CONDITIONAL EXPECTATIONS IN VON NEUMANN ALGEBRAS WITH RESPECT TO WEIGHTS

By

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1. Introduction.

The theory of conditional expectations developed by H. Umegaki depends upon the existence of traces; therefore, it is not applicable for von Neumann algebras of type III. By Takesaki conditional expectations with respect to weights were studied and the following theorem 2.1 was shown.

In this paper, we will study the properties and the characterization of conditional expectations. And it is the extension of the results [10] of H. Umegaki.

Throughout this paper the basic notations and results with respect to generalized Hilbert algebras follow to those of [7] and those with respect to weights to those of [5].

2. Conditional expectation.

Let M be a von Neumann algebra and let φ be a faithful normal semifinite weight on M_+ .

Let $\Sigma = \{\sigma_t; t \in \mathbb{R}\}$ be a modular automorphism group of M associated with φ . And we denote the above triple by (M, φ, Σ) .

THEOREM 2.1. [9] *If N is a Σ -invariant von Neumann subalgebra of M on which φ is semifinite, then there exists a linear mapping ε from M onto N such that*

$$(1) \quad \varepsilon(x^*) = \varepsilon(x)^*$$

$$(2) \quad \varepsilon(\varepsilon(x)) = \varepsilon(x)$$

- (3) $\varepsilon(x) \geq 0$ if $x \geq 0$
- (4) $\varepsilon(x) = 0, x \geq 0$ if and only if $x = 0$
- (5) $\varepsilon(x)^* \varepsilon(x) \leq \varepsilon(x^* x)$
- (6) $\varepsilon(\varepsilon(x_1)x_2) = \varepsilon(x_1)\varepsilon(x_2) = \varepsilon(x_1\varepsilon(x_2))$
- (7) ε is 6-weakly continuous
- (8) $\|\varepsilon(x)\| \leq \|x\|$
- (9) $\phi(\varepsilon(x)) = \phi(x)$ for all $x \in m_\varphi$

This ε is called the conditional expectation of M onto N with respect to φ and denoted by

$$\varepsilon = E_\varphi(\cdot | N).$$

Here we will simply state the provement of Takesaki [9].

Put $\mathfrak{A} = \mathfrak{A}_\varphi$ and $\mathfrak{H} = \mathfrak{H}_\varphi$. Let $\mathfrak{B}$ denote the intersection $\mathfrak{A} \cap N$. Then $\lambda_\varphi(\mathfrak{B})$ is the achieved generalized Hilbert algebra obtained by φ/N_+ ; hence $\mathfrak{L}(\mathfrak{B})$ is isomorphic to N . Let $\mathfrak{R}$ be the completion of $\lambda_\varphi(\mathfrak{B})$. Then $\mathfrak{R}$ is a closed subspace of $\mathfrak{H}$. Let E be the projection of $\mathfrak{H}$ onto $\mathfrak{R}$. Let $\Delta_{\mathfrak{A}}(\Delta_{\mathfrak{B}})$ be the modular operator of $\mathfrak{A}(\mathfrak{B})$ defined in $\mathfrak{H}(\mathfrak{R})$ and let $\Pi_{\mathfrak{A}}(\Pi_{\mathfrak{B}})$ be a left representation of $\mathfrak{A}(\mathfrak{B})$. Since N is Σ -invariant, the projection E commutes with $\Delta_{\mathfrak{A}}$ and we have

$$\begin{aligned} \Delta_{\mathfrak{B}} &= \Delta_{\mathfrak{A}}/\mathfrak{R} \\ \mathfrak{B} &= E\mathfrak{A}, \mathfrak{B}' = E\mathfrak{A}' \\ E(\xi \cdot \eta) &= \xi E\eta \quad \xi \in \mathfrak{B}, \eta \in \mathfrak{A} \\ E(\xi \cdot \eta) &= (E\xi)\eta \quad \xi \in \mathfrak{A}', \eta \in \mathfrak{B}'. \end{aligned}$$

Hence we have

$$\Pi_{\mathfrak{B}}(E\xi) = E\Pi_{\mathfrak{A}}(\xi)E, \xi \in \mathfrak{A}$$

and therefore,

$$E\mathfrak{L}(\mathfrak{A})E/\mathfrak{R} = \mathfrak{L}(\mathfrak{B})$$

Here we define

$$\varepsilon(x) = \Pi_N^{-1}(E\Pi_M(x)E), \quad x \in M.$$

$$\left(\begin{array}{l} \Pi_M(\Pi_N) \text{ is an isomorphism} \\ \text{of } M(N) \text{ onto } \mathfrak{L}(\mathfrak{A})(\mathfrak{L}(\mathfrak{B})) \end{array} \right)$$

We can prove that this ε satisfies the conditions of the theorem.

PROPOSITION 2.2. *Moreover, ε has the following properties;*

$$(10) \quad \varepsilon(\sigma_t(x)) = \sigma_t(\varepsilon(x)) \quad \text{for all } t \in R$$

$$(11) \quad \|\varepsilon(x)\|_2 \leq \|x\|_2 \quad \text{for all } x \in n_\varphi$$

(12) ε is uniquely extended to the projection of norm one from $L^2(\varphi, M)$ into $L^2(\varphi, N)$.

Particularly, if φ is finite, then

$$(13) \quad \|\varepsilon(x)\|_1 \leq \|x\|_1 \quad \text{for all } x \in M$$

(14) ε is uniquely extended to the projection of norm one from $L^1(\varphi, M)$ onto $L^1(\varphi, N)$.

(Proof) (10);

$$\begin{aligned} \varepsilon(\sigma_t(x)) &= \Pi_N^{-1}(E\Pi_M(\sigma_t(x))E) \\ &= \Pi_N^{-1}(E\Delta^{it}\Pi_M(x)\Delta^{-it}E) \\ &= \Pi_N^{-1}(\Delta^{it}E\Pi_M(x)E\Delta^{-it}) \\ &= \Pi_N^{-1}(\Delta^{it}\Pi_N(x)\Delta^{-it}) \\ &= \sigma_t(\varepsilon(x)) \end{aligned}$$

(11); For each $x \in n_\varphi$,

$$\begin{aligned} \|\varepsilon(x)\|_2^2 &= \varphi(\varepsilon(x)^*\varepsilon(x)) \\ &\leq \varphi(\varepsilon(x^*x)) \quad (\text{by (5)}) \\ &= \varphi(x^*x) \quad (\text{by (9)}) \\ &= \|x\|_2^2 \end{aligned}$$

Hence ε is extended to a continuous linear map from $L^2(\varphi, M)$ into

$L^2(\varphi, N)$ and is denoted by ε .

For each $x \in L^2(\varphi, M)$ we have

$$\|\varepsilon(x)\|_2 \leq \|x\|_2$$

(12); For each $x \in L^2(\varphi, M)$, there exists $\{x_n\} \subset n_\varphi$ such that

$$x_n \rightarrow x \quad \text{in } L^2(\varphi, M).$$

Since ε is continuous in L^2 -norm,

$$\varepsilon(x_n) \rightarrow \varepsilon(x) \quad \text{in } L^2(\varphi, N).$$

Therefore,

$$\varepsilon(\varepsilon(x_n)) \longrightarrow \varepsilon(\varepsilon(x))$$

On the other hand, by (2)

$$\varepsilon(\varepsilon(x_n)) = \varepsilon(x_n) \longrightarrow \varepsilon(x).$$

Hence we have

$$\begin{aligned} \varepsilon(\varepsilon(x)) &= \varepsilon(x) \\ \|\varepsilon\| &= \sup_{x \in n_\varphi} \frac{\|\varepsilon(x)\|_2}{\|x\|_2} \\ &\geq \sup_{x \in n_\varphi \cap N} \frac{\|\varepsilon(x)\|_2}{\|x\|_2} \\ &= \sup_{x \in n_\varphi \cap N} \frac{\|x\|_2}{\|x\|_2} = 1 \end{aligned}$$

On the other hand, by (11),

$$\|\varepsilon(x)\|_2 \leq \|x\|_2$$

Therefore, $\|\varepsilon\| = 1$

(13); If φ is finite, then by [[5] Corollary 3.10.] we have

$$M \subset L^1(\varphi).$$

For each $x \in M$,

$$\begin{aligned}
\|\varepsilon(x)\|_1 &= \sup_{\substack{a \in N \\ \|a\| \leq 1}} |\phi(a\varepsilon(x))| \\
&= \sup |\phi(\varepsilon(ax))| \\
&= \sup_{a \in N; \|a\| \leq 1} |\phi(ax)| \\
&\leq \sup_{a \in M; \|a\| \leq 1} |\phi(ax)| \\
&= \|x\|_1
\end{aligned}$$

Hence ε is extended to a continuous linear map from $L^1(\varphi, M)$ into $L^1(\varphi, N)$ and we denote by ε .

For each $x \in L^1(\varphi, M)$

$$\|\varepsilon(x)\|_1 \leq \|x\|_1$$

(14); For each $x \in L^1(\varphi, M)$ there exists $\{x_n\} \subset l_\varphi$ such that

$$x_n \rightarrow x \quad \text{in } L^1(\varphi, M).$$

Since ε is continuous in L^1 -norm,

$$\varepsilon(x_n) \rightarrow \varepsilon(x) \quad \text{in } L^1(\varphi, N)$$

$$\varepsilon(\varepsilon(x_n)) \longrightarrow \varepsilon(\varepsilon(x)).$$

On the other hand

$$\varepsilon(\varepsilon(x_n)) = \varepsilon(x_n) \longrightarrow \varepsilon(x).$$

Hence we have

$$\begin{aligned}
\varepsilon(\varepsilon(x)) &= \varepsilon(x) \\
\|\varepsilon\| &= \sup_{x \in M} \frac{\|\varepsilon(x)\|_1}{\|x\|_1} \\
&\geq \sup_{x \in N} \frac{\|\varepsilon(x)\|_1}{\|x\|_1} = 1
\end{aligned}$$

On the other hand

$$\|\varepsilon(x)\|_1 \leq \|x\|_1$$

Therefore, we have $\|\varepsilon\| = 1$.

3. Characterization

THEOREM 3.1. *If (M, φ, Σ) is given and ε' is a linear transform from M into M and satisfies the following conditions;*

$$\varepsilon'(1) = 1$$

$$(3)' \quad \varepsilon'(x) \geq 0 \quad \text{for } x \geq 0$$

$$(6)' \quad \varepsilon'(\varepsilon'(x_1)x_2) = \varepsilon'(x_1)\varepsilon'(x_2) = \varepsilon'(x_1\varepsilon'(x_2))$$

$$(7)' \quad \varepsilon' \text{ is } \sigma\text{-weakly continuous}$$

$$(9)' \quad \varphi(\varepsilon'(x)) \leq \varphi(x), \quad x \in M_+$$

$$(10)' \quad \varepsilon'(\sigma_t(x)) = \sigma_t(\varepsilon'(x)),$$

then $N = \{\varepsilon'(x); x \in M\}$ is a Σ -invariant von Neumann subalgebra and φ/N is semifinite. By Theorem 2.1 there exists a conditional expectation $\varepsilon = E_\varphi(|N)$ from M onto N with respect to φ and we have $\varepsilon = \varepsilon'$.

(Proof) For each $x \in M$ there exists x_1, x_2, x_3 and $x_4 \in M_+$ such that

$$x = x_1 - x_2 + ix_3 - ix_4.$$

Therefore,

$$x^* = x_1 - x_2 - ix_3 + ix_4$$

By (3)',

$$\begin{aligned} \varepsilon'(x^*) &= \varepsilon'(x_1) - \varepsilon'(x_2) - i\varepsilon'(x_3) + i\varepsilon'(x_4) \\ &= \varepsilon'(x)^* \end{aligned}$$

Moreover, since we have

$$\varepsilon'(x) + \varepsilon'(y) = \varepsilon'(x + y)$$

$$\lambda\varepsilon'(x) = \varepsilon'(\lambda x)$$

$$\varepsilon'(x)\varepsilon'(y) = \varepsilon'(\varepsilon'(x)y)$$

$$\varepsilon'(1) = 1,$$

N is a $*$ -subalgebra of M with 1.

By (7)', N is σ -weakly closed and by (10)', N is a Σ -invariant.

Hence N is a Σ -invariant von Neumann subalgebra of M .

For each $\varepsilon'(x) \in N$ there exists $\{x_n\} \subset m_\varphi$ such that

$$x_n \longrightarrow x \quad \sigma\text{-weakly.}$$

Since ε' is σ -weakly continuous,

$$\varepsilon'(x_n) \longrightarrow \varepsilon'(x) \quad \sigma\text{-weakly}$$

By (9)', $\varepsilon'(x_n) \in m_\varphi \cap N$

Hence $m_\varphi \cap N$ is σ -weakly dense in N i.e., φ/N is semifinite.

By Theorem 2.1 there exists a conditional expectation $\varepsilon = E_\varphi(\cdot|N)$.

We denote by

$$\varphi_1(x) = \varphi(\varepsilon'(x)) \quad x \in M_+.$$

By the linearity of ε' and (3)', φ_1 is a weight on M_+ and by the normality of ε' , φ_1 is normal.

Since $\varepsilon'(m_\varphi) \subset m_{\varphi_1}$, m_{φ_1} is σ -weakly dense in M i.e., φ_1 is semifinite.

$$\begin{aligned} \varphi_1(\sigma_t(x)) &= \varphi(\varepsilon'(\sigma_t(x))) \\ &= \varphi(\sigma_t(\varepsilon'(x))) \\ &= \varphi(\varepsilon'(x)) \\ &= \varphi_1(x) \end{aligned}$$

Hence φ_1 is a Σ -invariant normal semifinite weight on M_+ . By (9)'

$$\varphi_1 \leq \varphi$$

By [[6] Theorem 5.12; Radon-Nikodym Theorem], there exists uniquely $h \in M^+$ such that

$$0 \leq h \leq 1$$

$$\varphi_1(x) = \varphi(\varepsilon'(x)) = \varphi(h^{\frac{1}{2}} x h^{\frac{1}{2}}), \quad x \in M_+$$

For each $z \in m_\varphi \cap N$ and $x \in m_\varphi$,

$$\begin{aligned}
\phi(z\varepsilon'(x)) &= \phi(\varepsilon'(zx)) \\
&= \phi_1(zx) \\
&= \phi(\bar{h}^{\frac{1}{2}}zxh^{\frac{1}{2}}) \\
&= \phi(zxh) \\
&= \phi(\varepsilon(zxh)) \\
&= \phi(z\varepsilon(xh))
\end{aligned}$$

Therefore, for each $x \in m_\varphi$ we have

$$\varepsilon'(x) = \varepsilon(xh)$$

Since ε' and ε are σ -weakly continuous and m_φ is σ -weakly dense in M ,

$$\varepsilon'(x) = \varepsilon(xh) \quad \text{for all } x \in M \dots (*)$$

Particularly, from $\varepsilon'(1) = 1$,

$$\begin{aligned}
\varepsilon(h) &= 1 \\
\varepsilon(1-h) &= \varepsilon(1) - \varepsilon(h) \\
&= 1 - 1 = 0.
\end{aligned}$$

Since $0 \leq h \leq 1$ and ε is faithful, we have $h = 1$.

By (*),

$$\varepsilon'(x) = \varepsilon(x) \quad \text{for all } x \in M.$$

THEOREM 3.2. *(M, φ, Σ) is given and let φ be finite. If ε' is a linear transform from M into M and satisfies the following conditions;*

- (3)' $\varepsilon'(x) \geq 0 \quad \text{if } x \geq 0$
- (6)' $\varepsilon'(\varepsilon'(x_1)x_2) = \varepsilon'(x_1)\varepsilon'(x_2) = \varepsilon'(x_1\varepsilon'(x_2))$
- (10)' $\varepsilon'(\sigma_t(x)) = \sigma_t(\varepsilon'(x))$
- (13)' $\|\varepsilon'(x)\|_1 \leq \|x\|_1,$

then the weak closure N of $N_0 = \{z \in M; \varepsilon'(zx) = z\varepsilon'(x)$
 $\varepsilon'(xz) = \varepsilon'(x)z$
for all $x \in M\}$

is a Σ -invariant von Neumann subalgebra of M and φ/N is finite. By Theorem 2.1. there exists a conditional expectation $\varepsilon = E_\varphi(|N)$ from M onto N with respect to φ and moreover there exists uniquely a self-adjoint operator $h \in M^\Sigma$ such that

$$\|h\| \leq 1$$

$$\varepsilon(h) = \varepsilon'(1)$$

$$\varepsilon'(x) = \varepsilon(xh) = \varepsilon(hx) \quad \text{for all } x \in M$$

Particularly, if $\varepsilon'(1) = 1$, then we have

$$\varepsilon'(x) = \varepsilon(x) \quad \text{for all } x \in M.$$

(Proof) We have clearly that N is a von Neumann subalgebra of M . For each $z \in N_0$ and $x \in M$,

$$\begin{aligned} \varepsilon'(\sigma_t(z)x) &= \varepsilon'(\sigma_t(z\sigma_{-t}(x))) \\ &= \sigma_t(\varepsilon'(z\sigma_{-t}(x))) \\ &= \sigma_t(z\varepsilon'(\sigma_{-t}(x))) \\ &= \sigma_t(z)\sigma_t(\varepsilon'(\sigma_{-t}(x))) \\ &= \sigma_t(z)\varepsilon'(x) \end{aligned}$$

Similarly,

$$\varepsilon'(x\sigma_t(z)) = \varepsilon'(x)\sigma_t(z)$$

Therefore, we have

$$\sigma_t(N_0) \subset N_0$$

Hence

$$\sigma_t(N) \subset N$$

We shew that N is a Σ -invariant von Neumann subalgebra of M and clearly φ/N is finite. By Theorem 2.1 there exists $\varepsilon = E_\varphi(|N)$.

We denote by

$$\varphi_1(x) = \varphi(\varepsilon'(x)), \quad x \in M.$$

Then we have

$$\begin{aligned} |\varphi_1(x)| &= |\varphi(\varepsilon'(x))| \\ &\leq \sup_{a \in M; \|a\| \leq 1} |\varphi(a\varepsilon'(x))| \\ &= \|\varepsilon'(x)\|_1 \\ &\leq \|x\|_1 \end{aligned}$$

Hence φ_1 is extended to a continuous linear functional on $L^1(\varphi)$ and is denoted by φ_1 . By [5] Corollary 4.4., there exists uniquely $h \in M$ such that

$$\varphi_1(x) = \varphi(hx) \quad \text{for all } x \in M.$$

Moreover, for each $x \in M$

$$\begin{aligned} \varphi(\sigma_t(h)x) &= \varphi(\sigma_t(h\sigma_{-t}(x))) \\ &= \varphi(h\sigma_{-t}(x)) \\ &= \varphi_1(\sigma_{-t}(x)) \\ &= \varphi(\varepsilon'(\sigma_{-t}(x))) \\ &= \varphi(\sigma_{-t}(\varepsilon'(x))) \\ &= \varphi(\varepsilon'(x)) \\ &= \varphi_1(x) \\ &= \varphi(hx). \end{aligned}$$

Therefore, we have

$$\sigma_t(h) = h \quad \text{i.e., } h \in M^\Sigma.$$

For each $x \in M$ and $z \in N_0$

$$\begin{aligned} \varphi(z\varepsilon'(x)) &= \varphi(\varepsilon'(zx)) \\ &= \varphi(hzx) \end{aligned}$$

$$\begin{aligned}
&= \varphi(z x h) \\
&= \varphi(\varepsilon(z x h)) \\
&= \varphi(z \varepsilon(x h))
\end{aligned}$$

Since φ is σ -weakly continuous and N_0 is σ -weakly dense in N , we have

$$\varphi(z \varepsilon'(x)) = \varphi(z \varepsilon(x h))$$

for each $z \in N$.

By (6)' since $\varepsilon'(x)$ and $\varepsilon(x h) \in N$, we have

$$\varepsilon'(x) = \varepsilon(x h) \quad \text{for all } x \in M \dots (*)'$$

For all $x \in M$,

$$\begin{aligned}
\varphi(x h) &= \varphi(\varepsilon'(x)) \\
&= \varphi(\varepsilon'(x^*)^*) \\
&= \overline{\varphi(\varepsilon'(x^*))} \\
&= \overline{\varphi(x^* h)} \\
&= \varphi((x^* h)^*) \\
&= \varphi(h^* x) \\
&= \varphi(x h^*)
\end{aligned}$$

Therefore, we have $h = h^*$.

By [5] Corollary 4.4.]

$$\begin{aligned}
\|h\| &= \sup_{\substack{x \in M \\ \|x\|_1 \leq 1}} | \langle h, \omega(x) \rangle | \\
&= \sup_{\substack{x \in M \\ \|x\|_1 \leq 1}} |\varphi(x h)| \\
&= \sup |\varphi(\varepsilon'(x))| \\
&\leq \sup_{x \in M; \|x\|_1 \leq 1} \|x\|_1 \leq 1
\end{aligned}$$

Therefore, we have

$$h^2 \leq 1$$

For each $x \in M$,

$$\begin{aligned}\varepsilon(xh) &= \varepsilon'(x) \\ &= \varepsilon'(x^*)^* \\ &= \varepsilon'(x^*h)^* \\ &= \varepsilon'(h^*x) \\ &= \varepsilon'(hx)\end{aligned}$$

Particularly, if $\varepsilon'(1) = 1$, then by $(*)'$

$$\begin{aligned}\varepsilon(h) &= \varepsilon'(1) = 1 \\ 0 &\leq \varepsilon((1-h)(1-h)) \\ &= \varepsilon(1-2h+h^2) \\ &= \varepsilon(1) - 2\varepsilon(h) + \varepsilon(h^2) \\ &= -1 + \varepsilon(h^2) \\ &\leq 0\end{aligned}$$

Therefore, we have $h = 1$.

By $(*)'$, we have

$$\varepsilon'(x) = \varepsilon(x) \quad \text{for all } x \in M.$$

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