ON CERTAIN VALUES OF \(p\)-ADIC L-FUNCTIONS

By

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1. Introduction

Let \(\chi\) be a primitive Dirichlet character with conductor \(f\), whose values are contained in an algebraic closure \(\bar{Q}_p\) of the rational \(p\)-adic number field \(Q_p\). The ring of \(p\)-adic integers in \(Q_p\) is denoted by \(Z_p\) as usual.

For the \(p\)-adic \(L\)-function \(L_p(s, \chi)\) belonging to an even character \(\chi\) not equal to the principal character \(\chi^0\), the value \(L_p(1, \chi)\) of the function \(L_p(s, \chi)\) at \(s=1\) is especially of importance.

It is known that a formula \(L_p(1, \chi)=(1-\chi(p)p^{-1})\Omega_p(\chi)\), called Leopoldt’s formula, holds with a quantity \(\Omega_p(\chi)=-\frac{\tau(\chi)}{f}\sum a \tilde{\chi}(a) \log_p(1-\zeta_p^a)\), where \(\tau(\chi)\) means a normalized Gauss sum attached to \(\chi\) and \(\log_p(1-\zeta_p^a)\) denotes a generalized \(p\)-adic logarithm defined by a natural way [4], and finally \(\tilde{\chi}\) means the inverse character \(\chi^{-1}\) of \(\chi\).

This formula of Leopoldt is almost equivalent with the \(p\)-adic class number formula for real abelian number fields over the rationals \(Q\). The proof of Leopoldt’s formula has been given by Iwasawa [4] and by Amice-Fresnel [1] respectively.

The main purpose of the present paper is to give another proof of Leopoldt’s formula, which is slightly different from Iwasawa’s. At the same time we note that an exact approximation of \(L_p(1, \chi)\) by the generalized Bernoulli numbers can be obtained directly from our formula. It turns out that the classical congruences of the Ankeny-Artin-Chowla [2] and Slavutsky [9] for real quadratic number fields are extended to the case of real abelian number fields. In particular we shall derive a formula for such Dirichlet characters \(\chi\) that have prime power conductors \(f = p^n (n \geq 1)\).

In the last section we shall treat some congruences of Vandiver’s type for the relative class number of imaginary abelian number fields over \(Q\). This amounts to approximate the values \(L_p(0, \chi \omega)\) for odd characters \(\chi\) by Bernoulli
numbers. Here \( \omega \) means a Dirichlet character of conductor \( p \) \((p \equiv 2)\) or \(4\) \((p = 2)\) defined by \( \omega(x) = \lim_{n \to \infty} x^{p^n} \) for \( x \in \mathbb{Z}_p \), \( \omega(1, p) = 1 \) and \( \omega(x) = 0 \) for \( p | x \) \((p \neq 2)\) or \( \omega(x) = 1, -1 \) according as \( x \equiv 1 \pmod{4} \), \( x \equiv -1 \pmod{4} \) for \( x \in \mathbb{Z}_2 \), \( x, 2 = 1 \) and \( \omega(x) = 0 \) for \( 2 | x \) \((p = 2)\).

2. An approximation of logarithmic functions

Let \( x \) be a principal unit in \( \mathbb{Q}_p \), and then we know \( y = \log x \) exists. If \( v_p(y) > \frac{1}{p-1} \) by the exponential valuation \( v_p \) of \( \mathbb{Q}_p \) normalized such that \( v_p(p) = 1 \), then \( x = e^y \) can be defined and \( v_p(y) = v_p(x - 1) \) holds.

First we have

**Lemma 1.** For any natural number \( l \) such that \( v_p(x^{p^l} - 1) > \frac{1}{p-1} \) we have

\[
\log x \equiv \frac{x^{p^l} - 1}{p^l} \pmod{p^{l+2v_p(\log x)} - (p-1) / p^{l+1}}.
\]

In particular, if \( v_p(\log x) \geq \frac{1}{2(p-1)} \), then we have simply

\[
\log x \equiv \frac{x^{p^l} - 1}{p^l} \pmod{p^l}.
\]

**Proof.** By the assumption \( v_p(x^{p^l} - 1) > \frac{1}{p-1} \) we see \( v_p(\log x^{p^l}) = v_p(x^{p^l} - 1) > \frac{1}{p-1} \), and so it yields that

\[
x^{p^l} = e^{\log x^{p^l}} = e^{p^l \log x} = 1 + p^l \log x + \sum_{m=2}^{\infty} \frac{1}{m!} p^m i(\log x)^m.
\]

Here we see by making use of the sum \( s_p(m) \) of the coefficients of the canonical \( p \)-adic expansion of \( m \)

\[
v_p\left( \frac{1}{m!} p^m i(\log x)^m \right) = ml + mv_p(\log x) - \frac{m - s_p(m)}{p-1}
\]

\[
\geq m \left( l + v_p(\log x) - \frac{1}{p-1} \right) + \frac{1}{p-1}.
\]

Therefore, from the fact that \( l + v_p(\log x) > \frac{1}{p-1} \) we have
\[ v_p \left( \frac{1}{m!} P^{m!}(\log x)^m \right) \geq 2 \left( l + v_p(\log x) - \frac{1}{p-1} \right) + \frac{1}{p-1}. \]

Thus we obtain the first congruence. Moreover, if \( v_p(\log x) \geq \frac{1}{2(p-1)} \), then we conclude also the second congruence.

3. **Definition of \( \Omega_p(\chi) \)**

Let \( \chi \) be an even primitive Dirichlet character with conductor \( f \), and \( \chi \nmid \chi^0 \), namely \( f \nmid 1 \). Let \( \zeta_f \) be a primitive \( f \)-th root \( \zeta_f = e^{\frac{2\pi i}{f}} \) of unity, and \( \tau(\chi) \) be the Gauss sum \( \sum_a \chi(a)\zeta_f^a \) to \( \chi \).

We define \( \Omega_p(\chi) \) as usual as follows and denote them by \( \Omega_p(\chi) \):

1) For the case where \( f = f_\chi \) is not \( p \)-power

\[ \Omega_p(\chi) = - \frac{\tau(\chi)}{f} \sum_{a=1}^{f} \chi(a) \log(1 - \zeta_f^a). \]

2) For the case where \( f = p^n \) with \( n \geq 2 \)

\[ \Omega_p(\chi) = - \frac{\tau(\chi)}{f} \frac{1}{p} \sum_{a=1}^{f} \chi(a) \log \left\{ \frac{(1 - \zeta_f^a)^p}{(1 - \zeta_f^a)} \right\}. \]

3) For the case where \( f = p \), therefore \( p \nmid 2 \)

\[ \Omega_p(\chi) = - \frac{\tau(\chi)}{f} \frac{1}{p-1} \sum_{a=1}^{f} \chi(a) \log \left\{ \frac{(1 - \zeta_f^a)^{p-1}}{1 - \zeta_f^a} \right\}. \]

In the case 1) we see easily

\[ (1 - \chi(p)p^{r-1}) \Omega_p(\chi) = \frac{\tau(\chi)}{f} \frac{1}{p} \sum_{a=1}^{f} \chi(a) \log \left\{ \frac{(1 - \zeta_f^a)^p}{(1 - \zeta_f^a)} \right\}. \]

and this formula coincides with the formula of the case 2).

Now, we select a natural number \( c \), such that \( (c, pf) = 1 \), \( \chi(c) \neq 1 \), \( c > 1 \). Then we have

\[ (1 - \chi(p)p^{r-1}) \Omega_p(\chi) = \frac{\tau(\chi)}{f} \frac{1}{p} \sum_{a=1}^{f} \chi(ca) \log \left\{ \frac{(1 - \zeta_f^{ca})^p}{(1 - \zeta_f^c)^p} \right\}. \]

Therefore we have
\[(1 - \chi(p)p^{-1})\Omega_p(\chi) = \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \bar{\chi}(a) \log \left\{ \prod_{\mu=0}^{c-1} \frac{1 - \zeta_{f^\mu}^p \zeta_{c}^p}{1 - \zeta_{f^\mu}^p \zeta_c^p} \right\} \]

\[= \bar{\chi}(c) \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \bar{\chi}(a) \log \left\{ \frac{(1 - \zeta_f^p \zeta_c^p)}{(1 - \zeta_f^p \zeta_c^p)^p} \right\} \]

\[+ \bar{\chi}(c) \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \bar{\chi}(a) \sum_{\mu=1}^{c-1} \log \left\{ \frac{(1 - \zeta_f^p \zeta_c^p)^{\mu}}{(1 - \zeta_f^p \zeta_c^p)^p} \right\} \]

\[= \bar{\chi}(c)(1 - \chi(p)p^{-1})\Omega_p(\chi) + \bar{\chi}(c) \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \bar{\chi}(a) \sum_{\mu=1}^{c-1} \sum_{v=0}^{p-1} \log \left\{ \frac{1 - \zeta_f^p \zeta_c^p}{1 - \zeta_f^p \zeta_c^p} \right\}. \]

Namely, we obtain

\[(1 - \chi(p)p^{-1})(1 - \bar{\chi}(c))\Omega_p(\chi) \]

\[= \bar{\chi}(c) \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \bar{\chi}(a) \sum_{\mu=1}^{c-1} \sum_{v=0}^{p-1} \log \left( 1 + \frac{1 - \zeta_f^p \zeta_c^p}{\zeta_f^p \zeta_c^p - 1} \right). \]

In the case 3) we also have

\[\Omega_p(\chi) = -\frac{\tau(\chi)}{p} \frac{1}{p-1} \sum_a \bar{\chi}(a) \log \left\{ \frac{(1 - \zeta_c^p)^{p-1}}{-p} \right\} \]

\[= -\frac{\tau(\chi)}{p} \frac{1}{p-1} \sum_a \bar{\chi}(ca) \log \left\{ \frac{(1 - \zeta_c^p)^{p-1}}{-p} \right\} \]

\[= -\frac{\tau(\chi)}{p} \frac{1}{p-1} \sum_a \bar{\chi}(c) \bar{\chi}(a) \log \left\{ \frac{1 - \zeta_c^p}{1 - \zeta_c^p \zeta_f^p} \right\} \]

\[= -\frac{\tau(\chi)}{p} \frac{1}{p-1} \bar{\chi}(c) \sum_a \bar{\chi}(a) \log \left\{ \frac{1 - \zeta_c^p}{1 - \zeta_c^p \zeta_f^p} \right\}. \]

Thus we have

\[(1 - \bar{\chi}(c))\Omega_p(\chi) = \frac{\tau(\chi)}{p} \bar{\chi}(c) \sum_a \bar{\chi}(a) \log \left\{ \frac{c(1 - \zeta_c^p)}{1 - \zeta_c^p} \right\} \]

\[= \frac{\tau(\chi)}{p} \bar{\chi}(c) \sum_{\mu=1}^{c-1} \bar{\chi}(a) \log \left\{ \frac{1 - \zeta_c^p \zeta_f^p}{1 - \zeta_c^p \zeta_f^p \zeta_c^p} \right\} \]

\[= \frac{\tau(\chi)}{f} \bar{\chi}(c) \frac{1}{p} \sum_a \bar{\chi}(a) \sum_{\mu=1}^{c-1} \log \left\{ \frac{1 - \zeta_c^p \zeta_f^p \zeta_c^p}{(1 - \zeta_c^p \zeta_f^p \zeta_c^p)^p} \right\}. \]
\[ = \tilde{\chi}(c) \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \tilde{\chi}(a) \sum_{\mu=1}^{c-1} \sum_{\nu=0}^{p-1} \log \left\{ \frac{1 - \frac{r_p^\nu}{\zeta_f^\mu c^\nu}}{1 - \frac{1}{\zeta_f^\mu c^\nu}} \right\}. \]

Namely, we have

\[ (1 - \tilde{\chi}(c))\Omega_p(\chi) = \tilde{\chi}(c) \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \tilde{\chi}(a) \sum_{\mu=1}^{c-1} \sum_{\nu=0}^{p-1} \log \left( 1 + \frac{1 - \frac{r_p^\nu}{\zeta_f^\mu c^\nu}}{-1} \right). \]

Therefore we have the following formula, which is also contained in Iwasawa [4]:

\[ (1 - \tilde{\chi}(c))(1 - \chi(p)p^{-1})\Omega_p(\chi) \]

\[ = \tilde{\chi}(c) \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \tilde{\chi}(a) \sum_{\mu=1}^{c-1} \sum_{\nu=0}^{p-1} \log \left( 1 + \frac{1 - \frac{r_p^\nu}{\zeta_f^\mu c^\nu}}{-1} \right). \]

4. Leopoldt's formula

Now, set

\[ \xi(\chi, c, \mu) = \sum_a \tilde{\chi}(a) \sum_{\nu=0}^{p-1} \log \left( 1 + \frac{1 - \frac{r_p^\nu}{\zeta_f^\mu c^\nu}}{-1} \right) \]

and then we have

\[ (1 - \tilde{\chi}(c))(1 - \chi(p)p^{-1})\Omega_p(\chi) = \tilde{\chi}(c) \frac{\tau(\chi)}{f} \frac{1}{p} \sum_{\mu=1}^{c-1} \xi(\chi, c, \mu). \]

Note that \( \xi(\chi, c, \mu) = \xi(\chi, c, c - \mu) \) in the case \( p = 2 \).

Because we have here \( v_p \left( \frac{1}{\zeta_f^\mu c^\nu - 1} \right) \geq \frac{1}{p - 1} \), \( v_p \left( \log \left( 1 + \frac{1 - \frac{r_p^\nu}{\zeta_f^\mu c^\nu}}{-1} \right) \right) = v_p \left( \frac{1}{p} \log \left( 1 + \frac{1 - \frac{r_p^\nu}{\zeta_f^\mu c^\nu}}{-1} \right) \right) \geq \frac{1}{p - 1} \), we obtain by Lemma 1

\[ \xi(\chi, c, \mu) \equiv \sum_a \tilde{\chi}(a) \sum_{\mu=1}^{c-1} \frac{1}{p_{\chi a}^{\nu_0} \left( \begin{array}{c} p_{\chi a}^k \\ k \end{array} \right) \sum_{\nu=0}^{p-1} \left( \frac{1 - \frac{r_p^\nu}{\zeta_f^\mu c^\nu}}{-1} \right)^k \pmod{p^{\nu_0 + \delta_\chi}}. \]

The exponent \( \delta_\chi = v_p(\tau(\chi)) \) comes from the \( p \)-ordinal of the Gauss sums, and when we write \( \sum_{\nu=0}^{p-1} \log \left( 1 + \frac{1 - \frac{r_p^\nu}{\zeta_f^\mu c^\nu}}{-1} \right) \) such as
\[ \sum_{v=0}^{p^{-1}} \log \left( 1 + \frac{1 - \zeta_p^v}{\zeta_p^v - a \zeta_p^v} - 1 \right) = \sum_{v=0}^{p^{-1}} \frac{1}{p^{l_0}} - \left( 1 + \frac{1 - \zeta_p^v}{\zeta_p^v - a \zeta_p^v} - 1 \right) p^{l_0} A(\zeta_p^v), \]

then \( A(\zeta_p^v) \) turns out a polynomial of \( \zeta_p^v \) with coefficients in the integer ring of the field \( \mathbb{Q}_p(\zeta_p^v, \zeta_p^v) \), where \( f = p^l f_0, (p, f_0) = 1, a_1 \equiv a \pmod{p^n} \).

By multiplying \( \chi(1) \) with \( \chi = \chi \chi_0 \) and summing up over \( a_1 \) we see that the congruence holds modulo \( p^{l_0 + d} \).

By the way, for any sequence \( (a_n)(n = 0, 1, 2, \ldots) \), we define a linear difference operator \( \Delta_m \) by \( \Delta_n a_n = a_{n+m} - a_n \) as usual. Particularly the operator \( \Delta_1 \) is simply denoted by \( \Delta \). With this notation we have

**Lemma 2.** For an integer \( k \geq 1 \) it holds that

\[ \frac{1}{k} \sum_{v=0}^{p^{-1}} (1 - \zeta_p^v)^k \equiv (-1)^{k-1} p^{-1} \Delta^k 0 p^{l-1}(p-1) \pmod{p^{l+1}}, \]

where \( l \) means an arbitrary integer \( \geq 1 \) for odd \( p \), and \( \geq 3 \) for \( p = 2 \).

**Proof.** We see in fact

\[ \sum_{v=0}^{p^{-1}} (1 - \zeta_p^v)^k = \sum_j \binom{k}{j} (-1)^j \sum_{v=0}^{p^{-1}} \zeta_p^v \equiv p \sum_{j \equiv 0 \pmod{p}} \binom{k}{j} (-1)^j \quad \text{for} \quad k \geq 1. \]

Now, we know \( j p^{l-1}(p-1) \equiv 1 \pmod{p^l} \) for \( j \equiv 0 \pmod{p} \) to \( p \geq 3 \) and \( j^{2^{l-1}} \equiv 1 \pmod{2^{l+1}} \) for \( j \equiv 0 \pmod{2} \).

Thus we have first for an odd prime \( p \)

\[ \frac{1}{k} \sum_{j \equiv 0 \pmod{p}} \binom{k}{j} (-1)^j \equiv \sum_{j \equiv 0 \pmod{p}} \frac{1}{j} \binom{k}{j-1} (-1)^j j p^{l-1}(p-1) \pmod{p^l} \]

\[ \equiv (-1)^k \sum_j \binom{k-1}{j-1} (-1)^k j p^{l-1}(p-1)-1 \pmod{p^l} \]

for \( p^{l-1}(p-1)-1 \geq l \), that is, \( l \geq 1 \).

Namely we have

\[ \frac{1}{k} \sum_{v=0}^{p^{-1}} (1 - \zeta_p^v)^k \equiv (-1)^{k-1} p^{-1} \Delta^k 0 p^{l-1}(p-1) \pmod{p^{l+1}}. \]
For \( p = 2 \) we have similarly the same formula for \( p^{l-1}(p-1)-1 \geq l \), that is, \( p^{l-1}(p-1) \geq l+1 \), namely for \( l \geq 3 \).

When we want the congruences for \( p = 2, l = 2 \), we must write

\[
\frac{1}{k} \sum_{v=0}^{k-1} (1 - \zeta_2^v)^k \equiv (-1)^{k-1} 2 \cdot \frac{1}{k} \Delta^k 0^2 \pmod{2^2}.
\]

But, we have for \( l \geq 4 \) the better congruences

\[
\frac{1}{k} \sum_{v=0}^{k-1} (1 - \zeta_2^v)^k = \frac{1}{k} 2^k \equiv (-1)^{k-1} 2 \cdot \frac{1}{k} \Delta^k 0^{2^2} \pmod{2^{l+2}}.
\]

Namely,

\[
\frac{1}{k} \sum_{v=0}^{k-1} (1 - \zeta_2^v)^k = \frac{1}{k} 2^k \equiv (-1)^{k-1} 2 \cdot \sum_{j=0 \pmod{2}}^k \frac{k-1}{j-1} (-1)^{k-j} j^{2^l-1} \pmod{2^{l+2}}
\]

\[
\equiv (-1)^{k-1} 2 \cdot \sum_{j=0 \pmod{2}}^k \frac{k-1}{j-1} (-1)^{k-j} j^{2^l-1-1} \pmod{2^{l+2}}
\]

for \( 2^{l-1} - 1 \geq l+1 \), i.e., \( l \geq 4 \).

Thus we see

\[
\frac{1}{k} \sum_{v=0}^{k-1} (1 - \zeta_2^v)^k = \frac{1}{k} 2^k \equiv (-1)^{k-1} 2 \cdot \frac{1}{k} \Delta^k 0^{2^2} \pmod{2^{l+2}} \quad \text{for } l \geq 4.
\]

Next, we notice that

\[
\frac{1}{p^0 \binom{p^0}{k}} \equiv (-1)^{k-1} \frac{1}{k} \pmod{p^{0^2+k-s_p(k)}}.
\]

On the other hand from Lemma 2 we conclude that

\[
\sum_{v=0}^{p-1} (1 - \zeta_p^v)^k \equiv 0 \pmod{p^{0^2+k-s_p(k)}}
\]

holds. We have only to remark that \( \mathcal{S}(p^{l-1}(p-1), k) = \frac{1}{k!} \Delta^k 0^{p^{l-1}(p-1)} \) is the Stirling number of the second kind, which is always a rational integer, and in Lemma 2 we can take a sufficiently large number \( l \).

Therefore we obtain

\[
\frac{1}{p^0 \binom{p^0}{k}} \sum_{v=0}^{p-1} (1 - \zeta_p^v)^k \equiv (-1)^{k-1} \frac{1}{k} \sum_{v=0}^{p-1} (1 - \zeta_p^v)^k \pmod{p^{0^2+k}}.
\]

Thus we have from Lemma 2 also for \( l_0 \geq l+1 \)

\[
\frac{1}{p^0 \sum_{k=1}^{p^0} \binom{p^0}{k}} \frac{1}{(\zeta_p^r \zeta_p^{r-\mu} - 1)^k} \sum_{v=0}^{p-1} (1 - \zeta_p^v)^k
\]
\[\equiv \sum_{k=1}^{p^{l_0}} (-1)^{k-1} \frac{1}{k} \left(\frac{q}{\zeta_f} - \frac{\zeta_c^{p-\mu}}{1} - 1\right)^k \sum_{\nu=0}^{p-1} (1 - \zeta_p^\nu)^k \pmod{p^{l+1}}\]

\[\equiv \sum_{k=1}^{p^{l_0}} p \frac{1}{k} \left(\frac{q}{\zeta_f} - \frac{\zeta_c^{p-\mu}}{1} - 1\right)^k A_k Q^{l-1} (p-1) \pmod{p^{l+1}}\]

For \(p=2\) this congruence holds for \(l \geq 3\). For \(l=2\) we must replace the modulus \(2^3\) by \(2^2\). For \(l \geq 4\) we may take the modulus \(2^{l+2}\) instead of \(2^{l+1}\).

Therefore we obtain

\[\xi(\chi, c, \mu) = \sum_{a} \bar{\xi}(a) \ p^{l-1} (p-1) \sum_{k=1}^{p^{l_0}} \frac{1}{k} \left(\frac{q}{\zeta_f} - \frac{\zeta_c^{p-\mu}}{1} - 1\right)^k A_k Q^{l-1} (p-1) \pmod{p^{l+1+\delta}}.\]

Now, we compute the generating function of the numbers

\[\sum_{a} \bar{\xi}(a) \sum_{k=1}^{m} \frac{1}{k} \left(\frac{q}{\zeta_f} - \frac{\zeta_c^{p-\mu}}{1} - 1\right)^k (m=1, 2, 3, \ldots).\]

Set

\[F(t, \chi, \mu) = \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{a} \bar{\xi}(a) \sum_{k=1}^{m} \frac{1}{k} \left(\frac{q}{\zeta_f} - \frac{\zeta_c^{p-\mu}}{1} - 1\right)^k \right) t^m\]

with an indeterminate \(t\). Then we see

\[F(t, \chi, \mu) = \sum_{a} \bar{\xi}(a) \ \sum_{k=1}^{m} \frac{1}{k} \left(\frac{q}{\zeta_f} - \frac{\zeta_c^{p-\mu}}{1} - 1\right)^k (e^t - 1)^k\]

\[= \sum_{a} \bar{\xi}(a) \left\{ - \log \left(1 - \frac{e^t - 1}{\zeta_f - \zeta_c^{p-\mu} - 1}\right) \right\}\]

\[= - \sum_{a} \bar{\xi}(a) \ \log \left(\frac{\zeta_f - \zeta_c^{p-\mu} - e^t}{\zeta_f - \zeta_c^{p-\mu} - 1}\right).\]

Hence we see

\[F'(t, \chi, \mu) = \sum_{a} \bar{\xi}(a) \ \frac{e^t}{\zeta_f - \zeta_c^{p-\mu} - e^t}.\]

Because the generating function \(F(t, \chi)\) for the numbers

\[\sum_{a} \bar{\xi}(a) \sum_{k=1}^{m} \frac{1}{k} \left(\frac{q}{\zeta_f} - \frac{\zeta_c^{p-\mu}}{1} - 1\right)^k \ p \ p^{l_0} \ (m=1, 2, 3, \ldots)\]

is \(\sum_{a} F(t, \chi, \mu)\), we obtain

1) Note that the exponential function and the logarithmic function here and below mean the ordinary ones in complex variables.
\[ tF'_c(t, \chi) = \sum_a \tilde{\chi}(a) \sum_{\mu=1}^{c-1} \frac{te^t}{\zeta_f^a \zeta_c^{-\mu} - e^t} \]
\[ = \sum_a \tilde{\chi}(a) \left\{ \sum_{b=1}^f \frac{\zeta_f^{ab} te^{bt}}{e^{ft} - 1} - c \sum_{b=1}^f \frac{\zeta_f^{ab} te^{bc t}}{e^{ft} - 1} \right\} \]
\[ = \tau(\bar{\chi}) \left\{ e^{B_x t} - \chi(c) e^{B_x c t} \right\}. \]

In the above transformations we have used the identities:
\[ \sum_{\mu=1}^{c-1} \zeta_f^{-a} \zeta_c^{-\mu} - e^t = \frac{c e^t}{\zeta_f^{ac} - e^t} - \frac{e^t}{\zeta_f^{a} - e^t}, \]
\[ \frac{e^t}{\zeta_f^{a} - e^t} = \sum_{b=1}^f \frac{\zeta_f^{ab} e^{bt}}{1 - e^{ft}} \quad \text{for} \quad 0 \leq a \leq f - 1. \]

Therefore, if we compare the coefficients of \( t^m \) in the above formula, we have

**Lemma 3.** For \( m = 1, 2, 3, \ldots \) it holds that
\[ \sum_a \tilde{\chi}(a) \sum_{\mu=1}^{c-1} \sum_{k=1}^m \frac{1}{k} \frac{\Delta_k m}{(\zeta_f^{a} \zeta_c^{-\mu} - 1)^k} = \tau(\bar{\chi}) (1 - \chi(c) e^m) \frac{1}{m} B_x^m. \]

Thus we have
\[ \sum_{\mu=1}^{c-1} \xi(\chi, c, \mu) \equiv p \tau(\bar{\chi}) (1 - \chi(c) e^{p-1}(p-1)) \frac{1}{p^{l-1}(p-1)} B_x^{p^{l-1}(p-1)} \quad (\text{mod } p^{l+1+\delta}). \]

Hence we see
\[ (1 - \chi(c))(1 - \chi(p) p^{-1}) \chi \]
\[ \equiv - \frac{\tau(\bar{\chi})}{f} \frac{1}{p} p \tau(\bar{\chi}) (1 - \chi(c) e^{p-1}(p-1)) \frac{1}{p^{l-1}(p-1)} B_x^{p^{l-1}(p-1)} \quad (\text{mod } p^l). \]

Thus we obtain the following

**Theorem 1.** We have the congruences with the parameter \( c \)
\[ (1 - \chi(c))(1 - \chi(p) p^{-1}) \chi \equiv -(1 - \chi(c) e^{p-1}(p-1)) \frac{1}{p^{l-1}(p-1)} B_x^{p^{l-1}(p-1)} \quad (\text{mod } p^l). \]

for \( l \geq 1 \) in the case of \( p \neq 2 \) and for \( l \geq 3 \) in the case of \( p = 2 \).

For the case \( p = 2 \) we have
\[(1 - \chi(c))(1 - \chi(2)2^{-1}) \Omega_2(\chi) \equiv - (1 - \chi(c)2^{1 - 1}) \frac{1}{2^{l-1}} B^{2l-1}_x \pmod{2^{l+2}} \text{ for } l \geqslant 4,
\]
\[\equiv -(1 - \chi(c)2^{2}) \frac{1}{2^2} B^{2}_x \pmod{2^4}
\]
\[\equiv -(1 - \chi(c)2^{2}) \frac{1}{2} B^{2}_x \pmod{2^2}.
\]

From Theorem 1 Leopoldt's formula follows immediately.

**Corollary.** We have \(L_p(1, \chi) = (1 - \chi(p)p^{-1}) \Omega_p(\chi)\).

**Proof.** We know that the \(p\)-adic \(L\)-function \(L_p(s, \chi)\) with \(\chi\) not equal to the principal character \(\chi^0\) is continuous at \(s = 1\) \([5]\). Moreover, it holds that

\[
L_p(1, \chi) = \lim_{|m| \to 0} L_p(1 - m, \chi) = \lim_{|m| \to 0 \pmod{p^{-1}}} - (1 - \chi(p)p^{m-1}) \frac{1}{m} B^m_x
\]
\[= - \lim_{l \to \infty} (1 - \chi(p)p^{l-1}p^{-1}) \frac{1}{p^{l-1}(p-1)} B^{p^{l-1}(p-1)}_x
\]
\[= - \lim_{l \to \infty} \frac{1}{p^{l-1}(p-1)} B^{p^{l-1}(p-1)}_x.
\]

Herein note that the numbers \(\frac{p}{p^{l-1}(p-1)} B^{p^{l-1}(p-1)}_x\) with \(\chi \equiv \chi^0\) all are \(p\)-adic integers \([8]\).

Let \(l\) tend to the infinity in the above, and then yields

\[(1 - \chi(c))(1 - \chi(p)p^{-1}) \Omega_p(\chi) = (1 - \chi(c))L_p(1, \chi).
\]

By virtue of \(1 - \chi(c) \equiv 0\) we finally have

\[L_p(1, \chi) = (1 - \chi(p)p^{-1}) \Omega_p(\chi).
\]

5. A supplementary formula

In this section we supply a formula for the case where the conductor \(f_x\) is not a power of \(p\). In these cases we need not use the parameter \(c\).

If the conductor \(f\) is not a power of \(p\), then we have from the definition of \(\Omega_p(\chi)\) easily
\[ (1 - \chi(p)p^{-1})\Omega_p(\chi) = \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \tilde{\chi}(a) \log \frac{1 - \zeta_p^a}{1 - \zeta_p^a} \]

Hence we see

\[ (1 - \chi(p)p^{-1})\Omega_p(\chi) = \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \tilde{\chi}(a) \sum_{v=0}^{p-1} \log \left( \frac{1 - \zeta_p^{a+v}}{1 - \zeta_p^{a+v}} \right) \]

\[ = \frac{\tau(\chi)}{f} \frac{1}{p} \sum_a \tilde{\chi}(a) \sum_{v=1}^{p-1} \log \left( 1 + \frac{1 - \zeta_p^v}{\zeta_p^v - 1} \right). \]

Therefore, by a similar argument as before we obtain

**Theorem 2.** If the conductor \( f \) of \( \chi \) is not a power of \( p \), then we have

\[ (1 - \chi(p)p^{-1})\Omega_p(\chi) \equiv -\frac{1}{p^{l-1}(p-1)} B_{x}^{p^{l-1}(p-1)} \quad \text{(mod } p^l) \]

for \( l \geq 1 \) to odd \( p \). For \( p=2 \) we have

\[ (1 - \chi(2)2^{-1})\Omega_2(\chi) \equiv -\frac{1}{2^{l-1}} B_2^{2^{l-1}} \quad \text{(mod } 2^{l+2}) \text{ for } l \geq 4, \]

\[ \equiv -\frac{1}{2} B_2^{2} \quad \text{(mod } 2^{4}), \]

\[ \equiv -\frac{1}{2} B_2^{2} \quad \text{(mod } 2^{2}). \]

6. **Elimination of the parameter \( c \)**

If \( \chi \) is not of order \( p \)-power, then we can select the parameter \( c \) such as \( 1 - \chi(c) \) is a unit. Hence we have from our formula in Theorem 1

\[ (1 - \chi(p)p^{-1})\Omega_p(\chi) \equiv -\frac{1 - \chi(c)c^{p^{l-1}(p-1)}}{1 - \chi(c)} \frac{1}{p^{l-1}(p-1)} B_{x}^{p^{l-1}(p-1)} \quad \text{(mod } p^l). \]

Because \( \frac{1}{p^{l-1}(p-1)} B_{x}^{p^{l-1}(p-1)} \equiv 0 \) (mod \( p^0 \)) and \( \frac{1}{2^{l-1}} B_2^{2^{l-1}} \equiv 0 \) (mod 2) in this case [8] and \( c^{p^{l-1}(p-1)} \equiv 1 \) (mod \( p^l \)), we finally obtain the following

**Theorem 3.** If the order of \( \chi \) is not \( p \)-power, then it holds that

\[ (1 - \chi(p)p^{-1})\Omega_p(\chi) \equiv -\frac{1}{p^{l-1}(p-1)} B_{x}^{p^{l-1}(p-1)} \quad \text{(mod } p^l) \]
for \( l \geq 1 \) to the case of odd \( p \). For \( p = 2 \) we have

\[
(1 - \chi(2)2^{-1}) U_{2}(\chi) = \frac{1}{2^{l+1}} B_{x}^{2^{l-1}} \quad (\mod 2^{l+1}) \quad \text{for} \quad l \geq 4,
\]

\[
\equiv -\frac{1}{2^2} B_{x}^{2^2} \quad (\mod 2^4),
\]

\[
\equiv -\frac{1}{2} B_{x}^{2} \quad (\mod 2^2).
\]

Now, in the case where \( \chi \) is a character of the second kind\(^2\) and \( p \neq 2 \), we conclude from Theorem 1 similarly

\[
(1 - \chi(p)p^{-1}) U_{p}(\chi) \equiv -\frac{1}{p^{l+1}(p-1)} B_{x}^{p^{l-1}(p-1)} \chi(c) \frac{(1 - c^{p^{l-1}(p-1)})}{1 - \chi(c)}
\]

\[
\times \frac{1}{p^{l-1}(p-1)} B_{x}^{p^{l-1}(p-1)} \quad (\mod \ p^{l-1} p^{n-1}(p-1)),
\]

where we write \( f = p^n \) with \( n \geq 2 \).

On the one hand we see by making use of \( c = 1 + p \)

\[
\frac{\chi(c)(1 - c^{p^{l-1}(p-1)})}{1 - \chi(c)} \equiv 0 \quad (\mod \ p^{l-1} p^{n-1}(p-1))
\]

and

\[
\frac{1}{p^{l-1}(p-1)} B_{x}^{p^{l-1}(p-1)} \equiv \frac{u_p}{1 - \chi(l + p)} \quad (\mod p^0) \text{ with a unit } u_p
\]

[8], and we have

\[
\frac{\chi(c)(1 - c^{p^{l-1}(p-1)})}{1 - \chi(c)} \cdot \frac{1}{p^{l-1}(p-1)} B_{x}^{p^{l-1}(p-1)} \equiv 0 \quad (\mod p^{l-1} p^{n-1}(p-1)).
\]

Thus we have

\[
(1 - \chi(p)p^{-1}) U_{p}(\chi) = U_{p}(\chi) \equiv -\frac{1}{p^{l+1}(p-1)} B_{x}^{p^{l-1}(p-1)}
\]

\[
(\mod p^{l-1} p^{n-1}(p-1)).
\]

\(^2\) A Dirichlet character \( \chi \) is called to be of the second kind if its conductor is a power of \( p \) and each value \( \chi(x) \) for any integer \( x \) prime to \( p \) depends only on the unit \( \langle x \rangle \) in the decomposition \( x = a(x) \langle x \rangle \) of \( x \) as a unit in \( Z_p \).
In the case where $\chi$ is a character of the second kind and $p=2$, we similarly deduce from Theorem 1

\[
(1-\chi(2)2^{-1})\mathfrak{U}_2(\chi) \equiv -\frac{1}{2^{l-1}}B_2^{2^{l-1}} - \frac{\chi(c)(1-c^{2^{l-1}})}{1-\chi(c)} \frac{1}{2^{l-1}}B_2^{2^{l-1}} \pmod{2^{l+1/2^{n-3}}}
\]

for $l \geq 4$, to $f=2^n$ with $n \geq 4$ and

\[
(1-\chi(2)2^{-1})\mathfrak{U}_2(\chi) \equiv -\frac{1}{2^{l-1}}B_2^{2^{l-1}} - \frac{\chi(c)(1-c^{2^{l-1}})}{1-\chi(c)} \frac{1}{2^{l-1}}B_2^{2^{l-1}} \pmod{2^{l+1-1}}
\]

for $f=2^3$.

We take $c=1+2^2$ and then we see $1-(1+2^2)2^{l-1} \equiv 0 \pmod{2^{l+1}}$, and

\[
\frac{\chi(5)(1-5^{2^{l-1}})}{1-\chi(5)} \equiv 0 \quad (\text{mod } 2^{l+1-2^{n-3}}) \quad \text{for } n \geq 4,
\]

\[
\frac{\chi(5)(1-5^{2^{l-1}})}{1-\chi(5)} \equiv 0 \quad (\text{mod } 2^{l+1-1}) \quad \text{for } n=3.
\]

Because $\frac{1}{2^{l-1}}B_2^{2^{l-1}} \equiv 0 \pmod{2^{l+1-2^{n-3}}}$ to $n \geq 4$ and $\frac{1}{2^{l-1}}B_2^{2^{l-1}} \equiv 0 \pmod{2^0}$ to $n=3$ for $\chi \equiv \chi^0 [8]$, we finally have for $l \geq 4$, $n \geq 4$,

\[
(1-\chi(2)2^{-1})\mathfrak{U}_2(\chi) = \mathfrak{U}_2(\chi) \equiv -\frac{1}{2^{l-1}}B_2^{2^{l-1}} \pmod{2^{l+2-2^{n-3}}}
\]

and

\[
(1-\chi(2)2^{-1})\mathfrak{U}_2(\chi) = \mathfrak{U}_2(\chi) \equiv -\frac{1}{2^{l-1}}B_2^{2^{l-1}} \pmod{2^{l}} \quad \text{for } n=3.
\]

For $l=3$ we have

\[
(1-\chi(2)2^{-1})\mathfrak{U}_2(\chi) = \mathfrak{U}_2(\chi) \equiv -\frac{1}{2^{2}}B_2^{2^2} \pmod{2^{4-2^{n-3}}} \quad \text{for } n \geq 4,
\]

and

\[
(1-\chi(2)2^{-1})\mathfrak{U}_2(\chi) = \mathfrak{U}_2(\chi) \equiv -\frac{1}{2^{2}}B_2^{2^2} \pmod{2^{3}} \quad \text{for } n=3.
\]

In the case $l=2$ we see for $n \geq 4$
\[(1 - \chi(2)2^{-1})\Omega_2(\chi) = \Omega_2(\chi) \equiv -\frac{1}{2}B^2_k \quad \pmod{2^2 - 2^{n-2}} ,\]

and for \(n = 3\) after a direct calculation

\[(1 - \chi(2)2^{-1})\Omega_2(\chi) = \Omega_2(\chi) \equiv -\frac{1}{2}B^2_k \quad \pmod{2^2} .\]

**Theorem 4.** If \(\chi\) is a character of the second kind, then the congruences hold:

\[\Omega_p(\chi) \equiv -\frac{1}{p^{l-1}(p-1)}B^{2l-1}_{\chi} \quad \pmod{p^{l-1} - 2^{n-2}(p-1)} \quad \text{for } p \text{ odd},\]

\[\Omega_2(\chi) \equiv -\frac{1}{2^{l-1}}B^{2l-1}_{\chi} \quad \pmod{2^{l-1} - 2^{n-2}} \quad \text{for } l \geq 4, n \geq 4,\]

\[\equiv -\frac{1}{2^{l-1}}B^{2l-1}_{\chi} \quad \pmod{2^l} \quad \text{for } l \geq 4, n = 3,\]

\[\equiv -\frac{1}{2^2}B^{2}_{\chi} \quad \pmod{2^4 - 2^{n-2}} \quad \text{for } n \geq 4,\]

\[\equiv -\frac{1}{2^2}B^{2}_{\chi} \quad \pmod{2^3} \quad \text{for } n = 3,\]

\[\equiv -\frac{1}{2}B^{2}_{\chi} \quad \pmod{2^2 - 2^{n-2}} \quad \text{for } n \geq 4,\]

\[\equiv -\frac{1}{2}B^{2}_{\chi} \quad \pmod{2^2} \quad \text{for } n = 3.\]

### 7. Another computation in some cases

In this section we show a slightly different way to transform the quantities \(\Omega_p(\chi)\) with characters having prime power conductors \(p^n(n \geq 1)\).

We have easily from the definitions of \(\Omega_p(\chi)\) for \(n \geq 1\)

\[(1 - \chi(c))\Omega_p(\chi) = -\chi(c)\frac{\tau(\chi)}{p^n} \sum_a \bar{\omega}(a) \log \left( \frac{1}{c} \frac{1 - \zeta_{p^n}^{a/c}}{1 - \zeta_{p^n}} \right) .\]

Now, set \(\xi(\chi, c) = \sum_a \bar{\omega}(a) \log \left( \frac{1}{c} \frac{1 - \zeta_{p^n}^{a/c}}{1 - \zeta_{p^n}} \right) .\)

By Lemma 1 we have then
\[ \zeta(\chi, c) = \lim_{l \to \infty} \frac{1}{p^l} \sum_a \bar{\zeta}(a) \left\{ \frac{1}{c} \frac{1 - \zeta_{p^n}^{a_i}}{1 - \zeta_{p^n}^{p^n}} \right\}^k, \]

as we see \( \frac{1}{c} \frac{1 - \zeta_{p^n}^{a_i}}{1 - \zeta_{p^n}^{p^n}} \equiv 1 \pmod{p_a}, p_a = (1 - \zeta_{p^n}). \)

Namely, we have

\[ \zeta(\chi, c) = \lim_{l \to \infty} \frac{1}{p^l} \sum_a \bar{\zeta}(a) \sum_k \binom{p^l}{k} \left\{ \frac{1}{c} \frac{1 - \zeta_{p^n}^{a_i}}{1 - \zeta_{p^n}^{p^n}} \right\}^k \]

\[ = \lim_{l \to \infty} \frac{1}{p^l} \sum_a \bar{\zeta}(a) \sum_k \binom{p^l}{k} \left\{ \frac{1}{c} \sum_j \binom{c}{j} (-1)^{j-1} (1 - \zeta_{p^n}^{a_i})^{j-1} - 1 \right\}^k. \]

Therefore, we see further

\[ \zeta(\chi, c) = \lim_{l \to \infty} \frac{1}{p^l} \prod_{r=1}^{p^l} \sum_k \binom{p^l}{k} \sum_{a(k,r) \neq 0} \frac{k!}{k_1 \cdots k_{c-1}!} \frac{1}{c^k \binom{c}{2} \cdots \binom{c}{c}^{k_{c-1}}} \times \sum_a \bar{\zeta}(a) (-1)^r (1 - \zeta_{p^n,r}^a), \]

where the third summation is extended over all partitions \( a(k, r): k = k_1 + \cdots + k_{c-1}, r = k_1 + 2k_2 + \cdots + (c-1)k_{c-1}. \)

Now, we find that

\[ \sum_a \bar{\zeta}(a) (1 - \zeta_{p^n, r}^a) = \sum_j \binom{r}{j} (-1)^j \sum_a \bar{\zeta}(a) \zeta_{p^n}^{a_j} \]

\[ = \tau(\bar{\zeta}) \sum_j \binom{r}{j} (-1)^j \chi(j) \]

\[ = (-1)^r \tau(\bar{\zeta}) \sum_j \binom{r}{j} (-1)^{r-j} \chi(j). \]

We write symbolically the last sum by \( A^r \chi(0). \)

Thus we have

\[ \zeta(\chi, c) = \lim_{l \to \infty} \frac{1}{p^l} \prod_{r=1}^{p^l} \sum_k \binom{p^l}{k} \sum_{a(k,r) \neq 0} \frac{k!}{k_1 \cdots k_{c-1}!} \frac{1}{c^k \binom{c}{2} \cdots \binom{c}{c}^{k_{c-1}}} \times \tau(\bar{\zeta}) A^r \chi(0) \]

\[ = \lim_{l \to \infty} \frac{\tau(\bar{\zeta})}{p^l} \sum_k \binom{p^l}{k} \left\{ \frac{1}{c} \sum_{j=1}^{c} \binom{c}{j} A^{j-1} \right\}^k \chi(0) \]

\[ = \lim_{l \to \infty} \frac{\tau(\bar{\zeta})}{p^l} \sum_k \binom{p^l}{k} \left\{ \frac{1}{c} \frac{(1 + A)^{c-1} - 1}{A} \right\}^k \chi(0). \]
For the sake of simplicity we set \( \Gamma_c = \frac{(1 + A)^c - 1 - c}{A} \).

Then \( \Gamma_c \) is an operator which is a polynomial of \( A \) with integer coefficients. Furthermore, if we write \( P_c = \frac{(1 + A)^c - 1}{A} = \sum_{i=1}^{c-1} \binom{c}{i} A^{i-1} \), then we \( \Gamma_c^k = (P_c - c)^k \) \( \sum_j \binom{k}{j} (-c)^{k-j} P_c^j \). It follows from these that

\[
P_c \chi(0) = \sum_{i=1}^{c} \binom{c}{i} \frac{c-1}{i} \binom{c-1}{i-1} \binom{i-1}{j} (\sum_{j=0}^{i-1} S^j (-1)^{i-j} \chi(0)),
\]

where \( S \) means a shift operator on \( \chi(0) \), i.e., \( A + 1 = S \).

Consequently we have

\[
P_c \chi(0) = \sum_{i=0}^{c-1} \binom{c}{i} \binom{c-1}{i} \binom{c-j-1}{i-1-j} (-1)^{i-j-1} \chi(j)
\]

\[
= \sum_{i=0}^{c-1} \binom{c}{i} (-1)^{i-1} \chi(j) \sum_{i=j+1}^{c} \binom{c-1}{i} \binom{c-1-j}{i}.
\]

By the way it holds that

\[
\sum_{i=j+1}^{c} (-1)^j \frac{c}{i} \binom{c-1}{i} \binom{c-1-j}{i-1-j} = c \sum_{i=0}^{c-j-1} \frac{1}{i+j+1} (-1)^{i+j+1} \binom{c-1-j}{i}
\]

and

\[
\sum_{i=0}^{c-j-1} \frac{(-1)^i}{i+j+1} \binom{c-1-j}{i} = \frac{1}{c} \frac{1}{\binom{c-1}{j}} (j \geq 0).
\]

From these formulas yields

\[
P_c \chi(0) = \sum_{j=0}^{c-1} \binom{c}{j} (-1)^j \chi(j) \binom{c-1-j}{i} c (-1)^{j+1} \frac{1}{c} \frac{1}{\binom{c-1}{j}}
\]

\[
= \sum_{j=0}^{c-1} \chi(j).
\]

Thus we have

\[
P_c^j \chi(0) = \sum_{i_1=0}^{c-1} \cdots \sum_{i_j=0}^{c-1} \chi(i_1 + \cdots + i_j),
\]
and so for \( k \geq 1 \)
\[
\Gamma^k_c(0) = \sum_{j=1}^{\infty} \binom{k}{j} (-c)^{k-j} \sum_{i_1=0}^{c-1} \sum_{i_j=0}^{c-1} \chi(i_1 + \cdots + i_j).
\]
Furthermore we see
\[
\frac{1}{p^l} \sum_k \binom{p^l}{k} \frac{1}{c^k} \Gamma^k_c(0)
= \frac{1}{p^l} \sum_k \binom{p^l}{k} \frac{1}{c^k} \sum_j \binom{k}{j} (-c)^{k-j} \sum_{i_1=0}^{c-1} \sum_{i_j=0}^{c-1} \chi(i_1 + \cdots + i_j)
= \frac{1}{p^l} \sum_j \binom{p^l}{j} (-1)^j \sum_{i_1=0}^{c-1} \sum_{i_j=0}^{c-1} \chi(i_1 + \cdots + i_j) \sum_k \binom{p^l-j}{p^l-k} \frac{1}{c^k} (-c)^{k-j}
= \frac{1}{p^l} \sum_j \binom{p^l}{j} \frac{(-1)^j}{c^j} \sum_{i_1=0}^{c-1} \sum_{i_j=0}^{c-1} \chi(i_1 + \cdots + i_j) \sum_k \binom{p^l-j}{p^l-k} (-1)^k.
\]
By the way we see also easily
\[
\sum_k \binom{p^l-j}{p^l-k} (-1)^k = (-1)^{p^l} (1-(-1)^{p^l-j}) = \begin{cases} 0 & \text{for } p^l - j > 0, \\ (-1)^{p^l} & \text{for } p^l = j. \end{cases}
\]
Therefore we obtain finally
\[
\frac{1}{p^l} \sum_k \binom{p^l}{k} \frac{1}{c^k} \Gamma^k_c(0) = \frac{1}{p^l} \frac{1}{c^{p^l}} \sum_{i_1=0}^{c-1} \sum_{i_{p^l}=0}^{c-1} \chi(i_1 + \cdots + i_{p^l}).
\]
Thus we conclude that
\[
\xi(\chi, c) = \lim_{l \to \infty} \frac{1}{p^l} \frac{1}{c^{p^l}} \tau(\tilde{\chi}) \sum_{i_1=0}^{c-1} \sum_{i_{p^l}=0}^{c-1} \chi(i_1 + \cdots + i_{p^l}),
\]
and therefrom we obtain
\[
(1 - \chi(c)) \Omega_p(\chi) = \lim_{l \to \infty} \frac{1}{p^l} \frac{1}{c^{p^l}} \sum_{i_1=0}^{c-1} \sum_{i_{p^l}=0}^{c-1} \chi(i_1 + \cdots + i_{p^l}).
\]

**Theorem 5.** For such a parameter \( c \) that \( (c, p) = 1, \ c > 1, \ \chi(c) \equiv 1 \) hold, we have
\[
(1 - \chi(c)) \Omega_p(\chi) = \lim_{l \to \infty} \frac{1}{p^l} \frac{1}{c^{p^l}} \sum_{i_1=0}^{c-1} \sum_{i_{p^l}=0}^{c-1} \chi(i_1 + \cdots + i_{p^l}).
\]
From Theorem 1 this is equivalent to a formula:

$$L_p(1, \chi) = \frac{1}{1 - \chi(c)} \lim_{i \to \infty} \frac{1}{p^i} \frac{1}{c^\rho} \sum_{i=0}^{c-1} \cdots \sum_{i_p=0}^{c_p-1} \chi(i_1 + \cdots + i_{p^i}).$$

This formula shows an approximation of $L_p(1, \chi)$ by means of the character values instead of the Bernoulli numbers.

In the next section we shall clarify a meaning of this formula for characters of conductor $p$ in some extent.

8. Removal of $\omega$-factors

From Lemma 1 for the character $\chi$ of conductor $f_k = p$ we have

$$\zeta(\chi, c) \equiv \tau(\chi) \frac{1}{p^i} \sum_k \frac{1}{c^k} \Gamma_{c^k}[\chi(0)(\bmod p^{i+\delta i})].$$

If we denote $\chi(x) = \omega(x)^v$ with an even integer $v$ such that $2 \leq v \leq p - 3$, then we can verify in a similar way as before that

$$\frac{1}{c^k} \Gamma_{c^k}[\chi(0)(\bmod p^i)] = 1 \Gamma_{c^k}^v(0)^{p^{i-1} v} \quad (\bmod p^i).$$

From the fact that $\frac{1}{c^k} \Gamma_{c^k}[\chi(0)(\bmod p^{k-1} p^e(k)) = 0$, it follows that

$$\zeta(\chi, c) \equiv \tau(\chi) \sum_{k=1}^{p^i} \left(\frac{-1}{k}\right)^{k-1} \frac{1}{c^k} \Gamma_{c^k}^v(0)^{p^{i-1} v} \quad (\bmod p^i).$$

Therefore we have

$$(1 - \chi(c)) \zeta_p(\chi) \equiv \sum_{k=1}^{p^i} \left(\frac{-1}{k}\right)^{k-1} \frac{1}{c^k} \Gamma_{c^k}^v(0)^{p^{i-1} v} \quad (\bmod p^i).$$

In the sequel we calculate the generating function of the numbers appearing in the right-hand side of the congruence above.

$$G_c(t) = \sum_{m=1}^{c^\rho} \frac{1}{m!} \sum_{k=1}^{p^i} \frac{1}{c^k} \left(\frac{-1}{k}\right)^{k-1} \Gamma_{c^k}^v(0)^{p^{i-1} v}$$

$$= \sum_{k=1}^{p^i} \left(\frac{-1}{k}\right)^{k-1} \frac{1}{c^k} \sum_{m=1}^{c^\rho} \frac{1}{m!} \Gamma_{c^k}^v(0)^{p^{i-1} v}.$$
On certain values of \( p \)-adic \( L \)-functions

\[
\sum_{m=0}^{\infty} \frac{1}{m!} \mathcal{P} \omega_{m} = \sum_{i_{1}=0}^{c_{1}} \cdots \sum_{i_{j}=0}^{c_{j}} \sum_{m=0}^{\infty} \frac{1}{m!} (i_{1} + \cdots + i_{j})^{m} m
\]

\[
= \sum_{i_{1}=0}^{c_{1}} \cdots \sum_{i_{j}=0}^{c_{j}} e^{(i_{1} + \cdots + i_{j})t} \left( \frac{1 - e^{ct}}{1 - e^{t}} \right)^{j}.
\]

Consequently

\[
\sum_{m=0}^{\infty} \frac{1}{m!} \Gamma_{c} \omega_{m} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{j=0}^{k} \binom{k}{j} (-c)^{k-j} \mathcal{P} \omega_{m} \]

\[
= \sum_{j=0}^{k} \binom{k}{j} (-c)^{k-j} \sum_{m=0}^{\infty} \frac{1}{m!} \Delta_{c} \omega_{m} \]

\[
= \sum_{j=0}^{k} \binom{k}{j} (-c)^{k-j} \left( \frac{1 - e^{ct}}{1 - e^{t}} \right)^{j} \]

\[
= \left( \frac{1 - e^{ct}}{1 - e^{t}} - c \right)^{k}.
\]

Hence

\[
G_{c}(t) = \sum_{k=1}^{p} \frac{(-1)^{k-1}}{k} \frac{1}{c} \left( \frac{1 - e^{ct}}{1 - e^{t}} - c \right)^{k}
\]

\[
= \sum_{k=1}^{p} \frac{(-1)^{k-1}}{k} \left( \frac{1}{c} \frac{1 - e^{ct}}{1 - e^{t}} - 1 \right)^{k}
\]

\[
\sim \log \left( \frac{1}{c} \frac{1 - e^{ct}}{1 - e^{t}} \right).
\]

Namely we see

\[
tG_{c}(t) = \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \sum_{k=1}^{p} \frac{(-1)^{k-1}}{k} \frac{1}{c} \Gamma_{c} \omega_{m} \]

\[
\sim \frac{-ct e^{ct}}{1 - e^{ct}} + \frac{te^{t}}{1 - e^{t}} = e^{Bct} - e^{Bt},
\]

where \( \sim \) means an equality of two series up to terms of degree \( \leq p \).

By comparing the coefficients of \( t^{p-1} \) we obtain

**Lemma 4.** For \( m = 1, 2, \ldots, p \) hold
\[
\frac{\xi^k}{k} \prod_{k=1}^{\xi} \frac{(-1)^{k-1}}{k} T^k \alpha^m = \frac{1}{m} (\xi^m - 1) B^m.
\]

It follows from this that

\[
(1 - \chi(c)) \mathcal{Q}_p(\chi) \equiv - (1 - e^{p^{l-1}v}) \frac{1}{p^{l-1}v} B^{p^{l-1}v} \pmod{p^l}.
\]

But, \( \frac{1}{p^{l-1}v} B^{p^{l-1}v} \) is a \( p \)-adic integer and \( 1 - \chi(c) \equiv 1 - e^{p^{l-1}v} \pmod{p^l} \) gives us the following

**Theorem 6.** For a character \( \chi = \omega^v \) of conductor \( p \) we have

\[
\mathcal{Q}_p(\chi) \equiv - \frac{1}{p^{l-1}v} B^{p^{l-1}v} \pmod{p^l}.
\]

The formula to \( v = \frac{p-1}{2} \) is due to Slavutsky [9]. However, we can obtain similar formulas in general cases where \( \chi = \omega^v \chi_0 \) with \( \chi_0 \) a character without \( \omega \)-part, and \( 1 \leq v \leq p - 2 \). In fact, we can easily verify the congruences for odd \( p \) and \( l \geq 1 \)

\[
\frac{1}{p^{l-1}(p-1)} B^{p^{l-1}(p-1)} \equiv \frac{1}{p^{l-1}v} B^{p^{l-1}v} \pmod{p^l}.
\]

These congruences come out readily from the congruences of Kummer. Hence we can transform the congruences in Theorem 1 to the congruences similar to Theorem 6.

### 9. Approximation for \( L_p(0, \chi \omega) \)

Let \( K \) be in the sequel an imaginary abelian number field over \( Q \) of degree \( (K:Q) = 2g \), and \( \mathcal{X} \) the corresponding character group in the sense of the class field theory. Then we put \( \mathcal{X}^- = \{ \chi \in \mathcal{X}; \chi(-1) = -1 \} \). We know that the class number \( h_K \) of \( K \) can be decomposed in such as \( h_K = h_K h_{\overline{K}} \) with the first factor \( h_K \) and the second factor \( h_{\overline{K}} \).

For the second factor we have

\[
h_{\overline{K}} = \mathcal{Q}_{K_{\overline{K}}} \prod_{\chi \in \mathcal{X}^-} \left( -\frac{1}{2f_{\chi}} \sum_{x=1}^{f_{\chi}} \chi(x)x \right),
\]
where $Q_K$ means a unit-index $Q_K = (e: \delta e_0)$ with respect to $K/K^+$ with the maximal real subfield $K^+$ of $K$, and $w_K$ denotes the number of all roots of unity contained in $K$ [3].

Because we see $B_1^1 = \frac{1}{f_K} \sum_{x=1}^{f_K} \chi(x)x$ and $L_p(0, \chi \omega) = -(1 - \chi(p))B_1^1$, we have a $p$-adic class number formula for $K/K^+$:

$$h_K \prod_{\chi \in \hat{K}} (1 - \chi(p)) = Q_K w_K 2^{-s} \prod_{\chi \in \hat{K}, \chi(1) 
eq 1} L_p(0, \chi \omega).$$

When we approximate $L_p(0, \chi \omega)$ by the Bernoulli numbers of large index we obtain the so-called Vandiver congruences.

For example, in the case where $K_n = Q(\zeta_{p^{n+1}}), \zeta_{p^{n+1}} = e^{2\pi i/p^{n+1}}$, the following theorem holds:

**Theorem 7.** For $K_n = Q(\zeta_{p^{n+1}}), p \neq 2$ we have with $l \geq 1$

$$h_n \equiv 2^{1-n} p^{l-1} p^{n+1} \prod_{\nu, z, \chi} L_\nu(-vp^{l+1}, \chi \omega^{l+1}) \pmod{p^l},$$

where $\nu$ runs over all odd integers between 1 and $p - 1$, and $\chi \omega$ runs over all characters of the second kind defined modulo $p^{n+1}$.

For $K_n = Q(\zeta_{2^{n+1}})$ we have with $l \geq 1$

$$h_n \equiv 2^{n+1-2^{n-1}} \prod_{\chi} L_2(-2^{l+2^{n-1}} - 1, \chi \omega) \pmod{2^l},$$

where $\chi$ ranges over all odd characters defined modulo $2^{n+1}$.

**Proof.** We treat the case of odd $p$ first.

We know that $L_p(s, \chi \omega) = -\sum_{k=0}^{\infty} \binom{-s}{k} A^k a_1(\chi \omega)$ with $a_m(\chi \omega) = \frac{1}{m} D_{\chi \omega}^m$ for $s \in \mathbb{Z}_p$ to $\chi \omega \equiv \chi^0, \chi_1$, where we put $D_{\chi \omega}^m = (1 - \chi \omega^{-m}(p)p^{m-1}) B_{\chi \omega^{-m}, [8]}$. It follows from this that

$$L_p(-vp^l, \chi \omega) = -\sum_k \binom{vp^l}{k} A^k a_1(\chi \omega) = -a_1(\chi \omega) - \sum_{k=1} \binom{vp^l}{k} A^k a_1(\chi \omega).$$

For $k \geq 1$ we see

$$\sum_k \binom{vp^l}{k} A^k a_1(\chi \omega) \geq 1 - \sum_k \binom{vp^l}{k} A^k a_1(\chi \omega) \geq l + k \geq l + 1 + \frac{p - 2}{p - 1} + \frac{1}{p - 1}$$

$\geq l + 1$ and thus we obtain $L_p(0, \chi \omega) \equiv L_p(-vp^l, \chi \omega) \pmod{p^{l+1}}$.

In the case where $\chi \omega \equiv \chi_1 \equiv x^0$ we see also from the fact $L_p(-vp^l, \chi \omega)$
\[-(1+\Delta)^{\alpha}a_1(\omega) \quad \text{and} \quad \frac{1}{m} D_{\omega}^m = \frac{u_p}{1 - \omega(1+p)(1+p)^m} \pmod{p^0} \quad \text{that} \quad \Delta_{v_p}D_{\omega}^1 = \frac{u_p}{1 - \omega(1+p)(1+p)^{v_p+1}} - \frac{u_p}{1 - \omega(1+p)(1+p)} \pmod{p^{v_p+1}}. \]

From this we obtain easily

\[ (1 - \omega(1+p))^2 L_p(0, \omega) \equiv (1 - \omega(1+p))^2 L_p(-v_p, \omega) \pmod{p^{v_p+1}}. \]

In the case of \(\omega = \omega^0\) we have similarly

\[ pL_p(0, \omega) \equiv pL_p(-v_p, \omega) \pmod{p^v}. \]

Finally we see

\[ p^{v_1+2n} \prod_{\chi} L_p(0, \omega) \equiv p^{v_1+2n} \prod_{\chi} L_p(-v_p^{v_1+n}, \chi) \pmod{p^{v_1+n}}, \]

from which we conclude the congruence

\[ h_n = 2^{v_1-v} \prod_{\chi} L_p(-v_p^{v_1+n}, \chi) \pmod{p^v}. \]

We next prove the congruence in the case \(p=2\). In this case we have

\[ h_n = 2^{n+1} \prod_{\chi \in \mathbb{F}_2} \left( -\frac{1}{2^{v_1+n}} \sum_{x=1}^{f_1} \chi(x)x \right) = 2^{n+1} \prod_{\chi \in \mathbb{F}_2} \left( -\frac{1}{2} B_{\chi}^1 \right) = 2^{n+1} \prod_{\chi \in \mathbb{F}_2} L_2(0, \omega), \]

where \(\mathbb{F}_2 = \mathbb{F}^{-1} = \mathbb{F}^{-2}\) holds.

Note that \(\omega\) becomes always a character of the second kind. If \(\omega = \omega_1 \cong \omega^0\), then \(L_2(-2, \chi_1) = L_2(1 + (1+2^i), \chi_1) = -(1 - \chi_1 \omega^{-2i-1}(2)2^{2i-1}) \frac{1}{1 + 2i} B_{\chi_1}^{i+2} \pmod{2^{i+2}}\).

and \(\Delta_{2i}D_{\omega}^1 \equiv \frac{u_2}{1 + 2i} D_{\omega}^{i+2} - \frac{u_2}{1 - \chi_1(5)s^{i+2}} \pmod{2^{i+2}}\).

By means of \(5^{m-1}(5^i - 1) \equiv 0 \pmod{2^{i+2}}\), \(u_2 \equiv 0 \pmod{2}\) we see

\[ \frac{u_2}{1 - \chi_1(5)s^{i+2}} - \frac{u_2}{1 - \chi_1(5)s} = \frac{u_2}{(1 - \chi_1(5)s)2^{i+2}} \pmod{2^{i+3}} \]

for the order of \(\chi_1 \geq 2^2\), hence

\[ L_2(-2, \omega) \equiv L_2(0, \omega) \pmod{2^{i+2}} \] for the order of \(\omega \geq 2^2\), and
On certain values of $p$-adic $L$-functions

\[
\frac{u_2}{1 - \chi(5)^{1+2r}} - \frac{u_2}{1 - \chi(5)^5} \equiv 0 \pmod{2^{i+3-2}} \]
for the order of $\chi \omega = 2$, hence
\[
L_2(-2^i, \chi \omega) \equiv L_2(0, \chi \omega) \pmod{2^{i+1}}
\]
for the order of $\chi \omega = 2$.

If $\chi \omega = \chi^0$, then it follows that
\[
A_2 \chi L_2(0, \chi^0) \equiv \frac{1}{1 + 2i} D_2^{1+2i} - D_2^3 \equiv \frac{u_2}{1 - 5^{1+2i}} - \frac{u_2}{1 - 5} \pmod{2^{i+2}}
\]
\[
\frac{u_2}{1 - 5^{1+2i}} - \frac{u_2}{1 - 5} = \frac{u_2 5}{(1 - 5)^2} 2^{l+2} + \ldots \equiv 0 \pmod{2^{i+3-4}}.
\]

Thus we have $L_2(-2^i, \chi^0) \equiv L_2(0, \chi^0) \pmod{2^{l-1}}$ with $l \geq 1$. Noticing that
$L_2(-2^i, \chi \omega)$ for the order of $\chi \omega \geq 2$ and $2L_2(-2^i, \chi \omega)$ for the order of $\chi \omega = 1$
al are integers we have
\[
2 \prod_x L_2(-2^i, \chi \omega) \equiv 2 \prod_x L_2(0, \chi \omega) \pmod{2^i}.
\]

In this formula replacing $l$ by $l + 2^{n-1} - n$ and multiplying $2^{n-2^{n-1}}$ we obtain
\[
2^{n+1-2^{n-1}} \prod_x L_2(-2^{l+2^{n-1}-n}, \chi \omega) \equiv 2^{n+1-2^{n-1}} \prod_x L_2(0, \chi \omega) \pmod{2^i}.
\]

Therefore we conclude for $n \geq 1$
\[
h_n \equiv 2^{n+1-2^{n-1}} \prod_{x \in \mathbb{Q}} L_2(-2^{l+2^{n-1}-n}, \chi \omega) \pmod{2^i}.
\]

By the quite same method as above we can readily obtain similar congruences for arbitrary imaginary abelian fields $K/K^+$.

Bibliography

