ON A THEOREM OF C. HEIM

By

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C. Heim proved the following interesting theorem.

THEOREM (*Heim* [4]) *Let M be an n -dimensional complete Riemannian manifold with sectional curvature $K \geq 1$ and $\text{vol } M \geq \frac{1}{2} \text{vol } S^n$. Then all closed geodesics on M are not shorter than π , where $\text{vol } M$ is the volume of M and S^n is a sphere with constant curvature 1.*

In this note we give a simple proof of the above theorem.

1. Preliminaries

(a) **Volumes** (c.f. Berger-Gauduchon-Mazet [2]) Let M be a compact Riemannian manifold, g be its Riemannian metric and v_g be the volume element on M . For a point $m \in M$ let v_m be the volume element of the tangent space M_m to M at m , $\exp_m$ be the exponential mapping of M_m into M and let U_m be the maximal open neighborhood around the origin of M_m which $\exp_m$ maps diffeomorphically onto its image. We call U_m the injective neighborhood of m . Let $\theta(x)$ be the Jacobian determinant of $\exp_x$ for any point $x \in U_m$. Then we have

$$(1) \quad \text{vol } M = \int_{U_m} \theta \cdot v_m.$$

We use the Jacobi fields to calculate volume. Let $U_\varepsilon(m)$ be a normal neighborhood around m with radius $\varepsilon > 0$. For $0 < r < \varepsilon$ and a unit vector u on $U_\varepsilon(m)$ we give the expression of $\theta(ru)$. Let c_u be the geodesic starting from m , with the initial direction u . Let $\{y_2, \dots, y_n\}$ be an

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orthonormal basis of the orthogonal complement $u^\perp$ of u . Let $Y_i(s)$ be the Jacobi field along c_u such that $Y_i(0)=0$, $Y_i'(0)=y_i/r$. Then we have

$$(2) \quad \theta(ru) = \det(\langle Y_i(r), Y_j(r) \rangle)^{1/2}, \quad i, j = 2, \dots, n,$$

where $\langle, \rangle$ is the inner product on M .

(b) Rauch's comparison theorem (c.f. Gromoll-Klingenberg-Meyer [3]) For a point $m \in M$ let G_m be the set of all 2-dimensional linear subspaces of M_m and we put $G_M = \bigcup_{m \in M} G_m$. If $c: [0, l] \rightarrow M$ is a differentiable curve, we denote by G_c the set of all 2-dimensional linear subspaces of $M_{c(t)}$ each of which contains a tangent vector to c . For any $\sigma \in G_M$ let (u, v) be a basis of σ . Then we denote the sectional curvature by $K_\sigma = K(u, v)$.

THEOREM (Rauch) *Let M be an n -dimensional Riemannian manifold ($n \geq 2$) and S^n be an n -dimensional sphere with constant curvature 1. Let $c: [0, l] \rightarrow M$ and $\tilde{c}: [0, l] \rightarrow S$ be geodesics. Let Y (resp. $\tilde{Y}$) be the Jacobi field along c (resp. $\tilde{c}$) such that*

$$Y(0) = \tilde{Y}(0) = 0,$$

$$\langle Y', \dot{c} \rangle(0) = \langle \tilde{Y}', \dot{\tilde{c}} \rangle(0) = 0,$$

$$\|Y'(0)\| = \|\tilde{Y}'(0)\|,$$

where Y' is the covariant derivative with respect to the direction $\dot{c}$ and $\| \cdot \|$ is the norm. Furthermore we assume $K_\sigma \geq 1$ for all $t \in [0, l]$ and all $\sigma \in G_{c(t)}$ and c has no conjugate points on the interval $(0, l)$. Under these conditions we have the inequality

$$\|Y(t)\| \leq \|\tilde{Y}(t)\|$$

for all $t \in [0, l]$. If we have $\|Y(t_0)\| = \|\tilde{Y}(t_0)\| \neq 0$ for some $t_0 \in (0, l]$, the equality $K(Y(t), \dot{c}(t)) = 1$ holds good for all $t \in [0, t_0]$.

2. Proof of theorem.

We assume there exists a closed geodesic G of length $< \pi$. Let p be a point on G . Let r be an arbitrary point of M and q be a nearest

point on G from r . Then we have $d(r, q) \leq \frac{\pi}{2}$ (Berger [1]). A shortest geodesic rq is orthogonal to G at q . By the same argument as the proof of Berger's theorem (c.f. Nagayoshi-Tsukamoto [5]), we have $d(p, r) \leq \frac{\pi}{2}$. Hence the injective neighborhood U_p of p is contained in the open ball around with radius $\frac{\pi}{2}$. By (2), Rauch's comparison theorem and a property of Gramian determinant, we have

$$\text{vol } M \leq \frac{1}{2} \text{vol } S^n.$$

Therefore we have $\text{vol } M = \frac{1}{2} \text{vol } S^n$. By (2) and Rauch's comparison theorem, the set $\exp_p(U_p)$ is isometric to the open hemisphere of S^n . So M is of constant curvature 1. Hence M is isometric to a real projective space of constant curvature 1, because of $\text{vol } M = \frac{1}{2} \text{vol } S^n$. The length of G must be π . This fact contradicts to our assumption. Q.E.D.

Remark. In our previous paper [6] we proved the following theorem: *Let M be a complete Riemannian manifold with sectional curvature $K \geq 1$ and $\text{vol } M \geq \frac{1}{2} \text{vol } S^n$. If there exists a closed geodesic of minimal length π , then M is isometric to a real projective space with constant curvature 1.*

When we finished preparing this note, we heard from M.P. do Carmo K. Tenenblat gave another proof of the above theorem.

References

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