$ such that $g = ft$.

PROOF. (a) $\Rightarrow$ (b) Assume that $f \in \text{Reg}(\mathcal{C})$ and, let $ff^* \subset gg^*$ and $f \in \text{Coequ}(x, y)$. Without loss of the generality, we can assume that $xf = yf$ is a pullback in $\mathcal{C}$. Then we have $ff^* = x^*y \subset gg^*$ by 4.6 (d) and $yg \subset xx^*yg = xff^*g \subset xgg^*g = xg$ by 4.6 (e), (f).

Thus $yg = xg$ by 4.6 (g) and, since $f \in \text{Coequ}(x, y)$, there is a unique $t \in \mathcal{C}$ such that $g = ft$ as desired.

(b) $\Rightarrow$ (a) Assume (b) for a morphism $f \in \mathcal{C}$. Let $xf = yf$ be a pullback. We will show that $f \in \text{Coequ}(x, y)$. Let g be a morphism in $\mathcal{C}$ such that $xg = yg$, then we have

$$ff^* = y^*x \subset y^*xgg^* = y^*ygg^* \subset gg^* \quad \text{by 4.6 (d), (e)}$$

and therefore from the assumption (b), there exists a unique morphism $t \in \mathcal{C}$ such that $g = ft$, which shows that $f \in \text{Coequ}(x, y)$. This completes the proof.

A relation α such that $\mathbf{A} \subset \alpha\alpha^*$ and $\alpha^*\alpha \subset \mathbf{A}$ is called *proper* [12].

The next theorem gives a necessary and sufficient condition for proper relations to be morphisms.

THEOREM 5.2. *The following statements (a) and (b) are equivalent:*

- (a) $\mathcal{E} \subset \text{Reg}(\mathcal{C})$.
- (b) $\mathbf{A} \subset \alpha\alpha^*$ and $\alpha^*\alpha \subset \mathbf{A}$ always imply that there exists a unique morphism $t \in \mathcal{C}$ such that $\alpha = \Gamma(t)$.

PROOF. (a) $\Rightarrow$ (b) Assume that $\mathbf{A} \subset \alpha\alpha^*$ and $\alpha^*\alpha \subset \mathbf{A}$ and let $\alpha = f^*g$. Then we have

$$\mathbf{A} \subset \alpha\alpha^* = f^*g(f^*g)^* = (f^*gg^*)f.$$

By 4.6 (b), we have $f \in \mathcal{E}$ and so $f \in \text{Reg}(\mathcal{C})$ by (a). Now let $xf = yf$ be a pullback, then $ff^* = x^*y$ and $f \in \text{Coequ}(x, y)$. Thus we have

$$\alpha^*\alpha = g^*ff^*g = g^*x^*yg = [xg, yg] \subset \mathbf{A},$$

which implies $xg = yg$. By the regularity of f , there exists a unique $t \in \mathcal{C}$ such that $g = ft$. Hence we have

$$\alpha = f^*g = f^*ft = t = \Gamma(t) \quad \text{by } f \in \mathcal{E}.$$

(b) $\Rightarrow$ (a) Assume that $f \in \mathcal{E}$ and let $xf = yf$ be a pullback, then $f^*f = \mathbf{A}$ and $ff^* = x^*y$. We will show that $f \in \text{Coequ}(x, y)$.

Let g be a morphism such that $xg = yg$. Putting $\alpha = f^*g$, we have

$$\alpha\alpha^* = f^*gg^*f \supset f^*f = \mathbf{A}$$

and $\alpha^*\alpha = g^*ff^*g = g^*x^*yg = (xg)^*(yg) \subset \mathbf{A} \quad \text{by 4.6(e).}$

From the assumption (b), there exists a unique $t \in \mathcal{C}$ such that $\alpha = f^*g = t$. Consequently, we have $g \subset ff^*g = ft$ and so $g = ft$ by 4.6(g). As f is an epimorphism, the uniqueness of t is clear. Hence we have $f \in \text{Coequ}(x, y)$ and this completes the proof of the theorem.

As a corollary of 4.6 (b) and (c), we have the following

COROLLARY 5.3. *The following statements are equivalent:*

- (a) $\mathcal{E} \cap \text{Mon}(\mathcal{C}) = \text{Iso}(\mathcal{C})$.
- (b) $ff^* = \mathbf{1}$ and $f^*f = \mathbf{1} \Leftrightarrow f \in \text{Iso}(\mathcal{C})$.

The following two propositions are miscellaneous results.

PROPOSITION 5.4. *The category $\text{Rel}(\mathcal{C}; \mathcal{E})$ of relations is a groupoid if and only if $\mathcal{E} = \mathcal{C} = \text{Mon}(\mathcal{C}) = \text{Epi}(\mathcal{C})$.*

PROOF. Assume that $\mathcal{E} = \mathcal{C} = \text{Mon}(\mathcal{C}) = \text{Epi}(\mathcal{C})$, then for every $f \in \mathcal{C}$ $ff^* = \mathbf{1}$ by 4.6(b) and $f^*f = \mathbf{1}$ by 4.6(c), which proves that $\text{Rel}(\mathcal{C}; \mathcal{E})$ is a groupoid.

Conversely, assume that $\text{Rel}(\mathcal{C}; \mathcal{E})$ is a groupoid. Then we have $f \in \mathcal{E}$ for any $f \in \mathcal{C}$ (by 4.6(b)) and therefore $\mathcal{E} = \mathcal{C} = \text{Epi}(\mathcal{C})$, since every morphism $f \in \mathcal{C}$ is an isomorphism in $\text{Rel}(\mathcal{C}; \mathcal{E})$ and so a retraction in $\text{Rel}(\mathcal{C}; \mathcal{E})$. Also f^* is a left inverse of f by 4.6(b). Hence f^* is a right inverse of f , i.e. $ff^* = \mathbf{1}$, which implies $f \in \text{Mon}(\mathcal{C})$ by 4.6(c). This completes the proof.

PROPOSITION 5.5. *$\text{Rel}(\mathcal{C}; \mathcal{E}) = \mathcal{C}$ if and only if $\mathcal{C}$ is a groupoid.*

PROOF. Assume that $\mathcal{C}$ is a groupoid. Let α be an arbitrary relation in $\text{Rel}(\mathcal{C}; \mathcal{E})$ of the form f^*g . Since $\text{Iso}(\mathcal{C}) = \mathcal{C}$, we have

$$f^*g = [f, g] = [1, f^{-1}g] = f^{-1}g \in \mathcal{C},$$

since $(f, g) = f^*(1, f^{-1}g) \sim (1, f^{-1}g)$. Hence we have $\text{Rel}(\mathcal{C}; \mathcal{E}) = \mathcal{C}$.

Conversely, assume that $\text{Rel}(\mathcal{C}; \mathcal{E}) = \mathcal{C}$. Then for every $f \in \mathcal{C}$ we have $xf \subset \mathbf{1}$ and $\mathbf{1} \subset fx$ by 4.6(e), where $x = f^*$ and $x \in \text{Rel}(\mathcal{C}; \mathcal{E}) = \mathcal{C}$. Then we have $xf = 1$ and $fx = 1$ by 4.6(g), which proves that $\mathcal{C}$ is a groupoid and the proof is complete.

§ 6. Proofs of THEOREM B and THEOREM C

PROOF OF THEOREM B

(b) $\Rightarrow$ (a) Assume that $f \in \mathcal{E}$. Then $\Gamma(f)^*\Gamma(f) = \mathbf{1}$ by 4.3 and

$$\begin{aligned}
G(f)^*G(g) &= \tilde{G}(\Gamma(f))^*\tilde{G}(\Gamma(g)) = \tilde{G}(\Gamma(f)^*)\tilde{G}(\Gamma(g)) \\
&= \tilde{G}(\Gamma(f)^*\Gamma(g)) = \tilde{G}(\Delta) = \Delta,
\end{aligned}$$

which proves (a).

(a) $\Rightarrow$ (b) For any $\alpha = \Gamma(f)^*\Gamma(g)$, define $\tilde{G}(\alpha) = G(f)^*G(g)$.

First we will prove that this definition is well-defined. Let $(f, g), (f', g')$ be two spans in $\mathcal{C}$ such that $(f, g) < (f', g')$, then we have that $e^*(f, g) = s^*(f', g')$ for some $e \in \mathcal{E}, s \in \mathcal{C}$, and that

$$\begin{aligned}
G(f)^*G(g) &= G(f)^*G(e)^*G(s)G(g) \quad (\text{by the assumption (a)}) \\
&= G(ef)^*G(sg) \\
&= G(sf')^*G(sg') \\
&= G(f')^*G(s)^*G(s)G(g') \\
&\subset G(f')^*G(g') \quad (\text{by (RF.1)}).
\end{aligned}$$

Hence this indicates that the above definition is independent of the choice of representatives (f, g) of α . And from the definition it is clear that $\tilde{G}(\Delta) = \Delta$ and $\tilde{G}(\alpha)^* = \tilde{G}(\alpha^*)$ and $\tilde{G}(\Gamma(f)) = G(f)$ for $\alpha \in \text{Rel}(\mathcal{C}; \mathcal{E})$ and for $f \in \mathcal{C}$. Also $\tilde{G}$ preserves partial orders of I -categories.

To prove that $\tilde{G}$ is an I -functor, it suffices to show that $\tilde{G}(\alpha)\tilde{G}(\beta) = \tilde{G}(\alpha\beta)$ if $\alpha\beta$ is defined. Let $\alpha = \Gamma(a_1)^*\Gamma(a_2)$, $\beta = \Gamma(b_1)^*\Gamma(b_2)$ and let $xa_2 = yb_1$ be a pullback in $\mathcal{C}$. Since $G(a_2)G(b_1)^* = G(x)^*G(y)$ by (RF. 2), we have

$$\begin{aligned}
\tilde{G}(\alpha\beta) &= \tilde{G}(\Gamma(xa_1)^*\Gamma(yb_2)) = G(xa_1)^*G(yb_2) \\
&= G(a_1)^*G(x)^*G(y)G(b_2) = G(a_1)^*G(a_2)G(b_1)^*G(b_2) \\
&= \tilde{G}(\alpha)\tilde{G}(\beta)
\end{aligned}$$

which proves (b). Hence the proof of Theorem B is complete.

COROLLARY 6.1. *Let $\mathcal{C}, \mathcal{C}'$ be two categories with pullbacks and $\mathcal{E}, \mathcal{E}'$ retractive subcategories of $\mathcal{C}, \mathcal{C}'$, respectively.*

If a functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ preserves pullbacks and $F(\mathcal{E}) \subset \mathcal{E}'$, then there exists a unique I -functor $\tilde{F}: \text{Rel}(\mathcal{C}; \mathcal{E}) \rightarrow \text{Rel}(\mathcal{C}'; \mathcal{E}')$ making the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\
r \downarrow & & \downarrow r' \\
\text{Rel}(\mathcal{C}; \mathcal{E}) & \xrightarrow{\tilde{F}} & \text{Rel}(\mathcal{C}'; \mathcal{E}')
\end{array}$$

commutative.

COROLLARY 6.2. *The graphic embedding $\Gamma: \mathcal{C} \rightarrow \text{Rel}(\mathcal{C}; \mathcal{E})$ satisfies the axioms (REL. 1) and (REL. 2).*

PROOF. Proposition 4.3 means (REL. 1) for Γ and Theorem B means (REL. 2) for Γ .

PROOF OF THEOREM C

First we define a subclass $\mathcal{E}$ of $\mathcal{C}$ by $\{f \in \mathcal{C}; G(f)^*G(f) = \Delta\}$, then it is obvious that $\mathcal{E}$ is a subcategory of $\mathcal{C}$. If we can prove that $\mathcal{E}$ is a retractive subcategory of $\mathcal{C}$, then latter half of the proof immediately follows from the universality axiom (REL. 2) for the relational embedding $G: \mathcal{C} \rightarrow \mathcal{A}$ and the graphic embedding $\Gamma: \mathcal{C} \rightarrow \text{Rel}(\mathcal{C}; \mathcal{E})$.

For the proof of (E.1) of $\mathcal{E}$, let f be an isomorphism in $\mathcal{C}$ and g the inverse of f , namely $fg = 1$ and $gf = 1$. By (RF.1) and Lemma 4.5(a), we have $G(f)^*G(f) \subset \Delta$ and $\Delta \subset G(f)G(f)^*$, and so

$$G(f)^* = G(f)^*G(f)G(g) \subset G(g) \text{ and } G(g) \subset G(g)G(f)G(f)^* = G(f)^*.$$

Hence we have $G(f)^* = G(g)$ and therefore

$$G(f)^*G(f) = G(g)G(f) = G(gf) = G(1) = \Delta,$$

as desired. Thus we have shown that $\text{Iso}(\mathcal{C}) \subset \mathcal{E}$. Next, let $f \in \mathcal{E}$ and $fx = fy$, then

$$G(x) = G(f)^*G(f)G(x) = G(f)^*G(f)G(y) = G(y)$$

because $G(f)^*G(f) = \Delta$. Therefore we have $x = y$ since G is an embedding and so f is an epimorphism, which shows that $\mathcal{E} \subset \text{Epi}(\mathcal{C})$.

To prove (E.2), assume that $fg \in \mathcal{E}$, i.e. $G(fg)^*G(fg) = \Delta$. Then, applying the axiom (REL.1) to the fact that $\Delta = (G(fg)^*G(fg))G(g)$, we have $G(g)^*G(g) = \Delta$ and so $g \in \mathcal{E}$.

To prove (E.3), let $xf = yg$ be a pullback in $\mathcal{C}$ and $f \in \mathcal{E}$. Then we have $G(f)G(g)^* = G(x)^*G(y)$ by (RF.1) and

$$\Delta \subset G(g)G(g)^* = G(g)G(f)^*G(f)G(g)^* = G(g)G(f)^*G(x)^*G(y) \text{ (by 4.5(a)).}$$

Again applying the axiom (REL.1) to the fact that $\Delta \subset (G(g)G(f)^*G(x)^*)G(y)$, we obtain $G(y)^*G(y) = \Delta$ and therefore $y \in \mathcal{E}$. Hence $\mathcal{E}$ is, in fact, a retractive subcategory of $\mathcal{C}$, which completes the proof of Theorem C.

COROLLARY 6.3. *Let $\mathcal{C}$ be a category with pullbacks and pushouts.*

If $\text{Epi}(\mathcal{C})$ is a retractive subcategory of $\mathcal{C}$ and if $\text{Mon}(\mathcal{C})$ is a coretractive subcategory of $\mathcal{C}$, then there is an isomorphism

$$\text{Rel}(\mathcal{C}; \text{Epi}(\mathcal{C})) \approx \text{Rel}^{\text{op}}(\mathcal{C}; \text{Mon}(\mathcal{C}))$$

of I-categories, where $\text{Rel}^{\text{op}}(\mathcal{C}; \text{Mon}(\mathcal{C}))$ is the category of corelations relative to $\text{Mon}(\mathcal{C})$.

PROOF. It is immediate from Theorem C and its dual.

§ 7. $\mathcal{E}$ -subobjects and $\mathcal{E}$ -decompositions

Let $\mathcal{C}$ be a (fixed) category with pullbacks and $\mathcal{E}$ a (fixed) retractive subcategory of $\mathcal{C}$.

For two morphisms f, g in $\mathcal{C}$ with common range X , define $f \leq g$ if there exist $e \in \mathcal{E}$ and $s \in \mathcal{C}$ such that $ef = sg$. It is clear that $f \leq f$ for any $f \in \mathcal{C}$.

PROPOSITION 7.1. (a) $f \leq g \Rightarrow fh \leq gh$.
(b) $f \leq g$ and $g \leq h \Rightarrow f \leq h$.

PROOF. (a) is trivial and the proof of (b) is similar to that of Proposition 2.2.

Define $f \equiv g$ if $f \leq g$ and $g \leq f$, and this relation " $\equiv$ " is an equivalence relation on the class of morphisms with range X by the last proposition. The equivalence class of $f: A \rightarrow X$ is denoted by $\text{im } f$ and is called an $\mathcal{E}$ -subobject of X . The class of $\mathcal{E}$ -subobjects of X will be denoted by $S_{\mathcal{E}}(X)$. We introduce a partial order on $S_{\mathcal{E}}(X)$.

For $\text{im } f, \text{im } g \in S_{\mathcal{E}}(X)$, define $\text{im } f \leq \text{im } g$ if $f \leq g$. This definition is well-defined. Trivially we have

PROPOSITION 7.2. (a) *If $\mathcal{C}$ has finite products, then $\text{Rel}(A, B)$ can*

be identified with $S_{\mathcal{E}}(A \times B)$ as ordered classes.

- (b) If $xf = yg$ is a pullback for f and g , then
 $im\ xf = im\ yg = im\ f \wedge im\ g$ (where $\wedge$ denotes the intersection of $\mathcal{E}$ -subobjects).

COROLLARY 7.3. If $\mathcal{E}$ is (finitely) complete, then $Rel(\mathcal{E}; \mathcal{E})$ has (finite) intersections of relations.

Also the following proposition is obvious from definitions.

PROPOSITION 7.4. Let $\alpha = (a, b)$, $\alpha' = (a', b')$ be two spans in $\mathcal{E}$ and let $xf = yg$ be a pullback for f and g .

- (a) $\alpha < \alpha' \Rightarrow b \leq b'$.
 (b) $\alpha \sim \alpha' \Rightarrow im\ b = im\ b'$.
 (c) $f \leq g \Leftrightarrow x \in \mathcal{E}$.
 (d) $f \equiv g \Leftrightarrow x \in \mathcal{E}$ and $y \in \mathcal{E}$.

For any relation $\alpha = [a, b] \in Rel(\mathcal{E}; \mathcal{E})$, define $Im\ \alpha = im\ b$, then 7.4(b) shows that this definition is well-defined and it is easily seen that $Im\ f = Im\ \Gamma(f) = im\ f$ for any morphism $f \in \mathcal{E}$.

As a corollary of 7.4(a), we have

COROLLARY 7.5. Let α, α' be two relations in $Rel(\mathcal{E}; \mathcal{E})$. If $\alpha < \alpha'$, then $Im\ \alpha \leq Im\ \alpha'$.

LEMMA 7.6. Let x, x' be two morphisms in $\mathcal{E}$ and let α be a relation in $Rel(\mathcal{E}; \mathcal{E})$. If $x \leq x'$, then $Im\ x\alpha \leq Im\ x'\alpha$.

PROOF. Let $\alpha = [a, b]$ and $ex = sx'$ for some $e \in \mathcal{E}$ and $s \in \mathcal{E}$. Consider the following diagrams

$$\begin{array}{ccccc} \cdot & \xrightarrow{\hat{e}} & \cdot & \xrightarrow{y} & \cdot \\ \downarrow v & & \downarrow u & & \downarrow a \\ \cdot & \xrightarrow{e} & \cdot & \xrightarrow{x} & \cdot \end{array} \quad \text{and} \quad \begin{array}{ccccc} \cdot & \xrightarrow{\hat{s}} & \cdot & \xrightarrow{y'} & \cdot \\ \downarrow v' & & \downarrow u' & & \downarrow a \\ \cdot & \xrightarrow{s} & \cdot & \xrightarrow{x'} & \cdot \end{array}$$

where the squares (1)~(4) are pullbacks. Then we have

$$x\alpha = [1, x][a, b] = [u, yb], \quad x'\alpha = [1, x'][a, b] = [u', y'b]$$

and so $Im\ x\alpha = im\ yb$, $Im\ x'\alpha = im\ y'b$. To prove the lemma, it suffices to show that $y \leq y'$ in view of 7.1(b). Because both $(\hat{e}\ y)a = v(ex)$ and $(\hat{s}\ y')a = v'(sx')$ are pullbacks for a and $ex = sx'$, there exists a unique isomorphism ν such that $\nu v = v'$ and $\nu e y = s y'$.

Thus the equality $\nu e y = s y'$ with $\nu e \in \mathcal{E}$ implies that $y \leq y'$, which proves the lemma.

Let $\alpha \in Rel(\mathcal{C}; \mathcal{E})$ be any relation from X to Y . The last lemma enables us to define functions $\alpha_s: S_{\mathcal{E}}(X) \rightarrow S_{\mathcal{E}}(Y)$ and $\alpha^s: S_{\mathcal{E}}(Y) \rightarrow S_{\mathcal{E}}(X)$ by $(im\ x)\alpha_s = Im\ x\alpha$ for $im\ x \in S_{\mathcal{E}}(X)$ and $(im\ y)\alpha^s = Im\ y\alpha^*$ for $im\ y \in S_{\mathcal{E}}(Y)$.

It is natural that we call $(im\ x)\alpha_s$ the direct image of an $\mathcal{E}$ -subobject $im\ x$ of X under a relation α and $(im\ y)\alpha^s$ the inverse image of an $\mathcal{E}$ -subobject $im\ y$ of Y under a relation α . The functions α_s and α^s have the analogous properties to those of Grillet [4]. The details are omitted and will be discussed elsewhere.

An $\mathcal{E}$ -decomposition of a morphism $f \in \mathcal{C}$ is a pair (e, m) of $e \in \mathcal{E}$ and $m \in Mon(\mathcal{C})$ such that $f = em$. We say that $\mathcal{C}$ has $\mathcal{E}$ -decompositions if every morphism in $\mathcal{C}$ has an $\mathcal{E}$ -decomposition. An $\mathcal{E}$ -decomposition $f = em$ of $f \in \mathcal{C}$ is called an $\mathcal{E}$ -image decomposition if for any $\mathcal{E}$ -decomposition $f = e'm'$ of f there exists a unique morphism $k \in \mathcal{C}$ such that $e = e'k$ and $km = m'$. We say that $\mathcal{C}$ has $\mathcal{E}$ -image decompositions if every morphism in $\mathcal{C}$ has an $\mathcal{E}$ -image decomposition.

PROPOSITION 7.7. *Let $\mathcal{C}$ be a category with pullbacks and $\mathcal{E}$ a retractive subcategory of $\mathcal{C}$. Then each of the following statements implies the next one:*

- (a) $\mathcal{C}$ has $\mathcal{E}$ -decompositions and $\mathcal{E} \cap Mon(\mathcal{C}) = Iso(\mathcal{C})$.
- (b) Every morphism in $\mathcal{C}$ has a unique $\mathcal{E}$ -decomposition up to an isomorphism.
- (c) $\mathcal{C}$ has $\mathcal{E}$ -image decompositions.
- (d) There is an injective function $S_{\mathcal{E}}(X) \rightarrow S(X)$ for any object X in $\mathcal{C}$, where $S(X)$ denotes the class of the ordinary subobjects of X .

PROOF. (a) $\Rightarrow$ (b) Let $f = em = e'm'$ be two $\mathcal{E}$ -decompositions of f , i.e. $e, e' \in \mathcal{E}$ and $m, m' \in Mon(\mathcal{C})$. Consider a pullback $dm = d'm'$ for m and m' . Then we have that $d, d' \in Mon(\mathcal{C})$ by the pullback property and that $d, d' \in \mathcal{E}$ since there exists a unique morphism $s \in \mathcal{C}$ such that $sd = e$

and $sd' = e'$. From the assumption that $\mathcal{E} \cap \text{Mon}(\mathcal{C}) = \text{Iso}(\mathcal{C})$, we have that $d, d' \in \text{Iso}(\mathcal{C})$, which implies (b).

(b) $\Rightarrow$ (c) It is obvious.

(c) $\Rightarrow$ (d) Let $f = em$, $g = dn$ be $\mathcal{E}$ -image decompositions of f, g respectively and let $f = g$. Then there exist morphisms $e', d' \in \mathcal{E}$ such that $e'm = d'n$ and so m, n are equivalent as a subobject of X from the elementary property of $\mathcal{E}$ -image decompositions. Hence we have a well-defined function $\Psi: S_{\mathcal{E}}(X) \rightarrow S(X)$ $((imf)\Psi = [m]$, where $f = em$ is an $\mathcal{E}$ -image decomposition of f) and it is clear that Ψ is injective. This completes the proof.

REMARK. *As easily seen, the statement (b) implies that the function $\Psi: S_{\mathcal{E}}(X) \rightarrow S(X)$ is a natural isomorphism as order classes.*

Combining Proposition 7.7 with Proposition 7.2 (a), we have

COROLLARY 7.8. *Let $\mathcal{C}$ be a finite complete and locally small category and let $\mathcal{E}$ be a retractive subcategory of $\mathcal{C}$.*

If one of the statements (a), (b), (c) in 7.7 is valid, then the class of relations of $\text{Rel}(\mathcal{C}; \mathcal{E})$ from one object to another form a set.

§8. Examples

This section is devoted to illustrating examples of categories of relations.

EXAMPLE 8.1. Let $\mathcal{E}_{ns}$ be the category of sets and mappings between them. The category $\mathcal{E}_{ns}$ has the only one retractive subcategory, namely, $\text{Ret}(\mathcal{E}_{ns}) = \text{Reg}(\mathcal{E}_{ns}) = \text{Epi}(\mathcal{E}_{ns})$ and so has the only one category of relations in it. It is clear that the category $\text{Rel}(\mathcal{E}_{ns}; \text{Epi}(\mathcal{E}_{ns}))$ is equal to the category formed by the classical relations (or, *many valued* functions between sets).

EXAMPLE 8.2. Let $\mathcal{C}$ be an abelian category [3]. Then it is well-known that every epimorphism in $\mathcal{C}$ is a regular epimorphism, that is, $\text{Reg}(\mathcal{C}) = \text{Epi}(\mathcal{C})$. We can construct the category $\text{Rel}(\mathcal{C}; \text{Epi}(\mathcal{C}))$ of relations in $\mathcal{C}$ relative to $\text{Epi}(\mathcal{C})$, which is equal to the classical category of relations studied by S. MacLane [7] and D. Puppe [9].

In this case many other properties of relations are also valid (see

[7] and [9]).

EXAMPLE 8.3. Let $\mathbb{N}$ be the natural numbers. $\mathbb{N}$ can be considered as a category, i.e. the composition is the ordinary multiplication of natural numbers. Then the category $\mathbb{N}$ has pullbacks. For any $m, n \in \mathbb{N}$, let l be the least common multiple of m and n , then the square

$$\begin{array}{ccc} \bullet & \xrightarrow{\frac{l}{m}} & \bullet \\ \downarrow \frac{l}{n} & & \downarrow m \\ \bullet & \xrightarrow{n} & \bullet \end{array}$$

is a pullback for m and n .

It is obvious that $\mathbb{N} = \text{Epi}(\mathbb{N}) = \text{Mon}(\mathbb{N})$ and so $\mathbb{N}$ is a retractive subcategory of $\mathbb{N}$. Simple inspections show that $\text{Rel}(\mathbb{N}; \mathbb{N}) = \mathbb{Q}_+$, where $\mathbb{Q}_+$ is the positive rational numbers (cf. Proposition 5.4) and that if $\alpha < \beta$ for any relations $\alpha, \beta \in \text{Rel}(\mathbb{N}; \mathbb{N})$ then $\alpha = \beta$.

From this, $\text{Rel}(\mathbb{N}; \mathbb{N})$ has not intersections with respect to the ordering $<$. For example, if the intersection $\frac{2}{3} \wedge \frac{1}{4}$ of $\frac{2}{3}$ and $\frac{1}{4}$ does exist, we have $\frac{2}{3} = \frac{2}{3} \wedge \frac{1}{4} = \frac{1}{4}$ from the last assertion, which is a contradiction. Hence it proves that we can not define intersections of relations by defining them only by means of pairs of morphisms.

EXAMPLE 8.4. Let $\mathfrak{M}$ be the class of all cardinals. Since $\mathfrak{M}$ is (totally) ordered, we can regard it as a category. Adding to $\mathfrak{M}$ two extra objects A, B such that $m \leq A$ and $m \leq B$ for any cardinal m and A, B are not mutually comparable, we have a new category $\mathfrak{M}^{+2}$.

Clearly $\mathfrak{M}^{+2}$ has pullbacks, $\mathfrak{M}^{+2} = \text{Epi}(\mathfrak{M}^{+2}) = \text{Mon}(\mathfrak{M}^{+2})$ and $\text{Ret}(\mathfrak{M}^{+2}) = \text{Id}(\mathfrak{M}^{+2})$, where $\text{Id}(\mathfrak{M}^{+2})$ is the class of identities in $\mathfrak{M}^{+2}$.

From the definitions of §2 and §3, we have

$$\text{Rel}(A, B) = \text{Spn}(A, B) \approx \mathfrak{M} \quad \text{in } \text{Rel}(\mathfrak{M}^{+2}; \text{Ret}(\mathfrak{M}^{+2})).$$

Hence $\text{Rel}(A, B)$ is not a set. But $\text{Rel}(m, n)$ is a set (with a single element) for any $m, n \in \mathfrak{M}^{+2}$ in $\text{Rel}(\mathfrak{M}^{+2}; \mathfrak{M}^{+2})$.

EXAMPLE 8.5. Relations in bicategories were dealt with in A. Klein [6].

Let $(\mathcal{C}; \mathcal{S}, \mathcal{I})$ be a finitely complete bicategory (in the sense of Isbell) with the pullback axiom (E.3), which is called an Ore-like categorical condition (A) in [6]. Then $\mathcal{S}$ is a retractive subcategory of $\mathcal{C}$ (by (0.1) and (2.1) in [6]) and we can consider the category $Rel(\mathcal{C}; \mathcal{S})$ of relations in $\mathcal{C}$ relative to $\mathcal{S}$. In [6] Klein has constructed a category $\mathcal{R}_{\mathcal{C}}$ (of relations in $\mathcal{C}$), in which the axioms (REL. 1) and (REL. 2) are valid. The axiom (REL. 2) was verified as Theorem (3.5) in [6], but (REL. 1) is not explicitly proved there.

Furthermore $\mathcal{S} = \{f \in \mathcal{C} \mid \Gamma_f^- \Gamma_f = \mathcal{A}\}$ by (1.12) in [6] and therefore the category $\mathcal{R}_{\mathcal{C}}$ coincides with the category $Rel(\mathcal{C}; \mathcal{S})$ by Theorem C.

EXAMPLE 8.6. A regular category $\mathcal{C}$ is a finitely complete category, in which every morphism has a decomposition as a regular epimorphism followed by a monomorphism (called a regular decomposition or $Reg(\mathcal{C})$ -decomposition) and in which the pullback axiom (E.3) for $Reg(\mathcal{C})$ holds.

For example, the category of (Hausdorff) topological groups and continuous homomorphisms is a regular category.

In [4] P.A. Grillet investigated relations in regular categories and applied them to the arguments of sheaves with values in a suitable regular category. For a regular category $\mathcal{C}$, the class $Reg(\mathcal{C})$ of regular epimorphisms in $\mathcal{C}$ is a retractive subcategory of $\mathcal{C}$ (by Proposition I-1.10, 1.11), and the category $Rel(\mathcal{C}; Reg(\mathcal{C}))$ of relations in $\mathcal{C}$ and that of Grillet [4] are clearly identical. In this category of relations, any proper relation is a morphism in $\mathcal{C}$ by Theorem 5.2. The results of §5 give evidence of his assertion that regular categories have a definite advantage in considering relations for them. Note that in the present case $(\mathcal{C}; Reg(\mathcal{C}), Mon(\mathcal{C}))$ gives a bicategorical structure on $\mathcal{C}$ and that the method of A. Klein [6] defining relations somewhat differs from that of P.A. Grillet [4].

EXAMPLE 8.7. Let $\mathcal{T}_{op}\mathcal{P}$ be the category of topological spaces and continuous maps. It is clear that $\mathcal{T}_{op}\mathcal{P}$ has pullbacks and $Epi(\mathcal{T}_{op}\mathcal{P})$ is a retractive subcategory of $\mathcal{T}_{op}\mathcal{P}$ (cf. §1). Then we can construct the category $Rel(\mathcal{T}_{op}\mathcal{P}; Epi(\mathcal{T}_{op}\mathcal{P}))$ of relations, in which a proper relation is not always a morphism in $\mathcal{T}_{op}\mathcal{P}$, i.e. a continuous map, since it is not the case that $Epi(\mathcal{T}_{op}\mathcal{P}) \subset Reg(\mathcal{T}_{op}\mathcal{P})$ (cf. Theorem 5.2).

Let $\mathbb{R}$ be the real numbers (with the ordinary topology) and let $\mathbb{R}_0$ be the real numbers with the discrete topology. It is trivial that a map

$f: \mathbb{R}_0 \rightarrow \mathbb{R}$ ($f(x)=x$ for any $x \in \mathbb{R}_0$) is continuous, or a morphism in $\mathcal{T}_o\mathcal{P}$, and $f \in \text{Epi}(\mathcal{T}_o\mathcal{P}) \cap \text{Mon}(\mathcal{T}_o\mathcal{P})$, which shows that $ff^* = 1_{\mathbb{R}_0}$ and $f^*f = 1_{\mathbb{R}}$ in $\text{Rel}(\mathcal{T}_o\mathcal{P}; \text{Epi}(\mathcal{T}_o\mathcal{P}))$ by 4.6(b), (c).

Hence f^* is a proper relation. However f^* is not a morphism in $\mathcal{T}_o\mathcal{P}$. For, if f^* is a morphism in $\mathcal{T}_o\mathcal{P}$, then f has a (two sided) inverse in $\mathcal{T}_o\mathcal{P}$, namely f^* , and so f is an isomorphism in $\mathcal{T}_o\mathcal{P}$ (i.e. a homeomorphism), which is a contradiction.

Note that $\mathcal{T}_o\mathcal{P}$ has $\text{Epi}(\mathcal{T}_o\mathcal{P})$ -image decompositions.

EXAMPLE 8.8. In $\mathcal{T}_o\mathcal{P}$, $\text{Reg}(\mathcal{T}_o\mathcal{P})$ is just the class of identification maps and therefore it is trivial that $\text{Reg}(\mathcal{T}_o\mathcal{P})$ satisfies the axioms (E.1) and (E.2). By Proposition 1.2, we have a retractive subcategory $\text{Reg}^*(\mathcal{T}_o\mathcal{P})$ of $\mathcal{T}_o\mathcal{P}$ and the category $\text{Rel}(\mathcal{T}_o\mathcal{P}; \text{Reg}^*(\mathcal{T}_o\mathcal{P}))$ of relations relative to $\text{Reg}^*(\mathcal{T}_o\mathcal{P})$, in which a proper relation is always a morphism in $\mathcal{T}_o\mathcal{P}$ by Theorem 5.2.

EXAMPLE 8.9. Let $\mathcal{H}aus$ be the category of Hausdorff spaces and continuous maps. It is clear that $\mathcal{H}aus$ has pullbacks and $\text{Epi}(\mathcal{H}aus)$ satisfies the axioms (E.1) and (E.2) (cf. Corollary 1.3) but not the pullback axiom (E.3). Let $\mathbb{R}, \mathbb{Q}$ be the real numbers, the rational numbers, respectively and let $i, j: \mathbb{Q} \rightarrow \mathbb{R}$ be continuous maps such that $i(x)=x$, $j(x)=\lambda x$ for any $x \in \mathbb{Q}$, where λ is an irrational. Then $i, j: \mathbb{Q} \rightarrow \mathbb{R}$ are epimorphisms in $\mathcal{H}aus$, since a continuous map is an epimorphism in $\mathcal{H}aus$ if and only if it has a dense image. Let $0 \rightarrow \mathbb{Q}$ be the inclusion map, then the square

$$\begin{array}{ccc} 0 & \xrightarrow{\quad} & \mathbb{Q} \\ \downarrow & & \downarrow i \\ \mathbb{Q} & \xrightarrow{j} & \mathbb{R} \end{array}$$

is a pullback for i and j , and trivially $0 \rightarrow \mathbb{Q}$ is not an epimorphism in $\mathcal{H}aus$. Thus the pullback axiom (E.3) does not hold for $\text{Epi}(\mathcal{H}aus)$. From Corollary 1.3, however, we have the retractive subcategory $\text{Epi}^*(\mathcal{H}aus)$ and the category $\text{Rel}(\mathcal{H}aus; \text{Epi}^*(\mathcal{H}aus))$ of relations in $\mathcal{H}aus$ relative to $\text{Epi}^*(\mathcal{H}aus)$, in which a proper relation is not always a morphism in $\mathcal{H}aus$, (i.e. a continuous map), since it is not the case that $\text{Epi}^*(\mathcal{H}aus) \subset \text{Reg}(\mathcal{H}aus)$ (cf. Theorem 5.2).

Let $f: \mathbb{R}_0 \rightarrow \mathbb{R}$ be the map given in Example 8.7. Since $\mathbb{R}, \mathbb{R}_0$ are

clearly Hausdorff spaces, f is a morphism in $\mathcal{H}aus$ and furthermore $f \in Epi^*(\mathcal{H}aus) \cap Mon(\mathcal{H}aus)$ (since f is a surjective map).

By the same reason as in 8.7, f^* is a proper relation in $Rel(\mathcal{H}aus; Epi^*(\mathcal{H}aus))$ but not a morphism in $\mathcal{H}aus$.

Note that $\mathcal{H}aus$ has $Epi^*(\mathcal{H}aus)$ -image decompositions and $Epi^*(\mathcal{H}aus)$ includes the class of surjective maps in $\mathcal{H}aus$.

EXAMPLE 8.10. In $\mathcal{H}aus$, $Reg(\mathcal{H}aus)$ is just the class of identification maps, as in 8.8 and therefore it is trivial that $Reg(\mathcal{H}aus)$ satisfies the axioms (E.1) and (E.2). Again, by Proposition 1.2, we have a retractive subcategory $Reg^*(\mathcal{H}aus)$ of $\mathcal{H}aus$ and the category $Rel(\mathcal{H}aus; Reg^*(\mathcal{H}aus))$ of relations in $\mathcal{H}aus$ relative to $Reg^*(\mathcal{H}aus)$, in which a proper relation is always a morphism in $\mathcal{H}aus$ by Theorem 5.2.

Note that the class of surjective open maps in $\mathcal{H}aus$ is included in $Reg^*(\mathcal{H}aus)$.

REMARK. In 8.1, 8.7, 8.9, the relations from one object to another form a set and we do not know whether they form a set in 8.8, 8.10.

In 8.2, 8.5, 8.6, the class of relations among two objects is a set if the given category $\mathcal{C}$ is locally small.

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