EQUIVALENT MEASURES ON PRODUCT SPACES

By

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0. Introduction

Let $(\mathcal{Q}_n, \mathcal{F}_n, P_n)$ be a probability measure space for each $n \geq 1$ and let $(\mathcal{Q}, \mathcal{F}, P) = \prod_{n=1}^{\infty} (\mathcal{Q}_n, \mathcal{F}_n, P_n)$ be the product probability measure space. Assume that for each $n \geq 1$ there exists a σ -finite measure μ_n defined on the σ -algebra $\mathcal{F}_n$.

A σ -finite measure μ defined on the σ -algebra $\mathcal{F}$ is said to be a σ -finite *quasi-product* measure of $\{\mu_n\}_{n \geq 1}$ if for each $n \geq 1$

$$\mu = \prod_{i=1}^n \mu_i \times \mu_{n+1}^*$$

where μ_{n+1}^* is a σ -finite measure defined on the product space $\prod_{i=n+1}^{\infty} (\mathcal{Q}_i, \mathcal{F}_i)$ [6] [4].

Now the problem is that; Under what conditions on $\{P_n\}_{n \geq 1}$ and $\{\mu_n\}_{n \geq 1}$ does there exist a σ -finite, quasi-product measure μ equivalent with a product probability measure P ? (The equivalence of measures means that $P(A) = 0$ iff $\mu(A) = 0$). This problem was first discussed by S. Kakutani [5] who gave a necessary sufficient condition for the existence of an equivalent, *finite*, quasi-product measure μ , where $\mu(\mathcal{Q}) = \mu_n(\mathcal{Q}_n) = 1$ $n \geq 1$. The σ -finite case was discussed by C.C. MOORE [6] and O. TAKENOUCHI [8] when each $\mathcal{Q}_n$ is a finite or countable set and by D. HILL [4] when $\mathcal{Q}_n$ is a general probability measure space. And also it was shown in [6] and [4] that this existence problem is corresponding to the existence problem of equivalent σ -finite invariant measures under adding machine transformations.

Our purpose of this paper is to obtain a more simple and efficient, necessary sufficient condition for the existence of an equivalent σ -finite

quasi-product measure (see Theorem and Examples) and to remark the general D. Hill's condition is easily obtained from our condition with the use of the Kolmogorov's three series theorem.

1. Theorem and examples

If there exists an equivalent σ -finite quasi-product measure of $\{\mu_n\}_{n \geq 1}$ then μ_n and P_n are equivalent with each other ($\mu_n(A) = 0$ iff $P_n(A) = 0$). And so we may assume the equivalence of μ_n and P_n $n \geq 1$ through this paper. We denote the Radon-Nikodym derivative $\frac{d\mu_n}{dP_n}(\omega_n)$ by $X_n(\omega_n)$ $\omega_n \in \mathcal{Q}_n$, and the projection $\omega = (\omega_n)_{n \geq 1} \rightarrow \omega_n \in \mathcal{Q}_n$ by $\pi_n(\omega)$. We denote simply the sub σ -algebra $\pi_n^{-1}\mathcal{F}_n = \{\pi_n^{-1}A; A \in \mathcal{F}_n\}$ of $\mathcal{F}$ by $\mathcal{F}_n$ and the $\mathcal{F}_n$ -measurable random variable $X_n(\pi_n(\omega))$ $\omega \in \mathcal{Q}$ by $X_n(\omega)$, and then $X_1(\omega), X_2(\omega), \dots$ are positive valued, mutually independent random variables.

THEOREM *The following conditions are equivalent.*

- (1) *There exists a σ -finite, quasi-product measure μ of $\{\mu_n\}_{n \geq 1}$ equivalent with a product probability measure P .*
- (2) *There are positive constants $b_1, b_2, \dots$ such that*

$$\prod_{n=1}^{\infty} \frac{X_n(\omega)}{b_n}$$

converges with probability 1.

Proof. First we remark the condition (1) is equivalent with the following condition (1)' through the relation $\frac{d\mu}{dP}(\omega) = X(\omega)$.

- (1') *There exists a positive valued, $\mathcal{F}$ -measurable random variable $X(\omega)$ such that for each $n \geq 1$*

$$X(\omega) = \prod_{i=1}^n X_i(\omega) \times X_{n+1}^*(\omega)$$

where $X_{n+1}^*(\omega)$ is a $\bigvee_{i=n+1}^{\infty} \mathcal{F}_i$ -measurable random variable. (The sub

σ -algebra $\bigvee_{i=n+1}^{\infty} \mathcal{F}_i$ is the σ -algebra generated by $\mathcal{F}_{n+1}, \mathcal{F}_{n+2}, \dots$).

So, it is enough to show the equivalence of (1)' and (2). But this equivalence was already proved by J.L. Doob ([3] Chap. III Theorem 2.7 and

Theorem 2.8) using the Kolmogorov's three series theorem. Here we shall give a simple and short proof based on the Martingale convergence theorem.

LEMMA *If random variables $U_1(\omega), U_2(\omega), \dots$ satisfy*

$$|E\{e^{itU_n(\omega)}\}| \xrightarrow{n \rightarrow \infty} 1 \quad t \in R^1$$

then there are constants $a_1, a_2, \dots$ such that

$$U_n(\omega) - a_n \xrightarrow{n \rightarrow \infty} 0$$

in probability.

Proof. Let $\tilde{U}_n(\omega)$ be an independent random variable of $U_n(\omega)$ with the same distribution. Then

$$\int_{-1}^1 1 - |E\{e^{itU_n}\}|^2 dt = 2E\{1 - \phi(U_n - \tilde{U}_n)\}$$

$$\text{where } \phi(u) = \begin{cases} \frac{\sin u}{u} & \text{if } u \neq 0 \\ 1 & \text{if } u = 0. \end{cases}$$

Choosing small $\varepsilon > 0$,

$$\begin{aligned} E\{1 - \phi(U_n - \tilde{U}_n)\} &\geq (1 - \phi(\varepsilon))P(|U_n(\omega) - \tilde{U}_n(\omega)| > \varepsilon) \\ &\geq (1 - \phi(\varepsilon))P(U_n(\omega) - a_n > \varepsilon, \tilde{U}_n(\omega) \leq a_n) \\ &\quad + (1 - \phi(\varepsilon))P(U_n(\omega) - a_n < -\varepsilon, \tilde{U}_n(\omega) \geq a_n) \\ &\geq \frac{1}{2}P(|U_n(\omega) - a_n| > \varepsilon) \end{aligned}$$

where a_n is a median of $U_n(\omega)$;

$$P(U_n(\omega) \leq a_n) \geq \frac{1}{2}$$

$$P(U_n(\omega) \geq a_n) \geq \frac{1}{2}.$$

By the bounded convergence theorem

$$\int_{-1}^1 1 - |E\{e^{itU_n(\omega)}\}|^2 dt \xrightarrow{n \rightarrow \infty} 0$$

and so

$$U_n(\omega) - a_n \xrightarrow{n \rightarrow \infty} 0$$

in probability.

q.e.d.

Proof of (1)' $\rightarrow$ (2). Let $Y_n(\omega) = \log X_n(\omega)$, $Y_{n+1}^*(\omega) = \log X_{n+1}^*(\omega)$ and $Y(\omega) = \log X(\omega)$.

$$E\{e^{itY} \mid \bigvee_{j=1}^n \mathcal{F}_j\} = \prod_{j=1}^n e^{itY_j} E\{e^{itY_{n+1}^*}\}.$$

By the Martingale convergence theorem

$$E\{e^{itY} \mid \bigvee_{j=1}^n \mathcal{F}_j\} \xrightarrow{n \rightarrow \infty} e^{itY(\omega)}$$

with probability 1, and so

$$|E\{e^{itY_{n+1}^*}\}| \xrightarrow{n \rightarrow \infty} 1 \quad t \in R^1.$$

By the lemma there are constants $a_1, a_2, \dots$ such that

$$Y_{n+1}^*(\omega) - a_n \xrightarrow{n \rightarrow \infty} 0$$

in probability. Then for a sequence $\{c_n\}_{n \geq 1}$ with $\sum_{i=1}^n c_i = -a_n$ $n = 1, 2, \dots$

$$\sum_{i=1}^n Y_i(\omega) - c_i \xrightarrow{n \rightarrow \infty} Y(\omega)$$

in probability. From the independence of $\{Y_n\}_{n \geq 1}$

$$\sum_{i=1}^n Y_i(\omega) - c_i \xrightarrow{n \rightarrow \infty} Y(\omega)$$

with probability 1. If we set $b_n = e^{c_n}$ then

$$\prod_{n=1}^{\infty} \frac{X_n(\omega)}{b_n}$$

converges with probability 1.

Proof of (2) $\rightarrow$ (1)' is trivial.

q.e.d.

EXAMPLES. If there exists an equivalent σ -finite quasi-product mea-

sure μ then mutually independent random variables $X_1(\omega), X_2(\omega), \dots$ must satisfy that for a positive sequence $\{b_n\}_{n \geq 1}$

$$\frac{X_n(\omega)}{b_n} \xrightarrow{n \rightarrow \infty} 1$$

in probability. It is clear the following examples do not satisfy this necessary condition for the existence.

(1) ORNSTEIN's example [7] [1] [4].

$$\Omega_n = \{1, 2, \dots, k_n\} \quad k_n \geq 3 \quad P_n(i) = \begin{cases} \frac{1}{2} & \text{if } i=1 \\ \frac{1}{2(k_n-1)} & \text{if } i>1 \end{cases}$$

$$\mu_n(i) = \frac{1}{k_n} \quad X_n(\omega) = \begin{cases} \frac{2}{k_n} & \text{if } \omega_n=1 \\ \frac{2(k_n-1)}{k_n} & \text{if } \omega_n>1 \end{cases}$$

(2) BRUNEL's example [2] [1] [4].

$$\Omega_n = \{1, 2, 3\} \quad P_n(i) = \begin{cases} \frac{1}{4} & \text{if } i=1, 3 \\ \frac{1}{2} & \text{if } i=2 \end{cases}$$

$$\mu_n(i) = \frac{1}{3} \quad X_n(\omega) = \begin{cases} \frac{4}{3} & \text{if } \omega_n=1 \text{ or } \omega_n=3 \\ \frac{2}{3} & \text{if } \omega_n=2 \end{cases}$$

(3) ARNOLD's example [1] [4].

$$\Omega_n = \{1, 2\} \quad P_n(i) = \begin{cases} \frac{1}{\alpha+1} & \text{if } i=1 \\ \frac{\alpha}{\alpha+1} & \text{if } i=2 \end{cases} \quad 0 < \alpha < 1$$

$$\mu_n(i) = \frac{1}{2} \quad X_n(\omega) = \begin{cases} \frac{\alpha+1}{2} & \text{if } \omega_n = 1 \\ \frac{\alpha+1}{2\alpha} & \text{if } \omega_n = 2 \end{cases}$$

REMARK 1. An equivalent σ -finite quasi-product measure μ , if it exists, is unique up to constants from the 0-1 law.

REMARK 2. In the finite case ($\mu(\Omega) < \infty$) it can be more easily shown that *there exists an equivalent, finite, quasi-product measure if and only if there are positive constants $b_1, b_2, \dots$ such that*

$$\prod_{n=1}^{\infty} \sqrt{\frac{X_n(\omega)}{b_n}}$$

converges in the $L^2(\Omega, P)$ -sense and also, if and only if

$$\sum_{n=1}^{\infty} 1 - \frac{E(\sqrt{X_n})^2}{E(X_n)}$$

converges. The last condition is the Kakutani's condition [5], where $E(X_n) = 1$, $\mu(\Omega) = 1$.

REMARK 3. From the Kolmogorov's three series theorem the following lemma is proved.

LEMMA *For positive valued, mutually independent random variables $X_1(\omega), X_2(\omega), \dots, \prod_{n=1}^{\infty} X_n(\omega)$ converges with probability 1 if and only if there exists a $\mathcal{F}_n$ -measurable set A_n for each $n \geq 1$ such that the following three infinite products converge. (a) $\prod_{n=1}^{\infty} P(A_n)$ (b) $\prod_{n=1}^{\infty} E(X_n; A_n)$ (c) $\prod_{n=1}^{\infty} E(\sqrt{X_n}; A_n)$.*

And so from this Lemma and our Theorem it follows that *there exists a σ -finite quasi-product measure μ equivalent with P if and only if there exists a $\mathcal{F}_n$ -measurable set A_n for each $n \geq 1$ such that*

$$\sum_{n=1}^{\infty} 1 - \frac{E(\sqrt{X_n}; A_n)^2}{E(X_n; A_n)}$$

converges. The last condition is the D. Hill's condition [4].

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