ON THE CYLINDRICAL σ -ALGEBRA AND THE BOREL FIELD OF AN $\mathcal{LF}$ SPACE

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§1. Main results

Let (X, Y) be a dual system of topological vector spaces X and Y , $C(X, Y)$ be the minimal σ -algebra of subsets of X that makes all functionals in Y measurable and $B(X)$ be the minimal σ -algebra containing all open subsets of X . We call $C(X, Y)$ (resp. $B(X)$) the cylindrical σ -algebra (resp. Borel field) of X . S. Ahmad [1] proved that if X is a separable Fréchet space then $C(X, X')$ coincides with $B(X)$, where X' denotes the topological dual of X . In this paper we prove the following theorem.

THEOREM. Let $X = \bigcup_{n=1}^{\infty} X_n$ be an inductive limit of an increasing sequence of separable Fréchet spaces $\{X_n\}_{n=1}^{\infty}$ and (X, Y) be a dual system with a separately continuous bilinear form $\langle \cdot, \cdot \rangle$. Then we have $C(X, Y) = B(X)$.

COROLLARY. Let (X, Y) be the same as in the above theorem. Then the subspace X_n belongs to $C(X, Y)$ for every n and every probability measure on $C(X, Y)$ is Radon.

§2. Proof of the theorem

In the first place, we consider the case X is a Fréchet space.

LEMMA. Let X be a separable Fréchet space and (X, Y) be a dual system with a separately continuous bilinear form $\langle \cdot, \cdot \rangle$. Then we have $C(X, Y) = B(X)$.

PROOF. We may identify Y with a subspace of X' . Let $\{U_n\}_{n=1}^{\infty}$ be a countable base of neighborhoods of 0 in X . Then the polar U_n^0 is a compact metric

space with respect to the weak star topology $\sigma(X', X)$. For every n , put

$$W_n = U_n^0 \cap Y.$$

Each W_n is a separable metric space with respect to the topology $\sigma(X', X)$. Define seminorms $\{p_n\}_{n=1}^\infty$ on X as follows,

$$p_n(x) = \sup_{y \in W_n} |\langle x, y \rangle|.$$

The seminorms $\{p_n\}_{n=1}^\infty$ define a metrizable topology τ on X which is coarser than the initial topology. Moreover the seminorms are all measurable with respect to the cylindrical σ -algebra $C(X, Y)$. In fact, let $\{y_i^n\}_{i=1}^\infty$ be a countable dense subset of W_n with respect to the topology $\sigma(X', X)$, then we have

$$\begin{aligned} \{x \in X; p_n(x) \leq 1\} &= \{x \in X; \sup_i |\langle x, y_i^n \rangle| \leq 1\} \\ &= \bigcap_{i=1}^\infty \{x \in X; |\langle x, y_i^n \rangle| \leq 1\} \in C(X, Y). \end{aligned}$$

Therefore we have $C(X, Y) = B(X\tau)$, where $X\tau$ denotes the space X equipped with the topology τ .

Since X is a Polish space (i.e., a complete separable metric space) and τ is a Hausdorff topology coarser than the initial one of X , we have $B(X) = B(X\tau)$ (see for example N. Bourbaki [2] §6, X. Fernique [3]). This proves the lemma.

PROOF OF THE THEOREM. For each n , τ_n denotes the topology on X_n induced from X . Since X_n is a Polish space and τ_n is a Hausdorff topology coarser than the initial one of X_n , we have

$$\begin{aligned} B(X_n) &= B(X_n\tau_n) \\ &= B(X) \cap X_n. \end{aligned}$$

On the other hand, the pair $(X_n, Y|X_n)$ forms a dual system, hence by the above lemma we have

$$\begin{aligned} B(X_n) &= C(X_n, Y|X_n) \\ &= C(X, Y) \cap X_n. \end{aligned}$$

Therefore for every n , we have

$$B(X) \cap X_n = C(X, Y) \cap X_n.$$

Let A be any set in $B(X)$, then for every n , there exists an element C_n in $C(X, Y)$ such that

$$A \cap X_n = C_n \cap X_n.$$

Since $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence, the set A coincides with $\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} C_n$ which belongs to $C(X, Y)$. Thus we have proved $B(X) \subset C(X, Y)$. The converse is obvious. This proves the theorem.

References

- [1] S. AHMAD: *Éléments aléatoires dans les espaces vectoriels topologique*. Ann. Inst. Henri Poincaré, Vol. **II**, n°2, 1965, pp. 95–135.
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- [3] X. FERNIQUE: *Processus linéaires, processus généralisés*. Ann. Inst. Fourier, Grenoble. 17, 1, 1967, pp. 1–92.