CUBIC SPLINE FUNCTION AND DIFFERENCE METHOD

By

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(Received Oct. 29, 1973)

1. Introduction

Finite difference methods are of much use for approximating solutions of two-point boundary value problems. In the present paper, we shall give its mathematical foundations by using cubic splines ([4]).

Here we consider the following two-point boundary value problem:

$$(1) \quad \ddot{x}(t) = f(t, x(t)) \quad (0 \leq t \leq 1)$$

with boundary conditions

$$(2) \quad A_0 x(0) - B_0 \dot{x}(0) = a$$

$$(3) \quad A_1 x(1) + B_1 \dot{x}(1) = b,$$

where $f(t, x)$ is defined and twice continuously differentiable in a region D of (t, x) -space intercepted by two lines $t=0$ and $t=1$.

By applying the usual central difference method to (1)–(3), we obtain a system of nonlinear equations:

$$(4) \quad \frac{x_{p+1} - 2x_p + x_{p-1}}{h^2} = f(ph, x_p) \quad (1 \leq p \leq n-1),$$

$$(5) \quad \frac{x_1 - x_0}{h} - y_0 = \frac{h}{2} f(0, x_0),$$

$$(6) \quad A_0 x_0 - B_0 y_0 = a,$$

$$(7) \quad -\frac{x_n - x_{n-1}}{h} + y_n = \frac{h}{2} f(1, x_n),$$

$$(8) \quad A_1 x_n + B_1 y_n = b \quad ([5]).$$

Now by making use of B -spline $Q_4(t) = \frac{1}{6} \sum_{i=0}^4 (-1)^i \binom{4}{i} (t-i)_+^3$, we consider spline function of the form $x_h(t) = \sum_{p=-3}^{n-1} \alpha_p Q_4\left(\frac{t}{h} - p\right) (nh=1)$ with undetermined coefficients $\alpha_{-3}, \alpha_{-2}, \dots, \alpha_{n-1}$.

The above $x_h(t)$ will be an approximate solution to the problem (1)–(3) if it satisfies

$$(9) \quad \ddot{x}_h(t) = Pf(t, x_h(t)) \quad (0 \leq t \leq 1),$$

$$(10) \quad A_0 x_h(0) - B_0 \dot{x}_h(0) = a,$$

$$(11) \quad A_1 x_h(1) + B_1 \dot{x}_h(1) = b.$$

Here P is an operator defined by $(Pf)(t) = \sum_{p=0}^n \lambda_p L_p(t)$ such that

$$(12) \quad \sum_{p=0}^n \lambda_p \int_0^1 L_p(t) L_q(t) dt = \begin{cases} \frac{h}{2} f(t_q) & (q=0, n) \\ hf(t_q) & (q=1, 2, \dots, n-1), \end{cases}$$

where $L_p(t)$ is a piecewise linear function with property $L_p(t_q) = \delta_{p,q}$ ($t_q = qh$). Here we demonstrate two lemmas concerning the equivalence of the problem (4)–(8) and the problem (9)–(11).

LEMMA 1. *If the approximate solution $x_h(t)$ exists, $(x_h(0), x_h(h), \dots, x_h(nh), \dot{x}_h(0), \dot{x}_h(1))$ satisfy a system of nonlinear equations (4)–(8) with respect to $(x_0, x_1, \dots, x_n, y_0, y_n)$.*

Proof.

$$\text{Since } (Pf, L_q) = \begin{cases} \frac{h}{2} f(t_q) & (q=0, n) \\ hf(t_q) & (q=1, 2, \dots, n-1), \end{cases}$$

we have $(\ddot{x}_h, L_q) = (Pf(t, x_h(t)), L_q(t))$

$$(13) \quad = \begin{cases} \frac{h}{2} f(t_q, x_h(t_q)) & (q=0, n) \\ hf(t_q, x_h(t_q)) & (q=1, 2, \dots, n-1), \end{cases}$$

where we denote $\int_0^1 \varphi(t) \psi(t) dt$ by (φ, ψ) .

From (13), by integrating by parts, we have the desired result.

LEMMA 2. *If the solution $(x_0, x_1, \dots, x_n, y_0, y_n)$ of (4)–(8) exists, then a cubic spline function $x_h(t)$ such that*

$$x_h(t_p) = x_p \quad (p=0, 1, \dots, n), \quad \dot{x}_h(t_p) = y_p \quad (p=0, n)$$

satisfies the problem (9)–(11) ([1, p. 29]).

Proof. From (4), (5) and (7), it follows that

$$(14) \quad (\ddot{x}_h, L_q) = (Pf(t, x_h(t)), L_q(t)) \quad (q=0, 1, \dots, n).$$

Since $\ddot{x}_h(t) - Pf(t, x_h(t))$ is piecewise linear, we have

$$\ddot{x}_h(t) = Pf(t, x_h(t)), \text{ as desired.}$$

Now we construct a determining equation

$F(\alpha) = (F^{(1)}(\alpha), F^{(2)}(\alpha)) = 0$ of (9)–(11) with respect to α_p ($p = -3, -2, \dots, n-1$), where $F^{(2)}(\alpha) = (F_0^{(2)}(\alpha), F_1^{(2)}(\alpha), \dots, F_n^{(2)}(\alpha))$. From (9) (or (14)) we have

$$\begin{aligned} F_p^{(2)}(\alpha) &= \frac{1}{6h^2} [(\alpha_p + 4\alpha_{p-1} + \alpha_{p-2}) - 2(\alpha_{p-1} + 4\alpha_{p-2} + \alpha_{p-3}) \\ &\quad + (\alpha_{p-2} + 4\alpha_{p-3} + \alpha_{p-4})] - f\left(t_p, \frac{1}{6}(\alpha_{p-1} + 4\alpha_{p-2} + \alpha_{p-3})\right) \\ &= 0 \quad (p=1, 2, \dots, n-1), \end{aligned}$$

$$\begin{aligned} F_0^{(2)}(\alpha) &= \frac{1}{6h^2} [(\alpha_0 + 4\alpha_{-1} + \alpha_{-2}) - (\alpha_{-1} + 4\alpha_{-2} + \alpha_{-3})] \\ &\quad - \frac{1}{2h^2} (\alpha_{-1} - \alpha_{-3}) - \frac{1}{2} f\left(t_0, \frac{1}{6}(\alpha_{-1} + 4\alpha_{-2} + \alpha_{-3})\right) \\ &= 0, \end{aligned}$$

$$\begin{aligned} F_n^{(2)}(\alpha) &= \frac{1}{6h^2} [(\alpha_{n-1} + 4\alpha_{n-2} + \alpha_{n-3}) - (\alpha_{n-2} + 4\alpha_{n-3} + \alpha_{n-4})] \\ &\quad - \frac{1}{2h^2} (\alpha_{n-1} - \alpha_{n-3}) + \frac{1}{2} f\left(t_n, \frac{1}{6}(\alpha_{n-1} + 4\alpha_{n-2} + \alpha_{n-3})\right) \\ &= 0. \end{aligned}$$

We rewrite equations (10) and (11) in the following form:

$$F^{(1)}(\alpha) = \left\{ A_0 \frac{\alpha_{-3} + 4\alpha_{-2} + \alpha_{-1}}{6h} - B_0 \frac{\alpha_{-1} - \alpha_{-3}}{2h^2} - \frac{a}{h}, \right. \\ \left. A_1 \frac{\alpha_{n-3} + 4\alpha_{n-2} + \alpha_{n-1}}{6h} + B_1 \frac{\alpha_{n-1} - \alpha_{n-3}}{2h^2} - \frac{b}{h} \right\} = 0.$$

In the present paper we assume that the problem (1)–(3) has an isolated solution $\hat{x}(t)$ satisfying the internality condition

$$U = \{(t, x) \mid |x - \hat{x}(t)| \leq \delta, t \in [0, 1]\} \subset D$$

for some $\delta > 0$.

By making use of this assumption, in what follows, we can show the existence of approximate solution $\bar{x}_h(t)$ satisfying (9)–(11) and convergence to the exact solution $\hat{x}(t)$ for sufficiently small h , from which the solvability and convergence of (4)–(8) will be obtained. This condition is a little weaker than that of [4], where he assumes that the equation $\ddot{x} = 0$ has only a trivial solution with the homogeneous boundary conditions (2) and (3).

2. Some properties of operator P

In what follows, for any $\varphi(t) \in L^2(0, 1)$, we shall denote $\left[\int_0^1 \varphi^2(t) dt \right]^{\frac{1}{2}}$ by $\|\varphi\|$, and for any finite dimensional vector c , we shall denote its Euclidean norm by $|c|$.

LEMMA 3. *Let $f(t)$ be a continuously differentiable function on $[0, 1]$. Then the operator P is uniquely determined and satisfies $\|(I - P)f\| \leq C_1 h \|f'\|$ for some constant C_1 , independent of h .*

Proof.

Since the coefficient matrix of (12) is Gram matrix and $L_p(t)$ ($p = 0, 1, \dots, n$) are mutually independent, λ_p ($p = 0, 1, \dots, n$) are uniquely determined. To estimate $\|(I - P)f\|$, we rewrite

$$(I - P)f(t) = f(t) - \sum_{p=0}^n f(t_p) L_p(t) + \sum_{p=0}^n (f(t_p) - \lambda_p) L_p(t).$$

From above, it follows that

$$(16) \quad \|(I - P)f\| \leq \frac{h}{\sqrt{6}} \|f'\| + \sqrt{h} |A - F|,$$

where $A = (\lambda_0, \lambda_1, \dots, \lambda_n)$ and $F = (f(t_0), f(t_1), \dots, f(t_n))$.

Here $\lambda_p (p=0, 1, 2, \dots, n)$ satisfy the following equation

$$\frac{1}{6} \begin{pmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & 0 & 1 & 4 & 1 \\ & & & 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{n-1} \\ \lambda_n \end{pmatrix} = \begin{pmatrix} f(t_0)/2 \\ f(t_1) \\ \vdots \\ f(t_{n-1}) \\ f(t_n)/2 \end{pmatrix}$$

We denote the above coefficient matrix by A . The components of the vector $A(F - A)$ are

$$\begin{aligned} (A(F - A))_p &= \frac{1}{6} (f(t_{p+1}) - 2f(t_p) + f(t_{p-1})) \\ &= \frac{1}{6} \left(\int_{t_p}^{t_{p+1}} f(t) dt + \int_{t_p}^{t_{p-1}} f(t) dt \right) \leq \sqrt{\frac{2h}{6}} \int_{t_{p-1}}^{t_{p+1}} f^2(t) dt \\ &\quad (p=1, 2, \dots, n-1), \end{aligned}$$

$$\begin{aligned} (A(F - A))_0 &= \frac{1}{6} (f(t_1) - f(t_0)) = \frac{1}{6} \int_{t_0}^{t_1} f(t) dt \leq \sqrt{\frac{h}{6}} \int_{t_0}^{t_1} f^2(t) dt, \end{aligned}$$

and

$$\begin{aligned} (A(F - A))_n &= \frac{1}{6} (f(t_{n-1}) - f(t_n)) \\ &\leq \sqrt{\frac{h}{6}} \int_{t_{n-1}}^{t_n} f^2(t) dt. \end{aligned}$$

From above, we have

$$(17) \quad |A(A - F)| \leq \sqrt{\frac{h}{3}} \|f\|.$$

Since A is symmetric and its eigenvalues belong to $[1/6, 1]$, we see that

$$(18) \quad |A - F| \leq 2\sqrt{h} \|f\|.$$

By (16) and (18), we have

$$\|(I-P)f\| \leq \left(\frac{1}{\sqrt{6}} + 2\right)h\|f\|, \text{ as desired.}$$

LEMMA 4.

Let $\varphi(t) = \sum_{p=0}^n \gamma_p L_p(t)$. Then $\|P\varphi\| \leq C_2 \|\varphi\|$ for some constant C_2 independent of h .

Proof. Put $(P\varphi)(t) = \sum_{p=0}^n \mu_p L_p(t)$, and we have

$$(19) \quad \|P\varphi\| \leq \sqrt{h} |\mu|.$$

Since $\mu_p (p=0, 1, 2, \dots, n)$ satisfy the following

$$(20) \quad \frac{1}{6} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 & 1 \\ & & 1 & 2 \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = \begin{pmatrix} \gamma_0/2 \\ \gamma_1 \\ \vdots \\ \gamma_n/2 \end{pmatrix}, \text{ we have}$$

$$|\mu| \leq 6|\gamma|.$$

On the other hand, by simple calculation, we have

$$(21) \quad \|\varphi\| \geq \frac{\sqrt{h}}{\sqrt{6}} |\gamma|.$$

By (19), (20) and (21), we have $\|P\varphi\| \leq 6\sqrt{6} \|\varphi\|$, as desired.

3. Central difference scheme

Let $J(\alpha)$ be the Jacobian matrix of $F(\alpha)$ with respect to $\alpha_p (p=-3, -2, \dots, n-1)$. In order to investigate the properties of $J(\alpha)$, let us consider a linear system

$$(22) \quad J(\alpha)\xi + \eta = 0,$$

where $\xi = (\xi_{-3}, \xi_{-2}, \dots, \xi_{n-1})$ and $\eta = (\eta_{-1}, \eta_0, \dots, \eta_{n+1})$.

Corresponding to ξ and η , we consider cubic and piecewise linear functions $y_1(t)$ and $y_2(t)$ defined by

$$y_1(t) = \sum_{p=-3}^{n-1} \xi_p Q_4\left(\frac{t}{h} - p\right) \text{ and } y_2(t) = \eta_0 L_0(t) + \eta_n L_n(t) \\ + \sum_{p=0}^n \eta_p L_p(t).$$

From (22), we have:

$$\begin{aligned} & \frac{1}{h^2} (y_1(t_{p+1}) - 2y_1(t_p) + y_1(t_{p-1})) - f_x\left(t_p, \frac{\alpha_{p-1} + 4\alpha_{p-2} + \alpha_{p-3}}{6}\right) y_1(t_p) \\ & + y_2(t_p) = 0 \quad (1 \leq p \leq n-1), \\ & \frac{1}{h^2} (y_1(t_1) - y_1(t_0)) - \frac{1}{h} y_1(t_0) - f_x\left(t_0, \frac{\alpha_{-1} + 4\alpha_{-2} + \alpha_{-3}}{6}\right) y_1(t_0)/2 \\ & + \frac{1}{2} y_2(t_0) = 0, \\ & \frac{1}{h^2} (y_1(t_n) - y_1(t_{n-1})) - \frac{1}{h} y_1(t_n) + f_x\left(t_n, \frac{\alpha_{n-1} + 4\alpha_{n-2} + \alpha_{n-3}}{6}\right) y_1(t_n)/2 \\ & + \frac{1}{2} y_2(t_n) = 0, \end{aligned}$$

$$(23) \quad A_0 y_1(0) - B_0 \dot{y}_1(0) = -h\eta_{-1},$$

$$(24) \quad A_1 y_1(1) + B_1 \dot{y}_1(1) = -h\eta_{n+1}.$$

Here we consider a cubic spline function $\varphi(t) = \sum_{p=-3}^{n-1} \alpha_p Q_4\left(\frac{t}{h} - p\right)$, and then we have

$$\begin{aligned} & -(\ddot{y}_1(t), L_p(t)) + (P[f_x(t, \varphi(t))y_1(t)], L_p(t)) \\ & = (Py_2(t), L_p(t)) \quad (p=0, 1, 2, \dots, n). \end{aligned}$$

Since $\ddot{y}_1(t) - P[f_x(t, \varphi(t))y_1(t)] + Py_2(t)$ is linear, we have

$$(25) \quad \ddot{y}_1(t) = P[f_x(t, \varphi(t))y_1(t)] - Py_2(t) \quad (0 \leq t \leq 1).$$

Now corresponding to $\hat{x}(t)$, as easily seen, one can determine uniquely a cubic spline function $\hat{x}_h(t)$ of the form

$$\hat{x}_h(t) = \sum_{p=-3}^{n-1} \hat{\alpha}_p Q_4\left(\frac{t}{h} - p\right)$$

so that

$$\hat{x}_h(t_p) = \hat{x}(t_p) \quad (p=0, 1, \dots, n),$$

$$\dot{\hat{x}}_h(t_p) = \dot{\hat{x}}(t_p) \quad (p=0, n).$$

Here substitute $\hat{\alpha}$ for α in (25), then we have

$$(26) \quad \ddot{y}_1(t) = P[f_x(t, \hat{x}_h(t))y_1(t)] - Py_2(t) \quad (0 \leq t \leq 1).$$

Applying lemmas 3 and 4 to the system of equations (23), (24) and (26), we have

$$\|y_1\| \leq C_3 \|y_2\| \quad (|\xi| \leq C_4 |\eta|), \text{ from which}$$

$$|J^{-1}(\hat{\alpha})| \leq C_4^{-1} \text{ for sufficiently small } h.$$

For the details of the proof, see [3].

Since $|J(\alpha) - J(\hat{\alpha})| \leq C_5 |\alpha - \hat{\alpha}|$ and $|F(\hat{\alpha})| = 0(h)$, we have the following theorem.

THEOREM 1. *In a sufficiently small neighborhood of the isolated solution $\hat{x}(t)$ of (1)–(3), there is a cubic spline function $\bar{x}_h(t)$ of the form*

$$\bar{x}_h(t) = \sum_{p=-3}^{n-1} \bar{\alpha}_p Q_4\left(\frac{t}{h} - p\right) \text{ such that}$$

- (i) *the coefficient $\bar{\alpha} = (\alpha_{-3}, \alpha_{-2}, \dots, \alpha_{n-1})$ satisfies the determining equation $F(\alpha) = 0$*
- (ii) $\|x_h - x\| = 0(h^{3/2}), \|x_h - x\|_\infty = 0(h) \quad (h \rightarrow 0).$

COROLLARY. *For sufficiently small h , there exists a solution $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n, \bar{y}_0, \bar{y}_n)$ of (4)–(8) such that*

$$(28) \quad \left[\sum_{p=0}^n \left(\hat{x}(t_p) - \bar{x}_p \right)^2 \right]^{\frac{1}{2}} = 0(h) \quad (h \rightarrow 0).$$

Proof. We have only to combine theorem 1 and lemmas 1, 2.

Now by applying the analysis in [3], we have for $k=0, 1, 2$,

$$\|D^k(\bar{x}_h - \hat{x})\| = 0(h^{3/2}) \quad (h \rightarrow 0),$$

$$\|D^k(\bar{x}_h - \hat{x})\|_\infty = 0(h) \quad (h \rightarrow 0),$$

from which follow

$$(29) \quad \left[\sum_{p=1}^{n-1} \left(\frac{1}{2h} (\bar{x}_{p+1} - \bar{x}_{p-1}) - \dot{\hat{x}}(t_p) \right)^2 \right]^{\frac{1}{2}} = 0(h) \quad (h \rightarrow 0),$$

and

$$(30) \quad \left[\sum_{p=1}^{n-1} \left(\frac{1}{h^2} (\bar{x}_{p+1} - 2\bar{x}_p + \bar{x}_{p-1}) - \ddot{\hat{x}}(t_p) \right)^2 \right]^{\frac{1}{2}} = 0(h) \quad (h \rightarrow 0).$$

If $B_0=0$ or $B_1=0$, we have $\|\bar{x}_h - \hat{x}\|_\infty = 0(h^{3/2})$,
from which follows

$$\max_{0 \leq p \leq n} |\hat{x}(t_p) - \bar{x}_p| = 0(h^{3/2}).$$

It is easily seen that the asymptotic forms (28)–(29) are improved because of accurate treatment of boundary conditions (2) and (3) ([4]).

4. Some higher order differenc schems.

In this section, we assume $B_0=B_1=0$ and $A_0=A_1=1$.

In the first place we consider the following piecewise linear functions $\varphi_p(t)$ ($p=0, 1, \dots, n$)
so that

$$\varphi_p(t) = L_p(t) \quad (p=0, 1, n-1, n),$$

$$(31) \quad \varphi_p(t) = \frac{1}{12} (14L_p(t) - L_{p+1}(t) - L_{p-1}(t)) \quad (p=2, 3, \dots, n-2).$$

Since $\varphi_p(t)$ ($p=0, 1, \dots, n$) are linearly independent, we can define an operator $(pf)(t) = \sum_{p=0}^n \lambda_p \varphi_p(t)$ as follows:

$$(32) \quad \sum_{p=0}^n \lambda_p (\varphi_p, \varphi_q) = \begin{cases} \frac{h}{2} f(t_q) & (q=0, n) \\ hf(t_q) & (q=1, 2, \dots, n-1). \end{cases}$$

Then the equation (9) corresponds to a system of (5), (7) and the following equations.

$$(33) \quad \frac{1}{h^2} (x_{p+1} - 2x_p + x_{p-1}) = f(t_p, x_p) \quad (p=1, n-1),$$

$$(34) \quad \frac{-1}{12h^2} (x_{p+2} - 16x_{p+1} + 30x_p - 16x_{p-1} + x_{p-2}) = f(t_p, x_p) \\ (p=2, 3, \dots, n-2).$$

Here we notice that a system of equations (6), (8), (33) and (34) is the difference scheme considered by J. Bramble and H. Hubbard ([2]).

Now we define a system of equations $H(x) = (H^{(1)}(x), H^{(2)}(x), H^{(3)}(x)) = 0$:

$$H^{(1)}(x) = \left\{ \frac{1}{h} (x_0 - a), \frac{1}{h} (x_n - b) \right\} = 0, \\ H^{(2)}(x) = \left\{ \frac{1}{\sqrt{h}} (x_{p+1} - 2x_p + x_{p-1}) - \frac{h^2}{\sqrt{h}} f(t_p, x_p) \right\} = 0, \quad p=1, n-1 \\ H^{(3)}(x) = \left\{ \frac{-1}{12h^2} (x_{p+2} - 16x_{p+1} + 30x_p - 16x_{p-1} + x_{p-2}) - f(t_p, x_p) \right\} = 0 \\ p=2, 3, \dots, n-2.$$

To investigate the Jacobian matrix $J_H(x)$ of the determining equation $H(x)=0$, we consider a linear system:

$$(35) \quad J(x)z + w = 0,$$

where $x = (x(t_0), x(t_1), \dots, x(t_n))$, $z = (z_0, z_1, \dots, z_n)$ and $w = (w_0, w_1, \dots, w_n)$.

Corresponding to z and w , we construct a cubic spline function $z_h(t)$ and a piecewise linear function $w_h(t)$ so that

$$z_h(t_p) = z_p \quad (p=0, 1, 2, \dots, n), \\ \dot{z}_h(0) = \frac{1}{h} (z_1 - z_0) - \frac{h}{2} f_n(0, \hat{x}(0)) z_0$$

and

$$\dot{z}_h(1) = \frac{1}{h} (z_n - z_{n-1}) + \frac{h}{2} f_x(1, \hat{x}(1)) z_n;$$

$$w_h(t_p) = w_p \quad (p=2, 3, \dots, n-2),$$

$$w_h(t_p) = h^{-3/2} w_p \quad (p=1, n-1),$$

and

$$w_h(t_p) = 0 \quad (p=0, n).$$

From (35), we have

$$(36) \quad -\ddot{z}_h(t) + P[f_x(t, x(t))z_h(t)] = (Pw_h)(t) \quad (0 \leq t \leq 1),$$

with boundary conditions

$$z_h(0) = -hw_0, \quad z_h(1) = -hw_n.$$

In the first place, we consider a solution $\bar{z}_h(t)$ such that

$$(37) \quad -\ddot{z}_h(t) + \sigma(t)z_h(t) = 0 \quad (0 \leq t \leq 1),$$

with boundary conditions

$$z_h(0) = -hw_0, \quad z_h(1) = -hw_n,$$

where

$$\sigma(t) = f_x(t, \hat{x}(t)).$$

Since $\hat{x}(t)$ is isolated, by simple calculation, we have:

$$(38) \quad \|\bar{z}_h\|, \|\dot{\bar{z}}_h\| \leq C_6 h |(w_0, w_n)|.$$

Next we consider a solution $\psi_h(t)$ such that

$$(39) \quad -\ddot{\psi}_h(t) + \sigma(t)\psi_h(t) = (I - P)[\sigma(t)z_h(t)] + (Pw_h)(t),$$

with boundary conditions

$$\psi_h(0) = 0, \quad \psi_h(1) = 0.$$

Let G be an operator so that

$$(40) \quad (Gf)(t) = \int_0^t x(t-1)f(x)dx + \int_t^1 t(x-1)f(x)dx,$$

and from (39), we have

$$(41) \quad \psi_h(t) - (G\sigma\psi_h)(t) = G(I - P)[\sigma(t)z_h(t)] + GPw_h.$$

Since $\hat{x}(t)$ is isolated, by using Fredholm's alternating theorem, it follows that

$$(42) \quad \|\psi_h\| \leq C_7 (\|R\| + \|GPw_h\|),$$

and

$$(43) \quad \|\dot{\psi}_h\| \leq C_7 \left(\|\dot{R}\| + \left\| \frac{d}{dt} GPw_h \right\| \right)$$

where $R(t) = G(I - P)[\sigma(t)z_h(t)]$.

Finally we have

$$(44) \quad \|z_h\| \leq C_7(\|R\| + \|GPw_h\|) + C_6 h |(w_0, w_n)|,$$

and

$$(45) \quad \|\dot{z}_h\| \leq C_7 \left(\|\dot{R}\| + \left\| \frac{d}{dt} GPw_h \right\| \right) + C_6 h |(w_0, w_n)|.$$

Here we prepare two lemmas.

LEMMA 5. *If $g(t)$ is continuously differentiable, $\|(I-P)g\| \leq C_8 h \|\dot{g}\|$.*

Proof. If we set $(Pg)(t) = \sum_{p=0}^n \lambda_p L_p(t)$,

the coefficients λ_p ($p=0, 1, 2, \dots, n$) satisfy the following equations:

$$(46) \quad \frac{1}{6} \begin{pmatrix} 2 & 1 & & & \\ & 1 & 4 & 1 & 0 \\ & -\frac{1}{12} & \frac{10}{12} & \frac{54}{12} & \frac{10}{12} & -\frac{1}{12} \\ & & 0 & 1 & 4 & 1 \\ & & & & 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2}g(t_0) \\ g(t_1) \\ \vdots \\ g(t_{n-1}) \\ \frac{1}{2}g(t_n) \end{pmatrix}$$

Therefore we can demonstrate this lemma in the same way as in the proof of lemma 3.

LEMMA 6. *There hold the following estimations:*

$$\|GPw_h\| \leq C_9 \sqrt{h} |w|, \quad \left\| \frac{d}{dt} GPw_h \right\| \leq \frac{C_9}{\sqrt{h}} |w|.$$

Proof. Let $\psi(t) = (GPw_h)$ and $(Pw_h)(t) = \sum_{p=0}^n \lambda_p L_p(t)$, and $\psi(t)$ is a cubic spline function so that

$$\dot{\psi}(t_p) = \lambda_p \quad (p=0, 1, 2, \dots, n).$$

By making use of consistency relations, we have

$$(47) \quad \frac{h^2}{6} (\lambda_{p+1} + 4\lambda_p + \lambda_{p-1}) = \psi_{p+1} - 2\psi_p + \psi_{p-1},$$

where $\psi_p = \psi(t_p)$ ($p=0, 1, \dots, n$).

Here λ_p ($p=0, 1, 2, \dots, n$) satisfy the following

$$\begin{aligned}
 (48) \quad & \frac{1}{6}(2\lambda_0 + \lambda_1) = 0 \\
 & \frac{1}{6}(\lambda_0 + 4\lambda_1 + \lambda_2) = h^{-3/2}w_1 \\
 & \frac{1}{72}(-\lambda_{p-2} + 10\lambda_{p-1} + 54\lambda_p + 10\lambda_{p+1} - \lambda_{p+2}) = w_p \quad (p=2, 3, \dots, n-2), \\
 & \frac{1}{6}(\lambda_{n-2} + 4\lambda_{n-1} + \lambda_n) = h^{-3/2}w_{n-1}, \\
 (49) \quad & \frac{1}{6}(\lambda_{n-1} + 2\lambda_n) = 0.
 \end{aligned}$$

From above, we have a system of equations with respect to ψ_p ($p=0, 1, 2, \dots, n$):

$$\begin{aligned}
 & \psi_0 = 0, \\
 & \psi_2 - 2\psi_1 + \psi_0 = \sqrt{h} w_1, \\
 & \frac{-1}{12h^2}(\psi_{p+2} - 16\psi_{p+1} + 30\psi_p - 16\psi_{p-1} + \psi_{p-2}) = w_p \quad (p=2, 3, \dots, n-2), \\
 & \psi_{n-2} - 2\psi_{n-1} + \psi_n = \sqrt{h} w_{n-1}, \\
 & \psi_n = 0.
 \end{aligned}$$

Applying the analysis in [2], we have

$$(50) \quad |(\psi_0, \psi_1, \dots, \psi_n)| \leq C_{10}|(w_1, w_2, \dots, w_{n-1})|.$$

From (48) and (49), we have

$$\psi_1 - \psi_0 - h\dot{\psi}_0 = 0, \quad -(\psi_n - \psi_{n-1}) + h\dot{\psi}_n = 0 \quad ([1, p. 13]),$$

from which follow

$$(51) \quad |\psi_0| \leq \frac{1}{h}|(\psi_0, \psi_1)|, \quad |\dot{\psi}_n| \leq \frac{1}{h}|(\psi_n, \psi_{n-1})|.$$

Therefore we have

$$\|\psi\| \leq C_{11} \sqrt{h} |w|,$$

and $\|\dot{\psi}\| \leq \frac{C_{11}}{\sqrt{h}} |w|.$

Combining these lemmas yields the following

$$\|z_h\| \leq C_{12} \sqrt{h} |w|.$$

Since $\|z_h\| \geq C_{13} \sqrt{h} |z|$, it follows that

$$|J^{-1}(\hat{x})| \leq C_{14} \quad \text{for sufficiently small } h.$$

If $\hat{x}(t) \in C^6[0, 1]$, we can easily show that

$$|H(x_1) - H(x_2)| \leq C_{15} |x_1 - x_2|, \quad \text{and} \quad |H(\hat{x})| = O(h^{7/2}).$$

Thus we have

THEOREM 2. *If the isolated solution $\hat{x}(t)$ belongs to $C^6[0, 1]$, there exists a solution $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n)$ of $H(x) = 0$ such that*

$$(52) \quad \left[\sum_{p=0}^n \left(\hat{x}(t_p) - \bar{x}_p \right)^2 \right]^{\frac{1}{2}} = O(h^{7/2}) \quad (h \rightarrow 0)$$

for sufficiently small h .

Here we consider the following well known higher-order difference scheme:

$$(53) \quad \frac{1}{h^2} (x_{p+1} - 2x_p + x_{p-1}) = \frac{1}{12} \left(f(t_{p+1}, x_{p+1}) + 10f(t_p, x_p) \right. \\ \left. + f(t_{p-1}, x_{p-1}) \right) \quad (p = 1, 2, \dots, n)$$

$$x_0 = a, \quad x_n = b.$$

Let P be an operator defined by $(Pf)(t) = \sum_{p=0}^n \lambda_p L_p(t)$ such that

$$(Pf, L_q) = \begin{cases} \frac{h}{12} \left(f(t_{q+1}) + 10f(t_q) + f(t_{q-1}) \right) & (q = 1, 2, \dots, n-1) \\ \frac{h}{2} f(t_q) & (q = 0, n) \end{cases}$$

Then the equation (9) corresponds to (5), (7) and (53).

Hence likewise in the proof of theorem 2, we have

THEOREM 3. *Under the same condition of theorem 2, there exists a solution $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n)$ of (53) such that*

$$\left[\sum_{p=0}^n \left(\hat{x}(t_p) - \bar{x}_p \right)^2 \right]^{\frac{1}{2}} = O(h^{7/2}) \quad (h \rightarrow 0)$$

for sufficiently small h .

REMARK. Let P_1 be an operator defined by (12) and

$$(P_2 f)(t) = \sum_{p=1}^{n-1} \frac{1}{6} (f_{p+1} + 4f_p + f_{p-1}) L_p(t) + f_0 L_0(t) + f_n L_n(t).$$

Then we consider the following differential equation:

$$(54) \quad \ddot{x}(t) = P_1 f(t, x(t), P_2 \dot{x}(t)) \quad (0 \leq t \leq 1),$$

with boundary conditions (6) and (8).

By consistency relations, equation (54) corresponds to

$$\frac{1}{h^2} (x_{p+1} - 2x_p + x_{p-1}) = f\left(t_p, x_p, \frac{x_{p+1} - x_{p-1}}{2h}\right) \quad (p=1, 2, \dots, n-1),$$

$$\frac{1}{h} (x_1 - x_0) - y_0 = \frac{h}{2} f(0, x_0, y_0)$$

and

$$-\frac{1}{h} (x_n - x_{n-1}) + y_n = \frac{h}{2} f(1, x_n, y_n).$$

Finally we notice that the overall smoothness condition of the exact solution $\hat{x}(t)$ is indispensable to our conclusion (theorem 1). On the other hand, let P be an operator defined by $(Pf)(t) = \sum_{p=0}^n f(t_p) L_p(t)$, and then equation (9) is equivalent to collocation conditions

$$\ddot{x}_h(t_p) = f(t_p, x_h(t_p)) \quad (p=0, 1, \dots, n) \quad ([3]).$$

In this case, if $\hat{x}(t)$ is piecewise smooth on each $[x_p, x_{p+1}]$, we obtain lemmas 3 and 4, from which follows the same conclusion of theorem 1.

Thus it follows that collocation method is superior to difference method for piecewise smooth solution.

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