DETERMINATION OF 2-COMPONENTS OF THE 23 AND 24-STEMS IN HOMOTOPY GROUPS OF SPHERES

By

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§ 1. Introduction and results.

The present paper is concerned with the 2-component of the homotopy groups of spheres. Let π_*^n be the 2-component of the homotopy groups $\pi_*(S^n)$. Toda [T], Mimura and Toda [MT], and Mimura [MII] have completely determined π_{n+i}^n for $i \leq 22$. If n is large, π_{n+i}^n is the 2-component of the i -th stable homotopy group of sphere and much information has been obtained by making use of various types of the Adams spectral sequences. In this paper we are mainly concerned with the case of small n , namely we work in the unstable range. We can obtain much information from the unstable Adams spectral sequence. Although we can calculate and determine its E_2 -term algebraically, we have difficulty in determining the differentials in the spectral sequence and reconstructing the group extensions and products which are lost in passing to quotients. In fact, these are essentially geometric problems. For this reason π_*^n is still now vague beyond the 22-stem.

The purpose of this paper is to calculate the homotopy group π_{n+i}^n for $i=23$ and 24. We completely determine the group structure of π_{n+23}^n and π_{n+24}^n for all n , by constructing the generators of π_*^n geometrically. Our method is the so-called composition method established by Toda [T]. The basic tool is the EHA -exact sequence

$$\cdots \longrightarrow \pi_{i+2}^{2n+1} \xrightarrow{A} \pi_i^n \xrightarrow{E} \pi_{i+1}^{n+1} \xrightarrow{H} \pi_{i+1}^{2n+1} \xrightarrow{A} \pi_{i-1}^n \longrightarrow \cdots,$$

introduced by Whitehead and James, where E is the suspension homomorphism,

1) The present work was announced in [MMO]. However, some results in [MMO] which do not appear in this paper will be referred to [Mo].

H is the Hopf homomorphism and Δ is essentially the Whitehead product $[\iota_n, \]$. This enables us to calculate π_*^n inductively. The problem of calculating π_*^n is therefore divided into three steps; i) the construction of new elements of π_*^n , which are not represented by composition, suspension nor Whitehead product ii) the determination of the homomorphisms E , H and Δ , iii) the settlement of the group extension problem of the short exact sequence

$$0 \longrightarrow \text{Coker } \Delta \xrightarrow{E} \pi_*^n \xrightarrow{H} \text{Ker } \Delta \longrightarrow 0.$$

In order to settle each problem, many machineries have been developed in Toda's book [T]. In particular, the Toda bracket comes to play an important role. Although our calculation is very complicated in practice, we can carry it out by making use of various properties of Toda brackets.

We now summarize the results of our calculation in the following theorem. The namings of elements are given in the previous works [T], [MT] and [M], but new elements will be defined in the forthcoming sections.

THEOREM 1.1.

π_{n+23}^n and π_{n+24}^n are given by the table below.

(a) $\pi_{25}^2 = \{\eta_2 \circ \varepsilon_3 \circ \kappa_{11}\} \approx Z_2.$

$$\pi_{26}^3 = \{\bar{\alpha}\} \approx Z_4.$$

$$\pi_{27}^4 = \{E\bar{\alpha}\} \oplus \{v_4 \circ \bar{\kappa}_7\} \approx Z_4 \oplus Z_8.$$

$$\pi_{29}^5 = \{v_5 \circ \bar{\kappa}_8\} \oplus \{\bar{\rho}'''\} \oplus \{\phi_5\} \approx Z_8 \oplus Z_2 \oplus Z_2.$$

$$\pi_{28}^6 = \{v_6 \circ \bar{\kappa}_9\} \oplus \{\bar{\rho}''\} \oplus \{\phi_6\} \oplus \{\Delta(\lambda), \Delta(\xi)\} \approx Z_8 \oplus Z_4 \oplus Z_2 \oplus (Z_8 \oplus Z_4).$$

$$\pi_{30}^7 = \{v_7 \circ \bar{\kappa}_{10}\} \oplus \{\bar{\rho}'\} \oplus \{\phi_7\} \oplus \{\bar{\kappa}_7 \circ v_{27} - v_7 \circ \bar{\kappa}_{10}\} \oplus \{\sigma' \circ \sigma_{14} \circ \mu_{21}\}$$

$$\oplus \{\sigma' \circ \omega_{14}\} \approx Z_8 \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

$$\pi_{31}^8 = \{v_8 \circ \bar{\kappa}_{11}\} \oplus \{E\bar{\rho}'\} \oplus \{\phi_8\} \oplus \{\bar{\kappa}_8 \circ v_{28} - v_8 \circ \bar{\kappa}_{11}\} \oplus \{E\sigma' \circ \sigma_{15} \circ \mu_{22}\}$$

$$\oplus \{E\sigma' \circ \omega_{15}\} \oplus \{\sigma_8^2 \circ \mu_{22}\} \oplus \{\sigma_8 \circ \omega_{15}\} \oplus \{\sigma_8 \circ \eta^{*'}\}$$

$$\approx Z_8 \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

$$\pi_{32}^9 = \{\bar{\rho}_9\} \oplus \{v_9 \circ \bar{\kappa}_{12}\} \oplus \{\phi_9\} \oplus \{\bar{\kappa}_9 \circ v_{29} - v_9 \circ \bar{\kappa}_{12}\} \oplus \{\sigma_9^2 \circ \mu_{23}\}$$

$$\oplus \{\sigma_9 \circ \omega_{16}\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

$$\begin{aligned}\pi_{33}^{10} &= \{\bar{\rho}_{10}\} \oplus \{v_{10} \circ \bar{\kappa}_{13}\} \oplus \{\phi_{10}\} \oplus \{\bar{\kappa}_{10} \circ v_{30} - v_{10} \circ \bar{\kappa}_{13}\} \oplus \{\psi_{10}\} \\ &\approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_4.\end{aligned}$$

$$\begin{aligned}\pi_{n+23}^n &= \{\bar{\rho}_n\} \oplus \{v_n \circ \bar{\kappa}_{n+3}\} \oplus \{\phi_n\} \oplus \{\psi_n\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \\ &\text{for } n = 11, 12, 13, 14.\end{aligned}$$

$$\begin{aligned}\pi_{38}^{15} &= \{\bar{\rho}_{15}\} \oplus \{v_{15} \circ \bar{\kappa}_{18}\} \oplus \{\phi_{15}\} \oplus \{\psi_{15}\} \oplus \{\bar{\varepsilon}^{*'}\} \oplus \{\bar{v}^{*'}\} \\ &\approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.\end{aligned}$$

$$\begin{aligned}\pi_{39}^{16} &= \{\bar{\rho}_{16}\} \oplus \{v_{16} \circ \bar{\kappa}_{19}\} \oplus \{\phi_{16}\} \oplus \{\psi_{16}\} \oplus \{E\bar{\varepsilon}^{*'}\} \oplus \{E\bar{v}^{*'}\} \oplus \{\bar{\varepsilon}_{16}^*\} \\ &\oplus \{\bar{v}_{16}^*\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.\end{aligned}$$

$$\begin{aligned}\pi_{40}^{17} &= \{\bar{\rho}_{17}\} \oplus \{v_{17} \circ \bar{\kappa}_{20}\} \oplus \{\phi_{17}\} \oplus \{\psi_{17}\} \oplus \{\bar{\varepsilon}_{17}^*\} \oplus \{\bar{v}_{17}^*\} \\ &\approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.\end{aligned}$$

$$\begin{aligned}\pi_{41}^{18} &= \{\bar{\rho}_{18}\} \oplus \{v_{18} \circ \bar{\kappa}_{21}\} \oplus \{\phi_{18}\} \oplus \{\psi_{18}\} \oplus \{\bar{\varepsilon}_{18}^*\} \\ &\approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2.\end{aligned}$$

$$\begin{aligned}\pi_{n+23}^n &= \{\bar{\rho}_n\} \oplus \{v_n \circ \bar{\kappa}_{n+3}\} \oplus \{\phi_n\} \oplus \{\psi_n\} \\ &\approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \quad \text{for } n = 19, 20, 21.\end{aligned}$$

$$\pi_{n+23}^n = \{\bar{\rho}_n\} \oplus \{v_n \circ \bar{\kappa}_{n+3}\} \oplus \{\phi_n\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \quad \text{for } n = 22, 23.$$

$$\pi_{47}^{24} = \{\bar{\rho}_{24}\} \oplus \{v_{24} \circ \bar{\kappa}_{27}\} \oplus \{\phi_{24}\} \oplus \{\Delta_{49}\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2.$$

$$\pi_{n+23}^n = \{\bar{\rho}_n\} \oplus \{v_n \circ \bar{\kappa}_{n+3}\} \oplus \{\phi_n\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \quad \text{for } n \geq 25.$$

$$(b) \quad \pi_{26}^2 = \{\eta_2 \circ \bar{\alpha}\} \approx Z_4.$$

$$\pi_{27}^3 = \{\delta_3\} \oplus \{\bar{\mu}_3 \circ \sigma_{20}\} \oplus \{\varepsilon' \circ \kappa_{13}\} \approx Z_2 \oplus Z_2 \oplus Z_2.$$

$$\begin{aligned}\pi_{28}^4 &= \{\delta_4\} \oplus \{\bar{\mu}_4 \circ \sigma_{21}\} \oplus \{E\varepsilon' \circ \kappa_{14}\} \oplus \{v_4 \circ \eta_7 \circ \bar{\kappa}_8\} \oplus \{v_4 \circ \sigma' \circ \kappa_{14}\} \\ &\approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.\end{aligned}$$

$$\begin{aligned}\pi_{29}^5 &= \{\delta_5\} \oplus \{\bar{\mu}_5 \circ \sigma_{22}\} \oplus \{v_5 \circ \eta_8 \circ \bar{\kappa}_9\} \oplus \{v_5 \circ \sigma_8 \circ \kappa_{15}\} \\ &\approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.\end{aligned}$$

$$\pi_{30}^6 = \{\delta_6\} \oplus \{\bar{\mu}_6 \circ \sigma_{23}\} \oplus \{v_6 \circ \sigma_9 \circ \kappa_{16}\} \oplus \{\bar{\zeta}_6'\} \oplus \{\bar{\sigma}_6'\} \oplus \{\Delta(\lambda \circ \eta_{31})\}$$

$$\begin{aligned}
& \oplus \{ \Delta(\xi_{13} \circ \eta_{31}) \} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2. \\
\pi_{31}^7 &= \{ \delta_7 \} \oplus \{ \bar{\mu}_7 \circ \sigma_{24} \} \oplus \{ \nu_7 \circ \sigma_{10} \circ \kappa_{17} \} \oplus \{ \bar{\ell}'_7 \} \oplus \{ \bar{\sigma}'_7 \} \oplus \{ \sigma' \circ \bar{\mu}_{14} \} \\
& \oplus \{ \sigma' \circ \omega_{14} \circ \eta_{30} \} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \\
\pi_{32}^8 &= \{ \delta_8 \} \oplus \{ \bar{\mu}_8 \circ \sigma_{25} \} \oplus \{ \nu_8 \circ \sigma_{11} \circ \kappa_{18} \} \oplus \{ \bar{\ell}'_8 \} \oplus \{ \bar{\sigma}'_8 \} \oplus \{ E\sigma' \circ \bar{\mu}_{15} \} \\
& \oplus \{ E\sigma' \circ \omega_{15} \circ \eta_{31} \} \oplus \{ \sigma_8 \circ \bar{\mu}_{15} \} \oplus \{ \sigma_8^2 \circ \eta_{22} \circ \mu_{23} \} \\
& \oplus \{ \sigma_8 \circ \nu_{15} \circ \kappa_{18} \} \oplus \{ \sigma_8 \circ \varepsilon_{15}^* \} \oplus \{ \sigma_8 \circ \eta^{*'} \circ \eta_{31} \} \\
& \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \\
\pi_{33}^9 &= \{ \delta_9 \} \oplus \{ \bar{\mu}_9 \circ \sigma_{26} \} \oplus \{ \bar{\sigma}'_9 \} \oplus \{ \sigma_9 \circ \bar{\mu}_{16} \} \oplus \{ \sigma_9^2 \circ \eta_{23} \circ \mu_{24} \} \\
& \oplus \{ \sigma_9 \circ \nu_{16} \circ \kappa_{19} \} \oplus \{ \sigma_9 \circ \varepsilon_{16}^* \} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \\
\pi_{34}^{10} &= \{ \delta_{10} \} \oplus \{ \bar{\mu}_{10} \circ \sigma_{27} \} \oplus \{ \bar{\sigma}'_{10} \} \oplus \{ \tilde{\varepsilon}_{10} \} \oplus \{ \Delta \rho_{21} \} \\
& \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_4 \oplus Z_{32}. \\
\pi_{35}^{11} &= \{ \delta_{11} \} \oplus \{ \bar{\mu}_{11} \circ \sigma_{28} \} \oplus \{ \bar{\sigma}'_{11} \} \oplus \{ \tilde{\varepsilon}_{11} \} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_4. \\
\pi_{n+24}^n &= \{ \delta_n \} \oplus \{ \bar{\mu}_n \circ \sigma_{n+17} \} \oplus \{ \bar{\sigma}'_n \} \oplus \{ \tilde{\varepsilon}_n \} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \\
& \text{for } n=12, 13. \\
\pi_{38}^{14} &= \{ \delta_{14} \} \oplus \{ \bar{\mu}_{14} \circ \sigma_{31} \} \oplus \{ \bar{\sigma}'_{14} \} \oplus \{ \tilde{\varepsilon}_{14} \} \oplus \{ \zeta^* \} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_8. \\
\pi_{39}^{15} &= \{ \delta_{15} \} \oplus \{ \bar{\mu}_{15} \circ \sigma_{32} \} \oplus \{ \bar{\sigma}'_{15} \} \oplus \{ \tilde{\varepsilon}_{15} \} \oplus \{ E\zeta^* \} \oplus \{ \mu^{*'} \} \\
& \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \\
\pi_{40}^{16} &= \{ \delta_{16} \} \oplus \{ \bar{\mu}_{16} \circ \sigma_{33} \} \oplus \{ \bar{\sigma}'_{16} \} \oplus \{ \tilde{\varepsilon}_{16} \} \oplus \{ E^2 \zeta^* \} \oplus \{ E\mu^{*'} \} \oplus \{ \mu_{16}^* \} \\
& \oplus \{ \eta_{16}^* \circ \varepsilon_{32} \} \oplus \{ \eta_{16}^* \circ \bar{\nu}_{32} \} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \\
\pi_{41}^{17} &= \{ \delta_{17} \} \oplus \{ \bar{\mu}_{17} \circ \sigma_{34} \} \oplus \{ \bar{\sigma}'_{17} \} \oplus \{ \mu_{17}^* \} \oplus \{ \eta_{17}^* \circ \varepsilon_{33} \} \oplus \{ \eta_{17}^* \circ \bar{\nu}_{33} \} \\
& \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \\
\pi_{42}^{18} &= \{ \delta_{18} \} \oplus \{ \bar{\mu}_{18} \circ \sigma_{35} \} \oplus \{ \bar{\sigma}'_{18} \} \oplus \{ \Delta \sigma_{37} \} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_{16}. \\
\pi_{43}^{19} &= \{ \delta_{19} \} \oplus \{ \bar{\mu}_{19} \circ \sigma_{36} \} \oplus \{ \bar{\sigma}'_{19} \} \approx Z_2 \oplus Z_2 \oplus Z_2. \\
\pi_{n+24}^n &= \{ \delta_n \} \oplus \{ \bar{\mu}_n \circ \sigma_{n+17} \} \approx Z_2 \oplus Z_2 \quad \text{for } n=20, 21.
\end{aligned}$$

$$\pi_{46}^{22} = \{\delta_{22}\} \oplus \{\bar{\mu}_{22} \circ \sigma_{39}\} \oplus \{\Delta v_{45}\} \approx Z_2 \oplus Z_2 \oplus Z_4.$$

$$\pi_{47}^{23} = \{\delta_{23}\} \oplus \{\bar{\mu}_{23} \circ \sigma_{40}\} \oplus \{\tilde{\eta}'\} \approx Z_2 \oplus Z_2 \oplus Z_2.$$

$$\pi_{48}^{24} = \{\delta_{24}\} \oplus \{\bar{\mu}_{24} \circ \sigma_{41}\} \oplus \{E\tilde{\eta}'\} \oplus \{\tilde{\eta}\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

$$\pi_{49}^{25} = \{\delta_{25}\} \oplus \{\bar{\mu}_{25} \circ \sigma_{42}\} \oplus \{E\tilde{\eta}\} \approx Z_2 \oplus Z_2 \oplus Z_2.$$

$$\pi_{n+24}^n = \{\delta_n\} \oplus \{\bar{\mu}_n \circ \sigma_{n+17}\} \approx Z_2 \oplus Z_2 \quad \text{for } n \geq 26.$$

In the metastable range Theorem 1.1 overlaps with the work of Mahowald [Ma].

THEOREM 1.2.

$${}_2\pi_{23}^{\mathbb{S}} = \{\bar{\rho}\} \oplus \{v \circ \bar{\kappa}\} \oplus \{\eta^* \circ \sigma\} \approx Z_{16} \oplus Z_8 \oplus Z_2.$$

$${}_2\pi_{24}^{\mathbb{S}} = \{\bar{\mu} \circ \sigma\} \oplus \{\eta \circ \eta^* \circ \sigma\} \approx Z_2 \oplus Z_2.$$

REMARK 1.3. Although $\phi_5 \in \pi_{28}^{\mathbb{S}}$ and $\delta_3 \in \pi_{27}^{\mathbb{S}}$ are indecomposable, ϕ and δ are decomposable in the stable range (or after sufficient suspensions). According to Mahowald and Tangora [MaTa], the generators of ${}_2\pi_{23}^{\mathbb{S}}$ are ρ_3 , $v \circ \bar{\kappa}$, and $\langle \sigma^2, 2\iota, \varepsilon \rangle$ where ρ_3 generates $\text{Im } J$ in dimension 23 and coincides with our element $\bar{\rho}$, and $\langle \sigma^2, 2\iota, \varepsilon \rangle$ realizes $h_4 c_0$ in E_∞ -term of the Adams spectral sequence. We assert that although $h_4 c_0$ is indecomposable in E_∞ -term, the corresponding homotopy element is decomposable: that is, $\eta^* \circ \sigma \in \langle \sigma^2, 2\iota, \varepsilon \rangle \pmod{4v \circ \bar{\kappa}}$.

This paper is divided into six sections. In §2, we give some preliminaries needed for calculation. In §3, we define many new elements which appear in the 23, 24, 25-stem. In §4, and §5 we devote ourselves to calculation of the 23 and 24-stem, respectively. This leads us to Theorem 1.1. In §6, we prove Theorem 1.2.

Throughout of this paper we assume familiarity with the notations of [T], [MII] and [MT] upon which we draw freely.

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§2. Some preliminaries.

First we need to prepare some relations in π_*^n to construct some new elements.

These relations are mainly used to estimate the indeterminacy of Toda brackets.

The following may be found in Propositions 2.13 and 2.17 of [O2]

$$(2.1) \quad \mu_3 \circ \bar{v}_{12} \equiv 0 \pmod{2\bar{e}'}, \quad \mu_n \circ \bar{v}_{n+9} = 0 \quad (n \geq 5).$$

$$v_6 \circ \rho' \equiv 0 \pmod{2\zeta_6 \circ \sigma_{17}},$$

$$\mu_3 \circ \varepsilon_{12} \equiv \eta_3 \circ \mu_4 \circ \sigma_{13} \pmod{2\bar{e}'},$$

$$(2.2) \quad \sigma''' \circ \kappa_{12} = 0,$$

$$v_7 \circ \bar{e}_{10} = 0,$$

$$v_5^2 \circ E^2 \rho' = 0.$$

Proof. We have $E(\sigma''' \circ \kappa_{12}) = E\sigma''' \circ \kappa_{13} = 2\sigma'' \circ \kappa_{13} = \sigma'' \circ 2\kappa_{13} = 0$ by Lemma 5.14 of [T] and (15.5) of [MT]. By Lemma 6.5 of [T] $\Delta(\sigma''' \circ \kappa_{12}) = \Delta(\sigma''') \circ \kappa_{10} = 0$. Hence $\sigma''' \circ \kappa_{12} = 0$, since $\text{Ker } E \cap \text{Ker } \Delta$ is zero in π_{26}^5 .

$$v_7 \circ \bar{e}_{10} \in v_7 \circ \{\varepsilon_{10}, 2\iota_{18}, v_{18}^2\}_1$$

$$\subset \{v_7 \circ \varepsilon_{10}, 2\iota_{18}, v_{18}^2\}_1$$

$$= \pi_{19}^7 \circ v_{19}^2 = 0$$

by (7.18) of [T]. $v_5^2 \circ E^2 \rho' \equiv 0 \pmod{2v_5 \circ \zeta_8 \circ \sigma_{17}} = 0$ follows from (2.1). q.e.d.

$$(2.3) \quad \eta_9 \circ \mu_{10} \circ \varepsilon_{19} = 0,$$

$$\zeta_{11} \circ \sigma_{22} = 0,$$

$$\sigma_{10}^2 \circ \eta_{24} = 0,$$

$$\sigma_{10}^2 \circ \mu_{24} = 0.$$

Proof.

$$\eta_9 \circ \mu_{10} \circ \varepsilon_{19} = \eta_9 \circ \mu_{10} \circ (\bar{v}_{19} + \eta_{19} \circ \sigma_{20}) \quad (\text{by Lemma 6.4 of [T]})$$

$$= \eta_9^2 \circ \mu_{11} \circ \sigma_{20} \quad (\text{by (2.1)})$$

$$= 4\zeta_9 \circ \sigma_{20} \quad (\text{by (7.14) of [T]})$$

$$= 8\sigma_9 \circ \zeta_{16} \quad (\text{by Lemma 12.12 of [T]})$$

$$= 0.$$

$$\begin{aligned}
 \zeta_{11} \circ \sigma_{22} &= 2x\sigma_{11} \circ \zeta_{18} && \text{(by Lemma 12.12 of [T])} \\
 &= 0. && \text{(by (12.23) of [T])} \\
 \sigma_{10}^2 \circ \eta_{24} &= \sigma_{10} \circ (\bar{v}_{17} + \varepsilon_{17}) && \text{(by Lemma 6.4 of [T])} \\
 &= \sigma_{10} \circ \bar{v}_{17} + \sigma_{10} \circ \varepsilon_{17} \\
 &= 0. && \text{(by (10.18) of [T])} \\
 \sigma_{10}^2 \circ \mu_{24} &\in \sigma_{10}^2 \circ \{\eta_{24}, 2\varepsilon_{25}, 8\sigma_{25}\}_1 && \text{(by Lemma 6.5 of [T])} \\
 &\subset \{\sigma_{10}^2 \circ \eta_{24}, 2\varepsilon_{25}, 8\sigma_{25}\}_1 \\
 &= \pi_{26}^{10} \circ 8\sigma_{26} \\
 &= 0. && \text{q.e.d.}
 \end{aligned}$$

$$\begin{aligned}
 (2.4) \quad v_7 \circ \sigma_{10} \circ \mu_{17} &= 0, \\
 \bar{e}_3 \circ \sigma_{18} &= 0, \\
 \rho^{IV} \circ \sigma_{20} &= v_5 \circ \bar{\zeta}_8.
 \end{aligned}$$

Proof.

$$\begin{aligned}
 v_7 \circ \sigma_{10} \circ \mu_{17} &= x\sigma' \circ v_{14} \circ \mu_{17} && \text{(by (7.19) of [T])} \\
 &= 0. && \text{(by (7.25) of [T])}
 \end{aligned}$$

$$\begin{aligned}
 H(\bar{e}_3 \circ \sigma_{18}) &= H(\bar{e}_3) \circ \sigma_{18} \\
 &\equiv 0 \pmod{v_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}}. && \text{(by (7.20) and Lemma 10.2 of [T])}
 \end{aligned}$$

If $H(\bar{e}_3 \circ \sigma_{18}) = v_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}$, applying Δ we have $0 = \Delta H(\bar{e}_3 \circ \sigma_{18}) = \Delta(v_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}) = \eta_2 \circ v' \circ \eta_6 \circ \mu_7 \circ \sigma_{16}$ by Lemma 5.7 of [T]. This contradicts to a fact that $\eta_2 \circ v' \circ \eta_6 \circ \mu_7 \circ \sigma_{16}$ is a generator of π_{23}^2 . So $H(\bar{e}_3 \circ \sigma_{18}) = 0$, whence $\bar{e}_3 \circ \sigma_{18} = 0$ since $H: \pi_{25}^3 \rightarrow \pi_{25}^5$ is monic.

$$\begin{aligned}
 E(\rho^{IV} \circ \sigma_{20}) &= E\rho^{IV} \circ \sigma_{21} \\
 &= 2\rho''' \circ \sigma_{21} && \text{(by (10.15) of [T])} \\
 &= E(v_5 \circ \bar{\zeta}_8). && \text{(by (8.1) of [MII])}
 \end{aligned}$$

Since $E: \pi_{27}^5 \rightarrow \pi_{28}^6$ is monic, we have $\rho^{IV} \circ \sigma_{20} = v_5 \circ \bar{\zeta}_8$. q.e.d.

$$(2.5) \quad \begin{aligned} \mu_3 \circ \zeta_{12} &= v' \circ \eta_6 \circ \mu_7 \circ \sigma_{16}, \\ \varepsilon_7 \circ \zeta_{15} &= \bar{v}_7 \circ \zeta_{15} = 0. \end{aligned}$$

Proof.

$$\begin{aligned} H(\mu_3 \circ \zeta_{12}) &= H(\mu_3) \circ \zeta_{12} \\ &= \sigma'' \circ \zeta_{12} && \text{(by Lemma 6.5 of [T])} \\ &= 4\zeta_5 \circ \sigma_{16} && \text{(by Proposition (2.17) of [02])} \\ &= \eta_5^2 \circ \mu_7 \circ \sigma_{16} && \text{(by (7.14) of [T])} \\ &= H(v' \circ \eta_6 \circ \mu_7 \circ \sigma_{16}). && \text{(by (5.3) of [T])} \end{aligned}$$

Since $H: \pi_{2,3}^3 \rightarrow \pi_{2,3}^5$ is monic, we have $\mu_3 \circ \zeta_{12} = v' \circ \eta_6 \circ \mu_7 \circ \sigma_{16}$.

$$\begin{aligned} \varepsilon_7 \circ \zeta_{15} &\in \varepsilon_7 \circ \{v_{15}, 8\iota_{18}, 2\sigma_{18}\}_1 \\ &\subset \{\varepsilon_7 \circ v_{15}, 8\iota_{18}, 2\sigma_{18}\}_1 \\ &= \pi_{1,9}^7 \circ 2\sigma_{19} && \text{(by (7.13) of [T])} \\ &= 0. \end{aligned}$$

Similarly we have $\bar{v}_{10} \circ \zeta_{18} = 0$. Since $E^3: \pi_{2,6}^7 \rightarrow \pi_{2,9}^1$ is monic, $\bar{v}_7 \circ \zeta_{15} = 0$. q.e.d.

$$(2.6) \quad \theta' \circ \sigma_{23} = \xi' \circ \eta_{29}.$$

Proof.

$$\begin{aligned} H(\theta' \circ \sigma_{23}) &= H(\theta') \circ \sigma_{23} \\ &= \eta_{2,1}^2 \circ \sigma_{23} && \text{(by Lemma 7.5 of [T])} \\ &= v_{2,1}^3 + \eta_{2,1} \circ \varepsilon_{22} \\ &= H(\xi' \circ \eta_{29}). && \text{(by p. 165 of [T])} \end{aligned}$$

Hence $\theta' \circ \sigma_{23} - \xi' \circ \eta_{29} \in E\pi_{2,9}^1 = \{\bar{\zeta}_{11}\} \oplus \{\bar{\sigma}_{11}\}$. For the monomorphism $E^2: E\pi_{2,9}^1 \rightarrow \pi_{3,2}^1$, we have

$$E^2(\theta' \circ \sigma_{23} - \xi' \circ \eta_{29}) = E^2\theta' \circ \sigma_{25} - E^2\xi' \circ \eta_{31} = 0$$

by p. 80 and Lemma 12.19 of [T]. Thus $\theta' \circ \sigma_{23} - \xi' \circ \eta_{29} = 0$.

q.e.d.

$$(2.7) \quad \eta_6 \circ \bar{\sigma}_7 = \eta_5 \circ \zeta_6 = \eta_5 \circ \bar{\zeta}_6 = 0.$$

Proof.

$$\begin{aligned}
 \eta_6 \circ \bar{\sigma}_7 &\in \eta_6 \circ \{v_7, \varepsilon_{10} + \bar{v}_{10}, \sigma_{18}\}_1 \\
 &\subset \{\eta_6 \circ v_7, \varepsilon_{10} + \bar{v}_{10}, \sigma_{18}\}_1 \\
 &= \pi_{19}^6 \circ \sigma_{19} \\
 &= 0. \quad \text{(by (5.9), (7.20) of [T])}
 \end{aligned}$$

Similarly, $\eta_5 \circ \zeta_6 = \eta_5 \circ \bar{\zeta}_6 = 0$. q.e.d.

$$(2.8) \quad \bar{\mu}_5 \circ v_{22} = 0,$$

$$x\sigma_{12}^2 \in \{v_{12}, \sigma_{15}, v_{22}\}_1 \text{ for some odd integer } x.$$

Proof. Recall $\bar{\mu}_3 \circ v_{20} \in \pi_{23}^3 = \{v' \circ \bar{\mu}_6\} \oplus \{v' \circ \eta_6 \circ \mu_7 \circ \sigma_{16}\}$. Then $\bar{\mu}_5 \circ v_{22} \in E^2 \pi_{23}^3 = 0$.

Since $\{v_{12}, \sigma_{15}, v_{22}\}_1 \subset \pi_{26}^2 = \{\sigma_{12}^2, \kappa_{12}, \Delta(\iota_{25}) \circ v_{23}\}$, $\{v_{12}, \sigma_{15}, v_{22}\}_1$ contains $x\sigma_{12}^2 + y\kappa_{12}$ for some integers x, y . Then $\{v_{12}, \sigma_{15}, v_{22}\}_1 \circ \eta_{26} \subset \{v_{12}, \sigma_{15}, v_{22} \circ \eta_{25}\}_1 = v_{12} \circ E\pi_{26}^4 = 0$ by (5.9) of [T]. Hence $x\sigma_{12}^2 \circ \eta_{26} + y\kappa_{12} \circ \eta_{26} = 0$. But $\sigma_{12}^2 \circ \eta_{26} = \sigma_{12} \circ (\varepsilon_{19} + \bar{v}_{19}) = 0$ by Lemma 6.4 and (10.8) of [T] and $\kappa_{12} \circ \eta_{26} = \bar{\varepsilon}_{12}$ is non zero and of order 2. Hence we have $y \equiv 0 \pmod{2}$ and $x\sigma_{12}^2 \in \{v_{12}, \sigma_{15}, v_{22}\}_1$. Since $x\sigma_{12}^2$ is not divisible by 2 by Example 4 in p. 85 of [T], x must be an odd integer. q.e.d.

$$(2.9) \quad \mu_3 \circ \sigma_{12}^2 = 0.$$

Proof.

$$\begin{aligned}
 \mu_3 \circ \sigma_{12}^2 &\in \mu_3 \circ \{v_{12}, \sigma_{15}, v_{22}\}_1 \quad \text{(by (2.8))} \\
 &\subset \{\mu_3 \circ v_{12}, \sigma_{15}, v_{22}\}_1 \\
 &= \{v' \circ \eta_6 \circ \varepsilon_7, \sigma_{15}, v_{22}\}_1 \quad \text{(by Proposition 2.2 of [O2])} \\
 &\subset \{v' \circ \eta_6, \varepsilon_7 \circ \sigma_{15}, v_{22}\}_1 \\
 &= v' \circ \eta_6 \circ E\pi_{25}^6 + \pi_{23}^3 \circ v_{23} \quad \text{(by Lemma 10.7 of [T])} \\
 &= 0. \quad \text{(by (2.7), (2.8))} \quad \text{q.e.d.}
 \end{aligned}$$

§ 3. Some new elements.

In our calculation there appear many new elements which are not represented

by composition, suspension nor Whitehead product. In this section all of these elements are constructed by Toda brackets but a few elements that are defined during the course of calculation made in §4 and §5.

(3.1) *There exists an element $\bar{\alpha}$ of π_{26}^3 such that $2\bar{\alpha} = \eta_3 \circ \varepsilon_4 \circ \kappa_{12}$ and $H(\bar{\alpha}) = \alpha$.*

Proof. Choose $\bar{\alpha} \in \{\varepsilon_3, 2\iota_{11}, \kappa_{11}\}_1$, then

$$\begin{aligned} H(\bar{\alpha}) &\in H\{\varepsilon_3, 2\iota_{11}, \kappa_{11}\}_1 \\ &\subset \{H(\varepsilon_3), 2\iota_{11}, \kappa_{11}\}_1 \\ &= \{v_5^2, 2\iota_{11}, \kappa_{11}\}_1 \quad (\text{by Lemma 6.1 of [T]}) \\ &\equiv \alpha \end{aligned}$$

mod $v_5^2 \circ E\pi_{25}^1 + \pi_{12}^5 \circ \kappa_{12} = 0$ by (2.2), and

$$\begin{aligned} 2E\bar{\alpha} &\in 2\iota_4 \circ E\{\varepsilon_3, 2\iota_{11}, \kappa_{11}\} \\ &= \{2\iota_4, \varepsilon_4, 2\iota_{12}\}_1 \circ \kappa_{13} \\ &\equiv \varepsilon_4 \circ \eta_{12} \circ \kappa_{13} \quad (\text{by Corollary 3.7 of [T]}) \\ &= \eta_4 \circ \varepsilon_5 \circ \kappa_{13} \quad (\text{by (7.5) of [T]}) \end{aligned}$$

mod $2\iota_4 \circ E(\varepsilon_3 \circ \pi_{26}^{S^{11}}) + \pi_{12}^3 \circ \kappa_{12} = 0$. Since $E: \pi_{26}^3 \rightarrow \pi_{27}^4$ is monic, it follows that $2\bar{\alpha} = \eta_3 \circ \varepsilon_4 \circ \kappa_{12}$. q.e.d.

(3.2) *There exist elements $\bar{\rho}''' \in \pi_{28}^5$, $\bar{\rho}'' \in \pi_{29}^6$, $\bar{\rho}' \in \pi_{30}^7$ and $\bar{\rho}_9 \in \pi_{32}^9$ such that*

$$\begin{aligned} H(\bar{\rho}''') &= 4\bar{\iota}_9 = \eta_9^2 \circ \bar{\mu}_{11}, \\ H(\bar{\rho}'') &= \eta_{11} \circ \bar{\mu}_{12}, \\ H(\bar{\rho}') &\equiv \bar{\mu}_{13} \quad \text{mod } \sigma_{13} \circ \eta_{20} \circ \mu_{21}, \\ H(\bar{\rho}_9) &= 16\rho_{17}, \\ 2\bar{\rho}''' &= 0, \quad E(2\bar{\rho}'') = E^2\bar{\rho}''', \\ 2\bar{\rho}' &= E\bar{\rho}'' \quad \text{and} \quad E^2(2\bar{\rho}_9) = E^4\bar{\rho}'. \end{aligned}$$

Proof. Choose elements from the following secondary compositions.

$$\bar{\rho}''' \in \{\rho^{IV}, 2\iota_{20}, 8\sigma_{20}\}_1,$$

$$\begin{aligned}\bar{\rho}'' &\in \{-\rho''', 4\iota_{21}, 4\sigma_{21}\}_1, \\ \bar{\rho}' &\in \{\rho'', 8\iota_{22}, 2\sigma_{22}\}_1, \\ \bar{\rho}_9 &\in \{\rho', 16\iota_{24}, \sigma_{24}\}_1.\end{aligned}$$

Then we have

$$\begin{aligned}H(\bar{\rho}''') &\in H\{\rho^{IV}, 2\iota_{20}, 8\sigma_{20}\}_1 \\ &\subset \{H(\rho^{IV}), 2\iota_{20}, 8\sigma_{20}\}_1 \\ &= \{\eta_9^2 \circ \mu_{11}, 2\iota_{20}, 8\sigma_{20}\}_1 \\ &\supset \eta_9^2 \circ \{\mu_{11}, 2\iota_{20}, 8\sigma_{20}\}_1 \\ &\ni \eta_9^2 \circ \bar{\mu}_{11} \quad \text{mod } \eta_9^2 \circ \mu_{11} \circ E\pi_{27}^{19} + \pi_{21}^9 \circ 8\sigma_{21} = 0.\end{aligned}$$

Also

$$\begin{aligned}2\bar{\rho}''' &\in \{\rho^{IV}, 2\iota_{20}, 8\sigma_{20}\}_1 \circ 2\iota_{28} \\ &= \rho^{IV} \circ E\{2\iota_{19}, 8\sigma_{19}, 2\iota_{26}\} \\ &\subset \rho^{IV} \circ \{2\iota_{20}, 8\sigma_{20}, 2\iota_{27}\}_1 \\ &\ni \rho^{IV} \circ 8\sigma_{20} \circ \eta_{27} = 0 \quad (\text{by Corollary 3.7 of [T]}) \\ &\text{mod } \rho^{IV} \circ \{2\iota_{20} \circ E\pi_{27}^{19} + \pi_{28}^2 \circ 2\iota_{28}\} = 0.\end{aligned}$$

Similarly

$$\begin{aligned}H(\bar{\rho}'') &\in H\{-\rho''', 4\iota_{21}, 4\sigma_{21}\}_1 \\ &\subset \{H(-\rho'''), 4\iota_{21}, 4\sigma_{21}\}_1 \\ &= \{\eta_{11} \circ \mu_{12}, 4\iota_{21}, 4\sigma_{21}\}_1 \quad (\text{by (10.12) of [T]}) \\ &\supset \{\eta_{11} \circ \mu_{12}, 2\iota_{21}, 8\sigma_{21}\}_1 \\ &\supset \eta_{11} \circ \{\mu_{12}, 2\iota_{21}, 8\sigma_{21}\}_1 \ni \eta_{11} \circ \bar{\mu}_{12} \\ &\text{mod } \eta_{11} \circ \mu_{12} \circ E\pi_{28}^2 + \pi_{22}^{11} \circ 4\sigma_{22} = 0 \quad \text{by (2.1) and (2.3)}.\end{aligned}$$

Similarly

$$E(2\bar{\rho}'') \in 2\iota_7 \circ E\{-\rho''', 4\iota_{21}, 4\sigma_{21}\}_1$$

$$\begin{aligned}
&\subset E\{2\epsilon_6 \circ \rho''', 4\epsilon_{21}, 4\sigma_{21}\}_1 \\
&= E\{E\rho^{IV}, 4\epsilon_{21}, 4\sigma_{21}\}_1 \quad (\text{by (10.15) of [T]}) \\
&\equiv -E^2\bar{\rho}''' \\
&= E^2\bar{\rho}'''
\end{aligned}$$

and $E(2\epsilon_6 \circ \rho''' \circ E\pi_{28}^2 \circ \pi_{22}^6 \circ 4\sigma_{22}) = 0$.

Other relations are similarly obtained.

q.e.d.

$$\begin{aligned}
(3.3) \quad H(\Delta\lambda) &\equiv \pm\lambda' \pmod{2\lambda', 2\xi', \eta_{11} \circ \bar{\mu}_{12}}, \\
H(\Delta\xi_{13}) &\equiv \pm\xi' \pmod{2\lambda', 2\xi', \eta_{11} \circ \bar{\mu}_{12}}.
\end{aligned}$$

Proof. Apply Corollary 5.10 of [MI] to the case $\alpha = \Delta\epsilon_{13}$, $\beta = \lambda'$, $n = 5$, $p = 10$, $i = 28$. Then

$$\begin{aligned}
H\Delta(2\lambda) &= H\Delta(E^2\lambda') \\
&= H(\Delta(\epsilon_{13}) \circ \lambda') \\
&= H\Delta(\epsilon_{13}) \circ \lambda' + (E^5\Delta\epsilon_{13}) \circ (E^{10}\Delta\epsilon_{13}) \circ H(\lambda') \\
&= 2\lambda'.
\end{aligned}$$

Note that $\pi_{29}(S^{11}) = \pi_{29}^1$. The indeterminacy is

$$\sum_{k=3}^6 f_k^* \pi_{29}^{5k+1} + \text{Ker}(E: \pi_{29}^1 \longrightarrow \pi_{30}^2) \subset \{4\lambda', 4\xi'\},$$

where $f_k: S^{5k+1} \rightarrow S^{11}$. Thus we have

$$H(\Delta\lambda) \equiv \lambda' \pmod{2\lambda', 2\xi', \eta_{11} \circ \bar{\mu}_{12}}.$$

Since we can replace λ, λ' by ξ_{13}, ξ' , we have the second statement. q.e.d.

(3.4) *There exist elements $\bar{e}^*, \bar{v}^*$ of π_{38}^1 and $\bar{e}_{16}^*, \bar{v}_{16}^*$ of π_{39}^1 such that*

$$\begin{aligned}
H(\bar{e}^*) &= \eta_{29} \circ \epsilon_{30}, & H(\bar{v}^*) &= v_{29}^3, \\
H(\bar{e}_{16}^*) &= \epsilon_{31}, & H(\bar{v}_{16}^*) &= \bar{v}_{31}, \\
\Delta(\epsilon_{33}) &= E\bar{e}^*, & \Delta(\bar{v}_{33}) &= E\bar{v}^*.
\end{aligned}$$

Moreover $\bar{e}_{16}^*$ and $\bar{v}_{16}^*$ are of order 2.

Proof. Choose $\bar{e}_{16}^* \in \{\sigma_{16}^2, 2\epsilon_{30}, \epsilon_{30}\}_1$, then

$$\begin{aligned} H(\bar{e}_{16}^*) &\in H\{\sigma_{16}^2, 2\epsilon_{30}, \epsilon_{30}\}_1 \\ &= -\Delta^{-1}(2\sigma_{15}^2) \circ \epsilon_{31} \\ &= \epsilon_{31}. \end{aligned} \quad (\text{by (10.10) of [T]})$$

We have

$$\begin{aligned} 2\bar{e}_{16}^* &= \bar{e}_{16}^* \circ 2\epsilon_{39} \in \{\sigma_{16}^2, 2\epsilon_{30}, \epsilon_{30}\}_1 \circ 2\epsilon_{39} \\ &= \sigma_{16}^2 \circ E\{2\epsilon_{29}, \epsilon_{29}, 2\epsilon_{37}\} \\ &\subset \sigma_{16}^2 \circ \{2\epsilon_{30}, \epsilon_{30}, 2\epsilon_{38}\}_1 \\ &\ni \sigma_{16}^2 \circ \epsilon_{30} \circ \eta_{38} = 0 \end{aligned} \quad \begin{aligned} &(\text{by Corollary 3.7 and} \\ &\text{Lemma 10.7 of [T]}) \end{aligned}$$

$$\text{mod } \sigma_{16}^2 \circ \{2\epsilon_{30} \circ \pi_{39}(S^{30}) + \pi_{39}(S^{30}) \circ 2\epsilon_{39}\} = 0.$$

Define $\bar{v}_{16}^* = \bar{e}_{16}^* + \eta_{16}^* \circ \sigma_{32}$. Then

$$\begin{aligned} H(\bar{v}_{16}^*) &= H(\bar{e}_{16}^*) + H(\eta_{16}^*) \circ \sigma_{32} \\ &= \epsilon_{31} + \eta_{31} \circ \sigma_{32} = \bar{v}_{31} \end{aligned} \quad (\text{by Lemmas 12.14, 6.4 of [T]}),$$

$$\text{and } 2\bar{v}_{16}^* = 2\bar{e}_{16}^* + 2\eta_{16}^* \circ \sigma_{32} = 0.$$

By Proposition 11.6 of [T] we have the rest of the statement. q.e.d.

(3.5) *There exists an element δ_3 of π_{27}^3 such that $H(\delta_3) \equiv v_5 \circ \bar{\sigma}_8 \text{ mod } v_5 \circ \bar{\zeta}_8$.*

Proof. Choose $\delta_3 \in \{\epsilon_3, \epsilon_{11} + \bar{v}_{11}, \sigma_{19}\}_1$, then

$$\begin{aligned} H(\delta_3) &\in H\{\epsilon_3, \epsilon_{11} + \bar{v}_{11}, \sigma_{19}\}_1 \\ &\subset \{H(\epsilon_3), \epsilon_{11} + \bar{v}_{11}, \sigma_{19}\}_1 \\ &= \{v_5^2, \epsilon_{11} + \bar{v}_{11}, \sigma_{19}\}_1 \quad (\text{by Lemma 6.1 of [T]}) \\ &\supset v_5 \circ \{v_8, \epsilon_{11} + \bar{v}_{11}, \sigma_{19}\}_1 \\ &\ni v_5 \circ \bar{\sigma}_8 \end{aligned}$$

$$\text{mod } v_5^2 \circ E\pi_{26}^1 + \pi_{20}^2 \circ \sigma_{20} = \{v_5 \circ \bar{\zeta}_8\} \text{ by (2.4).} \quad \text{q.e.d.}$$

(3.6) Put $\beta'' = \{2\sigma_{11}, v_{18}, \mu_{21}\}_1 \in \pi_{31}^{11}$, then we have $H(\beta'') = \eta_{21} \circ \mu_{22}$, $\Delta(\mu_{25}) = E\beta''$ and $2\beta'' = 0$. So we can define β'' by this Toda bracket instead of the definition given by Lemma 16.2 of [MT].

Proof. Since $\{2\sigma_{11}, v_{18}, \mu_{21}\}_1$ is a coset of $2\sigma_{11} \circ E\pi_{30}^{17} + \pi_{22}^{12} \circ \mu_{22} = 0$ by (2.5), this is a single element. Then we have

$$\begin{aligned} H(\beta'') &= H\{2\sigma_{11}, v_{18}, \mu_{21}\} = -\Delta^{-1}(2\sigma_{10} \circ v_{17}) \circ \mu_{22} \\ &= \eta_{21} \circ \mu_{22} \end{aligned} \quad (\text{by (7.21) of [T]})$$

and

$$\begin{aligned} E\beta'' &= E\{2\sigma_{11}, v_{18}, \mu_{21}\}_1 \subset \{2\sigma_{12}, v_{19}, \mu_{22}\}_1 \\ &\subset \{\sigma_{12}, 2v_{19}, \mu_{22}\}_1 \supset \{\sigma_{21}, v_{19}, 2\mu_{22}\}_1 \equiv 0 \end{aligned}$$

mod $\sigma_{12} \circ E\pi_{31}^{18} + \pi_{23}^{12} \circ \mu_{23} = \{\Delta\mu_{25}\} \approx Z_2$. But $E\beta'' \neq 0$, since $E: \pi_{32}^{11} \rightarrow \pi_{31}^{12}$ is monic.

Hence we have $E\beta'' = \Delta\mu_{25}$. Finally

$$\begin{aligned} 2\beta'' &= \beta'' \circ 2\epsilon_{31} \in \{2\sigma_{11}, v_{18}, \mu_{21}\} \circ 2\epsilon_{31} \\ &\subset \{2\sigma_{11}, v_{18}, 2\mu_{21}\} = 2\sigma_{11} \circ E\pi_{30}^{17} = 0. \end{aligned} \quad \text{q.e.d.}$$

The following element is in the 25-stem.

(3.7) There exists an element $\tilde{\beta}''$ of π_{31}^6 such that

$$H(\tilde{\beta}'') = \beta'' \quad \text{and} \quad 2\tilde{\beta}'' = 0.$$

Proof. Choose $\tilde{\beta}''$ from $\{\Delta\sigma_{13}, v_{18}, \mu_{21}\}_1$, then

$$\begin{aligned} H(\tilde{\beta}'') &\in H\{\Delta\sigma_{13}, v_{18}, \mu_{21}\}_1 \\ &\subset \{H\Delta\sigma_{13}, v_{18}, \mu_{21}\}_1 \\ &= \{2\sigma_{11}, v_{18}, \mu_{21}\}_1 \quad (\text{by Proposition 2.7 of [T]}) \\ &= \beta''. \quad (\text{by (3.6)}) \end{aligned}$$

$$\begin{aligned} 2\tilde{\beta}'' &= \tilde{\beta}'' \circ 2\epsilon_{31} \in \{\Delta\sigma_{31}, v_{18}, \mu_{21}\} \circ 2\epsilon_{31} \\ &\subset \{\Delta\sigma_{31}, v_{18}, 2\mu_{21}\}_1 = \Delta\sigma_{31} \circ E\pi_{30}^{17} = 0. \end{aligned} \quad \text{q.e.d.}$$

Remark. (3.7) will be used to show

$$\Delta\beta'' = \Delta H(\beta'') = 0.$$

(3.8) *There exist elements $\bar{\sigma}'_6, \bar{\sigma}_6, \bar{\zeta}'_6, \bar{\zeta}_6$ of π_{30} such that $2\bar{\sigma}'_6 = 8\bar{\zeta}'_6 = 8\bar{\zeta}_6 = 0$,*

$$H(\bar{\sigma}'_6) = \bar{\sigma}_{11},$$

$$H(\bar{\sigma}_6) \equiv \bar{\sigma}_{11} \pmod{\zeta' \circ \eta_{29}},$$

$$H(\bar{\zeta}'_6) \equiv \bar{\zeta}_{11} \pmod{2\bar{\zeta}_{11}},$$

$$H(\bar{\zeta}_6) \equiv \bar{\zeta}_{11} \pmod{2\bar{\zeta}_{11}},$$

$$\Delta(\bar{\zeta}_{13}) \equiv 2\bar{\zeta}'_6 \pmod{4\bar{\zeta}'_6}.$$

Proof. Choose elements from the following secondary compositions

$$\bar{\sigma}'_6 \in \{v_6, \eta_9, \bar{\sigma}_{10}\}_3,$$

$$\bar{\sigma}_6 \in \{\bar{v}_6, e_{14} + \bar{v}_{14}, \sigma_{22}\}_1,$$

$$\bar{\zeta}'_6 \in \{v_6, \eta_9, \bar{\zeta}_{10}\}_3,$$

$$\bar{\zeta}_6 \in \{\zeta', 8\epsilon_{22}, 2\sigma_{22}\}_1.$$

Then we have

$$H(\bar{\sigma}'_6) \in H\{v_6, \eta_9, \bar{\sigma}_{10}\}_1$$

$$= -\Delta^{-1}(v_5 \circ \eta_8) \circ \bar{\sigma}_{11} = \bar{\sigma}_{11}. \quad (\text{by (5.10) of [T]})$$

$$2\bar{\sigma}'_6 = \bar{\sigma}'_6 \circ 2\epsilon_{30} \in \{v_6, \eta_9, \bar{\sigma}_{10}\}_3 \circ 2\epsilon_{30}$$

$$\subset \{v_6, \eta_9, 2\bar{\sigma}_{10}\}_3 = v_6 \circ E^3 \pi_{27}^6 = 0.$$

$$H(\bar{\zeta}'_6) \in H\{\zeta', 8\epsilon_{22}, 2\sigma_{22}\}_1$$

$$\subset \{H(\zeta'), 8\epsilon_{22}, 2\sigma_{22}\}_1$$

$$= \{x\zeta_{11}, 8\epsilon_{22}, 2\sigma_{22}\}_1 \quad (\text{by Lemma 12.1 of [T]})$$

$$\ni x\bar{\zeta}_{11} \quad \text{for some odd integer } x$$

$$\pmod{x\zeta_{11} \circ E\pi_{29}^2 + \pi_{23}^1 \circ 2\sigma_{23}} = 0. \quad (\text{by (2.5)})$$

$$8\bar{\zeta}'_6 = \bar{\zeta}'_6 \circ 8\epsilon_{30} \in \{\zeta', 8\epsilon_{22}, 2\sigma_{22}\}_1 \circ 8\epsilon_{30}$$

$$\begin{aligned}
&= \zeta' \circ E\{8\epsilon_{21}, 2\sigma_{21}, 8\epsilon_{28}\} \\
&\subset \zeta' \circ \{8\epsilon_{22}, 2\sigma_{22}, 8\epsilon_{29}\}_1 \ni 0 && \text{(by Corollary 3.7 of [T])} \\
&\text{mod } \zeta' \circ 8\epsilon_{22} \circ \pi_{30}(S^{2,2}) = 0. \\
&H(\bar{\sigma}'_6) \in H\{\bar{v}_6, \epsilon_{14} + \bar{v}_{14}, \sigma_{22}\}_1 \\
&\subset \{H(\bar{v}_6), \epsilon_{14} + \bar{v}_{14}, \sigma_{22}\}_1 \\
&= \{xv_{11}, \epsilon_{14} + \bar{v}_{14}, \sigma_{22}\}_1 && \text{(by Lemma 6.2 of [T])} \\
&\ni x\bar{\sigma}_{11} = \bar{\sigma}_{11} \quad \text{for some odd integer } x \\
&\text{mod } xv_{11} \circ E\pi_{29}^{1,3} + \pi_{23}^{1,1} \circ \sigma_{23} = \{\zeta' \circ \eta_{29}\}. && \text{(by (2.6))} \\
&H(\bar{\zeta}'_6) \in H\{v_6, \eta_9, \bar{\zeta}_{10}\}_1 \\
&= -\Delta^1(v_5 \circ \eta_8) \circ \bar{\zeta}_{11} = \bar{\zeta}_{11} + \{2\bar{\zeta}_{11}\}. && \text{(by (5.10) of [T])} \\
&8\bar{\zeta}'_6 = \bar{\zeta}'_6 \circ 8\epsilon_{39} \in \{v_6, \eta_9, \bar{\zeta}_{10}\}_3 \circ 8\epsilon_{30} \\
&\subset \{v_6, \eta_9, 8\bar{\zeta}_{10}\}_3 = v_6 \circ E^3\pi_{27}^6 = 0. \\
&\Delta(\bar{\zeta}_{13}) = \Delta(\epsilon_{13}) \circ \bar{\zeta}_{11} \\
&\in \{v_6, \eta_9, 2\epsilon_{10}\}_3 \circ \bar{\zeta}_{11} && \text{(See Lemma 5.10 of [T])} \\
&\subset \{v_6, \eta_9, 2\bar{\zeta}_{10}\}_3 \subset \{v_6, \eta_9, \bar{\zeta}_{10}\} \circ 2\epsilon_{30} \\
&\ni \bar{\zeta}'_6 \circ 2\epsilon_{30} = 2\bar{\zeta}'_6 \\
&\text{mod } v_6 \circ E^3\pi_{27}^6 + \pi_{11}^6 \circ 2\bar{\zeta}_{11} = \{\Delta(2\bar{\zeta}_{13})\} \\
&\text{or } \Delta(\bar{\zeta}_{13}) \equiv 2\bar{\zeta}'_6 \pmod{4\bar{\zeta}'_6}. && \text{q.e.d.}
\end{aligned}$$

Remark The elements $\bar{\sigma}'_6$ and $\bar{\zeta}'_6$ will be used to estimate the order of $\bar{\sigma}'_6$ and $\bar{\zeta}'_6$. (e.g., $\Delta(\bar{\zeta}_{13}) = 2\bar{\zeta}'_6$ and also c.f. p. 32).

(3.9) *There exists an element $\bar{\epsilon}_{10}$ of $\pi_{34}^{1,0}$ such that*

$$H(\bar{\epsilon}_{10}) = \bar{\epsilon}_{19} \quad \text{and} \quad 2\bar{\epsilon}_{10} = \sigma_{10} \circ v_{17} \circ \kappa_{20}.$$

Proof. Choose $\bar{\epsilon}_{10}$ from $\{\sigma_{10}, \bar{\epsilon}_{17}, \eta_{32}\}_5$. Then

$$H(\bar{\epsilon}_{10}) \in H\{\sigma_{10}, \bar{\epsilon}_{17}, \eta_{32}\}_5$$

$$\begin{aligned}
 &= -\Delta^{-1}(\sigma_9 \circ \bar{e}_{16}) \circ \eta_{33} \\
 &= \kappa_{19} \circ \eta_{33} = \bar{e}_{19}. \quad (\text{by (8.4) of [MII] and (15.3) of [MT]})
 \end{aligned}$$

Also

$$\begin{aligned}
 2\bar{e}_{10} &= \bar{e}_{10} \circ 2\iota_{34} \in \{\sigma_{10}, \bar{e}_{17}, \eta_{32}\}_5 \circ 2\iota_{34} \\
 &= \sigma_{10} \circ E^5\{\bar{e}_{12}, \eta_{27}, 2\iota_{28}\} \\
 &= \sigma_{10} \circ E^5\{\eta_{12} \circ \kappa_{13}, \eta_{27}, 2\iota_{28}\} \quad (\text{by (15.3) of [MT]}) \\
 &\subset \sigma_{10} \circ E^5\{\eta_{12}, \kappa_{13} \circ \eta_{27}, 2\iota_{28}\} \\
 &\equiv \sigma_{10} \circ \nu_{17} \circ \kappa_{20} \quad (\text{by Lemma 15.1 of [MT]}) \\
 \text{mod } \sigma_{10} \circ \eta_{17} \circ \mu_{18} \circ \sigma_{27} &= 0. \quad (\text{by (2.1)}) \quad \text{q.e.d.}
 \end{aligned}$$

(3.10) *There exist elements ζ^* and $\zeta^{*'}$ of π_{38}^{14} such that $H(\zeta^*) \equiv \zeta_{27} \text{ mod } 2\pi_{38}(S^{27})$, $\Delta(\eta_{33} \circ \varepsilon_{34}) = E^2\zeta^*$ and $H(\zeta^{*'}) = \zeta_{27}$ and $8\zeta^{*'} = 0$.*

Proof. Apply Proposition 11.11, i) of [T] to the case $\alpha = \eta_{26} \circ \varepsilon_{27}$, $n = 13$, $i = 37$. Then there exists an element $\zeta^* \in \pi_{38}^{14}$ such that $\Delta(\eta_{33} \circ \varepsilon_{34}) = E^2\zeta^*$ and $H(\zeta^*) \in \{2\iota_{27}, \eta_{27}, \eta_{28} \circ \varepsilon_{29}\}_2$. It is easily checked that $\{2\iota_{27}, \eta_{27}, \eta_{28} \circ \varepsilon_{29}\}_2 \ni \zeta_{27} \text{ mod } 2\pi_{38}(S^{27})$. Choose an element $\zeta^{*'}$ from $\{\omega_{14}, 8\iota_{30}, 2\sigma_{30}\}_1$. Then

$$\begin{aligned}
 H(\zeta^{*'}) &\in H\{\omega_{14}, 8\iota_{30}, 2\sigma_{30}\}_1 \\
 &\subset \{H(\omega_{14}), 8\iota_{30}, 2\sigma_{30}\}_1 \\
 &= \{\nu_{27}, 8\iota_{30}, 2\sigma_{30}\}_1 \quad (\text{by Lemma 12.15 of [T]}) \\
 &\ni \zeta_{27} \quad \text{mod } \nu_{27} \circ E\pi_{37}^{29} + \pi_{31}^{27} \circ 2\sigma_{31} = 0.
 \end{aligned}$$

$$\begin{aligned}
 8\zeta^{*'} &= \zeta^{*'} \circ 8\iota_{38} \in \{\omega_{14}, 8\iota_{30}, 2\sigma_{30}\}_1 \circ 8\iota_{38} \\
 &= \omega_{14} \circ E\{8\iota_{29}, 2\sigma_{29}, 8\iota_{36}\} \\
 &\subset \omega_{14} \circ \{8\iota_{30}, 2\sigma_{30}, 8\iota_{37}\}_1 \\
 &\ni 0 \quad (\text{by Corollary 3.7 of [T]})
 \end{aligned}$$

$$\text{mod } \omega_{14} \circ 8\iota_{30} \circ \pi_{38}(S^{30}) = 0. \quad \text{q.e.d.}$$

(3.11) *There exist elements μ^* of π_{40}^{16} and $\mu^{*'}$ of π_{39}^{15} such that $H(\mu^*) = \mu_{31}$,*

$$2\mu^* = 0, \text{ and } H(\mu^*) = \eta_{29} \circ \mu_{30}, \Delta(\mu_{33}) = E\mu^*.$$

Proof. Choose an element μ^* from $\{\sigma_{16}^2, 2\epsilon_{30}, \mu_{30}\}_1$. Then

$$\begin{aligned} H(\mu^*) &\in H\{\sigma_{16}^2, 2\epsilon_{30}, \mu_{30}\}_1 \\ &= -\Delta^{-1}(2\sigma_{15}^2) \circ \mu_{31} \\ &= \mu_{31}. \end{aligned} \quad (\text{by (10.10) of [T]})$$

$$\begin{aligned} 2\mu^* &= \mu^* \circ 2\epsilon_{40} \in \{\sigma_{16}^2, 2\epsilon_{30}, \mu_{30}\}_1 \circ 2\epsilon_{40} \\ &= \sigma_{16}^2 \circ E\{2\epsilon_{29}, \mu_{29}, 2\epsilon_{38}\} \\ &\subset \sigma_{16}^2 \circ \{2\epsilon_{30}, \mu_{30}, 2\epsilon_{39}\}_1 \\ &\ni \sigma_{16}^2 \circ \mu_{30} \circ \eta_{30} = 0 \quad (\text{by Corollary 3.7 of [T]}) \\ \text{mod } \sigma_{16}^2 \circ 2\epsilon_{30} \circ \pi_{40}(S^{30}) &= 0. \end{aligned}$$

The other statement is proved by making use of Proposition 11.10, i) of [T].
q.e.d.

(3.12) *There exists an element $\hat{\epsilon}_{11}$ of π_{36}^{11} such that $H(\hat{\epsilon}_{11}) = \bar{\epsilon}_{21}$ and $2\hat{\epsilon}_{11} = 0$.*

Proof. Apply Proposition 11.10, i) of [T] to the case $\alpha = \kappa_{20}$, $n = 10$, $i = 35$. Then there exists $\hat{\epsilon}_{11} \in \pi_{36}^{11}$ such that $H(\hat{\epsilon}_{11}) = \bar{\epsilon}_{21}$. We can show that $\hat{\epsilon}_{11}$ is of order 2. But we omit a proof because it is not used in this paper. Actually the element $\hat{\epsilon}_{11}$ lies in the 25-stem.
q.e.d.

§ 4. Calculation of the 23-stem.

First we have by (5.2) of [T] and [MII]

$$(4.1) \quad \pi_{25}^2 = \{\eta_2 \circ \epsilon_3 \circ \kappa_{11}\} \approx Z_2.$$

Consider the exact sequence

$$\pi_{27}^5 \xrightarrow{\Delta} \pi_{25}^2 \xrightarrow{E} \pi_{26}^3 \xrightarrow{H} \pi_{26}^5 \xrightarrow{\Delta} \pi_{24}^2,$$

where $\pi_{26}^5 = \{\epsilon_5 \circ \kappa_{13}\} \oplus \{v_5 \circ \bar{\epsilon}_8\} \oplus \{v_5 \circ \bar{\sigma}_8\} \approx Z_2 \oplus Z_2 \oplus Z_2$ and $\pi_{26}^5 = \{\alpha\} \oplus \{v_5 \circ \eta_8 \circ \bar{\mu}_9\} \approx Z_2 \oplus Z_2$. By Lemma 5.2 of [T] $\Delta(\epsilon_5 \circ \kappa_{13}) = 0$. By Lemma 5.7 of [T] and the fact in p. 313 of [MII] $\Delta(v_5 \circ \bar{\epsilon}_8) = \Delta(v_5 \circ \bar{\sigma}_8) = \eta_2 \circ v' \circ \bar{\epsilon}_6 = 0$. Similarly we have $\Delta(v_5 \circ \bar{\sigma}_8) = 0$. Hence we have

$$(4.2) \quad \Delta\pi_{27}^5 = 0.$$

Now by Lemma 5.7 of [T] $\Delta(v_5 \circ \eta_8 \circ \bar{\mu}_9) = \Delta(v_5) \circ \eta_6 \circ \bar{\mu}_7 = \eta_2 \circ v' \circ \eta_6 \circ \bar{\mu}_7$. So (3.1), (4.1) and (4.2) yield

$$(4.3) \quad \pi_{26}^3 = \{\bar{\alpha}\} \approx Z_4.$$

By (5.6) of [T] and [MT] we have

$$(4.4) \quad \pi_{27}^4 = \{v_4 \circ \bar{\kappa}_7\} \oplus \{E\bar{\alpha}\} \approx Z_8 \oplus Z_4.$$

Next we show

$$(4.5) \quad \Delta(\bar{\kappa}_9) = \pm 2v_4 \circ \bar{\kappa}_7 \pm E\bar{\alpha}.$$

Proof. First by (7.5) and (7.6) of [T] and (3.1) we have

$$\begin{aligned} v' \circ \bar{\kappa}' &\in v' \circ \{v_6, \eta_9, \eta_{10} \circ \kappa_{11}\} = -\{v', v_6, \eta_9\} \circ \eta_{11} \circ \kappa_{12} \\ &= \varepsilon_3 \circ \eta_{11} \circ \kappa_{12} = \eta_3 \circ \varepsilon_4 \circ \kappa_{12} = 2\bar{\alpha}. \end{aligned}$$

Then

$$\begin{aligned} \Delta(2\bar{\kappa}_9) &= \pm \Delta(E^3 \bar{\kappa}') && \text{(by (16.7) of [MT])} \\ &= \pm \Delta(\iota_9) \circ E\bar{\kappa}' \\ &= \pm (2v_4 - Ev') \circ E\bar{\kappa}' && \text{(by (5.8) of [T])} \\ &= \pm (2v_4 \circ 2\bar{\kappa}_7 - 2E\bar{\alpha}) && \text{(by (16.7) of [MT])} \\ &= 2(2v_4 \circ \bar{\kappa}_7 - E\bar{\alpha}). \end{aligned}$$

Hence we have $\Delta(\bar{\kappa}_9) = \pm 2v_4 \circ \bar{\kappa}_7 \pm E\bar{\alpha}$. q.e.d.

Consider the exact sequence

$$\pi_{29}^9 \xrightarrow{\Delta} \pi_{27}^4 \xrightarrow{E} \pi_{28}^5 \xrightarrow{H} \pi_{28}^9 \xrightarrow{\Delta} \pi_{26}^4,$$

where $\pi_{29}^9 = \{\bar{\kappa}_9\} \approx Z_8$. Then $E\pi_{27}^4 = \{v_4 \circ \bar{\kappa}_8\} \approx Z_8$ by (4.4) and (4.5). We have $\text{Ker}(\Delta: \pi_{28}^9 \rightarrow \pi_{26}^4) = \{4\bar{\epsilon}_9\} \oplus \{\bar{\sigma}_9\} \approx Z_2 \oplus Z_2$ by the facts in pp. 313–4 of [MII]. Hence there are two possibilities for the structure of π_{28}^5 ; $Z_8 \oplus Z_2 \oplus Z_2$ or $Z_{16} \oplus Z_2$. We prove that $\pi_{28}^5 \approx Z_8 \oplus Z_2 \oplus Z_2$ by showing the following.

(4.6) *There exists an element ϕ_5 of π_{28}^5 such that $H(\phi_5) = \bar{\sigma}_9$ and $2\phi_5 = 0$.*

Proof. Since $\{v_5, 2v_8, \varepsilon_{11} + \bar{v}_{11}\}_1 = 0$ and $\{2v_7, \varepsilon_{10} + \bar{v}_{10}, \sigma_{18}\} \ni 0$, the ter-

tiary Toda bracket $\{v_5, 2v_8, \varepsilon_{11} + \bar{v}_{11}, \sigma_{19}\}_1$ is well-defined. Choose $\phi'_5 \in \{v_5, 2v_8, \varepsilon_{11} + \bar{v}_{11}, \sigma_{19}\}_1$, then

$$\begin{aligned} H(\phi'_5) &\in H\{v_5, 2v_8, \varepsilon_{11} + \bar{v}_{11}, \sigma_{19}\}_1 \\ &\subset \{\Delta^{-1}(2v_4^2), \varepsilon_{12} + \bar{v}_{12}, \sigma_{20}\} && \text{(by Proposition (7.18) of [O1])} \\ &= \{v_9, \varepsilon_{12} + \bar{v}_{12}, \sigma_{20}\} && \text{(by (5.13) of [T])} \\ &\ni \sigma_9 \end{aligned}$$

$\text{mod } v_9 \circ \pi_{28}(S^{12}) + \pi_{21}^9 \circ \sigma_{21} = 0$. Since $\Delta \pi_{30}^1 = 0$ holds by (3.3) and (3.8), $E: \pi_{28}^5 \rightarrow \pi_{29}^6$ is monic and so $8\{v_5, 2v_8, \varepsilon_{11} + \bar{v}_{11}, \sigma_{19}\}_1 = 8\varepsilon_5 \circ \{v_5, 2v_8, \varepsilon_{11} + \bar{v}_{11}, \sigma_{19}\}_1$. This Toda bracket and $\{8\varepsilon_5, v_5, 2v_8, \varepsilon_{11} + \bar{v}_{11}\} \circ \sigma_{21}$ have a common element, where $\{8\varepsilon_5, v_5, 2v_8, \varepsilon_{11} + \bar{v}_{11}\} \circ \sigma_{21} \subset \pi_{21}^3 \circ \sigma_{21} = 0$ by (2.4) and (2.9). Hence we have $2\phi'_5 = 2yv_5 \circ \bar{\kappa}_8$ for some integer y . Put $\phi_5 = \phi'_5 - yv_5 \circ \bar{\kappa}_8$, then $2\phi_5 = 0$ and $H(\phi_5) = \bar{\sigma}_9$. q.e.d.

Therefore by (3.2) we have

$$(4.7) \quad \pi_{28}^5 = \{v_5 \circ \bar{\kappa}_8\} \oplus \{\bar{\rho}'''\} \oplus \{\phi_5\} \approx Z_8 \oplus Z_2 \oplus Z_2.$$

$$(4.8) \quad H(\sigma' \circ \omega_{14} \circ \eta_{30}) = 4\xi_{13} + 4y\lambda \quad \text{for } y=0 \text{ or } 1.$$

Proof. $H(\sigma' \circ \omega_{14} \circ \eta_{30}) = H(\sigma' \circ \omega_{14}) \circ \eta_{30} = \eta_{13} \circ \omega_{14} \circ \eta_{30}$ by Example 5.12 of [MI]. This element lies in $\pi_{31}^3 = \{\xi_{13}\} \oplus \{\lambda\} \oplus \{\eta_{13} \circ \bar{\mu}_{14}\} \approx Z_8 \oplus Z_8 \oplus Z_2$ and clearly of order 2. Hence we can put $\eta_{13} \circ \omega_{14} \circ \eta_{30} = 4x\xi_{13} + 4y\lambda + z\eta_{13} \circ \bar{\mu}_{14}$ for some integers x, y, z . Then $E^\infty(\eta_{13} \circ \omega_{14} \circ \eta_{30}) = \eta \circ \omega \circ \eta = \omega \circ \eta^2 \equiv \eta^* \circ \eta^2 = 4v^* \text{ mod } \sigma \circ \mu \circ \eta^2 = 0$ by Proposition 12.20 and Proposition 14.1 of [T]. Further $E^\infty(4x\xi_{13}) = 4x\xi = 4xv^*$ by Lemma 12.24 of [T], $E^\infty(4y\lambda) = 4yE^\infty\lambda = 4y(2v^*) = 0$ by Lemma 12.18 of [T], and $E^\infty(z\eta_{13} \circ \bar{\mu}_{14}) = z\eta \circ \bar{\mu}$. Thus we have $4v^* = 4xv^* + z\eta \circ \bar{\mu}$. Hence x is an odd integer and z is even. Thus (4.8) is proved. q.e.d.

$$(4.9) \quad \begin{aligned} \pi_{29}^6 &= \{v_6 \circ \bar{\kappa}_9\} \oplus \{\bar{\rho}''\} \oplus \{\phi_6\} \oplus \{\Delta(\lambda), \Delta(\xi_{13})\} \\ &\approx Z_8 \oplus Z_4 \oplus Z_2 \oplus (Z_8 \oplus Z_4). \end{aligned}$$

Proof. Consider the exact sequence $\pi_{28}^5 \xrightarrow{E} \pi_{29}^6 \xrightarrow{H} \pi_{29}^1$, where E is monic by (3.3) and (3.8), and $\pi_{29}^1 = \{\eta_{11} \circ \bar{\mu}_{12}\} \oplus \{\lambda', \xi'\} \approx Z_2 \oplus (Z_8 \oplus Z_4)$. Then $4(\Delta(\xi_{13}) + y\Delta\lambda) = \Delta H(\sigma' \circ \omega_{14} \circ \eta_{30}) = 0$ by (4.8). Hence by (2.3) the group $\{\Delta(\lambda), \Delta(\xi_{13})\} \approx Z_8 \oplus Z_4$ is a direct summand of π_{29}^6 . Consider next the exact sequence $\pi_{31}^3 \xrightarrow{\Delta} \pi_{29}^6 \xrightarrow{E} \pi_{30}^7$, where $\pi_{31}^3 = \{\eta_{13} \circ \bar{\mu}_{14}\} \oplus \{\lambda\} \oplus \{\xi_{13}\} \approx Z_8 \oplus Z_8 \oplus Z_2$, in

which $\Delta(\eta_{13} \circ \bar{\mu}_{14}) = \Delta(\eta_{13}) \circ \bar{\mu}_{12} = 0$. Hence the suspension $E: E\pi_{28}^5 \rightarrow \pi_{30}^7$ is monic, and so we have $2\bar{\rho}'' = E\bar{\rho}'''$ by (3.2). Consider again the first exact sequence, then (4.9) is proved. q.e.d.

Before determining the structure of π_{n+23}^n for $n \geq 7$, we need some information about direct summands in π_{n+23}^n .

$$\begin{aligned}
 (4.10) \quad & H(\sigma' \circ \omega_{14}) = e_{13}^*, \\
 & H(\sigma' \circ \sigma_{14} \circ \mu_{21}) = \sigma_{13} \circ \eta_{20} \circ \mu_{21}, \\
 & H(\bar{\kappa}_7 \circ v_{27}) = v_{13} \circ \kappa_{16}, \\
 & 2(\sigma' \circ \omega_{14}) = 2(\sigma' \circ \sigma_{14} \circ \mu_{21}) = 0.
 \end{aligned}$$

Proof. The first relation appears in Example 5.12 of [MI]. The others are obtained as follows. By Lemma 5.14 of [T], $H(\sigma' \circ \sigma_{14} \circ \mu_{21}) = H(\sigma') \circ \sigma_{14} \circ \mu_{21} = \eta_{13} \circ \sigma_{14} \circ \mu_{21} = \sigma_{13} \circ \eta_{20} \circ \mu_{21}$. Similarly we have $H(\bar{\kappa}_7 \circ v_{27}) = v_{13} \circ \kappa_{16}$. Since μ_{21} is of order 2, we get $2(\sigma' \circ \sigma_{14} \circ \mu_{21}) = 0$. Since $E: \pi_{30}^7 \rightarrow \pi_{31}^8$ is monic, $2(\sigma' \circ \omega_{14}) = 0$ follows from $2E(\sigma' \circ \omega_{14}) = E\sigma' \circ 2\omega_{15} = 0$. q.e.d.

Remark 4.11. $\{\sigma' \circ \omega_{14}\} \oplus \{\sigma' \circ \sigma_{14} \circ \mu_{21}\} \approx Z_2 \oplus Z_2$ is a direct summand of π_{30}^7 .

Then (5.15) of [T] gives rise to

$$\begin{aligned}
 (4.12) \quad & \{E\sigma' \circ \omega_{15}\} \oplus \{E\sigma' \circ \sigma_{15} \circ \mu_{22}\} \oplus \{\sigma_8^2 \circ \mu_{22}\} \oplus \{\sigma_8 \circ \omega_{15}\} \oplus \{\sigma_8 \circ \eta^{*'}\} \\
 & \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \text{ is a direct summand of } \pi_{31}^8.
 \end{aligned}$$

Consider the exact sequence $\pi_{33}^{17} \xrightarrow{A} \pi_{31}^8 \xrightarrow{E} \pi_{32}^9$, where $\pi_{33}^{17} = \{\eta_{17}^*\} \oplus \{\omega_{17}\} \oplus \{\sigma_{17} \circ \mu_{24}\} \approx Z_2 \oplus Z_2 \oplus Z_2$. We show

$$\begin{aligned}
 \Delta(\omega_{17}) &= E\sigma' \circ \omega_{15}, \\
 \Delta(\sigma_{17} \circ \mu_{24}) &= E\sigma' \circ \sigma_{15} \circ \mu_{22}, \\
 \Delta(\eta_{17}^*) &\equiv \sigma_8 \circ \eta^{*'} \pmod{\sigma_8^2 \circ \mu_{22}, 4v_8 \circ \bar{\kappa}_{11}, E\sigma' \circ \omega_{15}, E\sigma' \circ \sigma_{15} \circ \mu_{22}}.
 \end{aligned}$$

By (5.16) of [T],

$$\begin{aligned}
 \Delta(\omega_{17}) &= \Delta(\iota_{17}) \circ \omega_{15} = \pm(2\sigma_8 - E\sigma') \circ \omega_{15} = E\sigma' \circ \omega_{15}, \\
 \Delta(\sigma_{17} \circ \mu_{24}) &= \Delta(\iota_{17}) \circ \sigma_{15} \circ \mu_{22} = \pm(2\sigma_{18} - E\sigma') \circ \sigma_{15} \circ \mu_{22} \\
 &= E\sigma' \circ \sigma_{15} \circ \mu_{22}.
 \end{aligned}$$

Since σ_3^3 is of order 2, we have

$$\begin{aligned}\sigma_9 \circ E\eta^{*'} &\in \sigma_9 \circ \{\sigma_{16}, 4\sigma_{23}, \eta_{30}\}_1 \\ &\subset \{\sigma_9^2, 4\sigma_{23}, \eta_{30}\}_1 \\ &\supset \{2\sigma_9^3, 2\iota_{30}, \eta_{30}\}_1 \\ &\equiv 0\end{aligned}$$

$\text{mod } \sigma_9^2 \circ E\pi_{31}^{22} + \pi_{31}^9 \circ \eta_{31} = \{\sigma_9^2 \circ \mu_{23}, 4v_9 \circ \bar{\kappa}_{12}\}$. Now we note that $\text{Im } \Delta: \pi_{33}^{17} \rightarrow \pi_{31}^8$ has at most three direct summands of Z_2 . Hence we have $\Delta(\eta_{17}^*) \equiv \sigma_8 \circ \eta^{*'} \text{ mod } \sigma_8^2 \circ \mu_{22}, 4v_8 \circ \bar{\kappa}_{11}, E\sigma \circ \omega_{15}, E\sigma' \circ \sigma_{15} \circ \mu_{22}$.

Therefore we have shown

$$(4.13) \quad \{\sigma_9 \circ \omega_{16}\} \oplus \{\sigma_9^2 \circ \mu_{23}\} \approx Z_2 \oplus Z_2 \text{ is a subgroup of } \pi_{32}^9.$$

Consider next the exact sequence $\pi_{34}^{19} \xrightarrow{\Delta} \pi_{32}^9 \xrightarrow{E} \pi_{33}^{10}$, where $\pi_{34}^{19} = \{\rho_{19}\} \oplus \{\bar{e}_{19}\} \approx Z_{32} \oplus Z_2$. We show:

$$\Delta(\rho_{19}) = \sigma_9^2 \circ \mu_{23}, \quad \Delta(2\rho_{19}) = 0, \quad \Delta(\bar{e}_{19}) = 0.$$

By Proposition 2.7 of [T],

$$\Delta(2\rho_{19}) = \Delta(2\iota_{19}) \circ \rho_{17} = \Delta(H\Delta\iota_{21}) \circ \rho_{17} = 0.$$

By (3.9),

$$\Delta(\bar{e}_{19}) = \Delta H\bar{e}_{10} = 0.$$

Hence $\text{Ker}(E: \pi_{32}^9 \rightarrow \pi_{33}^{10})$ contains at most one non zero element. However $\sigma_9^2 \circ \mu_{23} \neq 0$ by (4.13) and $\sigma_{10}^2 \circ \mu_{24} = 0$ by (2.3). Therefore we must have $\Delta(\rho_{19}) = \sigma_9^2 \circ \mu_{23}$.

In the above argument we have shown

$$(4.14) \quad E^3: E\pi_{29}^6 \longrightarrow \pi_{33}^{10} \text{ is monic.}$$

$$(4.15) \quad \text{Ker}(\Delta: \pi_{33}^{17} \longrightarrow \pi_{31}^8) = 0,$$

$$\text{Ker}(\Delta: \pi_{34}^{19} \longrightarrow \pi_{32}^9) = \{2\rho_{19}\} \oplus \{\bar{e}_{19}\} \approx Z_{16} \oplus Z_2.$$

We now return to determining π_{30}^7 . Consider the exact sequence $\pi_{29}^6 \xrightarrow{E} \pi_{30}^7 \xrightarrow{H} \pi_{30}^3$, where

$$\pi_{30}^3 = \{e_{13}^*\} \oplus \{\sigma_{13} \circ \eta_{20} \circ \mu_{21}\} \oplus \{v_{13} \circ \kappa_{16}\} \oplus \{\bar{\mu}_{13}\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

As is shown in the proof of (4.9), $E\pi_{29}^6 = \{v_7 \circ \bar{\kappa}_{10}\} \oplus \{E\bar{\rho}'''\} \oplus \{\phi_7\} \approx Z_8 \oplus Z_4 \oplus Z_2$. Now apply Proposition 3.2 of [T] to the case $\alpha = \bar{\kappa}_7$, $\beta = v_4$, then we have

$$(4.16) \quad \Delta(\kappa_{21}) = \bar{\kappa}_{10} \circ v_{30} - v_{10} \circ \bar{\kappa}_{13}.$$

Hence $2(\bar{\kappa}_{10} \circ v_{30} - v_{10} \circ \bar{\kappa}_{13}) = \Delta(2\kappa_{21}) = 0$. However since $E^3: E\pi_{29}^6 \rightarrow \pi_{33}^1$ is monic by (4.14), this implies $2(\bar{\kappa}_7 \circ v_{27} - v_7 \circ \bar{\kappa}_{10}) = 0$. Therefore by (3.2) and (4.10), we have

$$(4.17) \quad \begin{aligned} \pi_{30}^7 &= \{v_7 \circ \bar{\kappa}_{10}\} \oplus \{\bar{\rho}'\} \oplus \{\phi_7\} \oplus \{\bar{\kappa}_7 \circ v_{27} - v_7 \circ \bar{\kappa}_{10}\} \\ &\quad \oplus \{\sigma' \circ \sigma_{14} \circ \mu_{21}\} \oplus \{\sigma' \circ \omega_{14}\} \approx Z_8 \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

Hence by (5.15) and Theorem 12.16 of [T], and by (4.17), we have

$$(4.18) \quad \begin{aligned} \pi_{31}^8 &= \{v_8 \circ \bar{\kappa}_{11}\} \oplus \{E\bar{\rho}'\} \oplus \{\phi_8\} \oplus \{\bar{\kappa}_8 \circ v_{28} - v_8 \circ \bar{\kappa}_{11}\} \\ &\quad \oplus \{E\sigma' \circ \sigma_{15} \circ \mu_{22}\} \oplus \{E\sigma' \circ \omega_{15}\} \oplus \{\sigma_8^2 \circ \mu_{22}\} \oplus \{\sigma_8 \circ \omega_{15}\} \\ &\quad \oplus \{\sigma_8 \circ \eta^{*'}\} \approx Z_8 \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

Before going to compute π_{32}^9 we need some preparations:

$$(4.19) \quad \begin{aligned} \{2\iota_{18}, \sigma_{18}^2, \eta_{32}\}_1 &\ni \omega_{18} + y\sigma_{18} \circ \mu_{25} \quad \text{for some integer } y, \\ \{\sigma', \sigma_{14}^2, \eta_{28}\}_1 &\subset E\pi_{29}^6 + \{\sigma' \circ \sigma_{14} \circ \mu_{21}\}. \end{aligned}$$

Proof. Observe that $\{2\iota_{18}, \sigma_{18}^2, \eta_{32}\}_1 \subset \pi_{34}(S^{18}) = \{\omega_{18}\} \oplus \{\sigma_{18} \circ \mu_{25}\} \approx Z_2 \oplus Z_2$. Hence this Toda bracket contains $x\omega_{18} + y\sigma_{18} \circ \mu_{25}$ for some integers x, y . So $\langle 2\iota, \sigma^2, \eta \rangle \ni x\omega + y\sigma \circ \mu$. By (3.9) of [T], $\langle 2\iota, \sigma^2, \eta \rangle + \langle \sigma^2, \eta, 2\iota \rangle + \langle \eta, 2\iota, \sigma^2 \rangle \ni 0$. Since $\{\sigma_{13}^2, \eta_{27}, 2\iota_{28}\}_1$, which is in $\pi_{29}(S^{13}) = \pi_{13}^3 = \{\sigma_{13} \circ \mu_{20}\}$, is clearly a single element, we have $\langle \sigma^2, \eta, 2\iota \rangle = z\sigma \circ \mu$ for some integer z . Now $\langle \eta, 2\iota, \sigma^2 \rangle \subset \langle \eta, 2\sigma, \sigma \rangle \ni \eta^* \equiv \omega \pmod{\sigma \circ \mu}$, where the indeterminacy is $\eta \circ_2 \pi_{15}^5 + {}_2\pi_9^5 \circ \sigma = \{\mu \circ \sigma\}$. So $\langle \eta, 2\iota, \sigma^2 \rangle \subset \omega + \{\mu \circ \sigma\}$. Therefore $\langle 2\iota, \sigma^2, \eta \rangle + z\sigma \circ \mu + \omega + \{\sigma \circ \mu\} \ni 0$. It follows that x must be an odd integer and $\{2\iota_{18}, \sigma_{18}^2, \eta_{32}\}_1 \ni \omega_{18} + y\sigma_{18} \circ \mu_{25}$. Next we show the second statement. By Lemma 5.5 and Lemma 5.14 of [T], we have

$$\begin{aligned} H\{\sigma', \sigma_{14}^2, \eta_{28}\}_1 &\subset \{H(\sigma'), \sigma_{14}^2, \eta_{28}\}_1 \\ &= \{\eta_{13}, \sigma_{14}^2, \eta_{28}\}_1 \\ &\ni \sigma_{13}^2 \circ v_{27} \\ &= 0 \end{aligned}$$

$$\text{mod } \eta_{13} \circ E \pi_{29}^{13} + \pi_{29}^{13} \circ \eta_{29} = \{\eta_{13} \circ \sigma_{14} \circ \mu_{21}\}.$$

Thus $H\{\sigma', \sigma_{14}^2, \eta_{28}\} \subset \{\eta_{13} \circ \sigma_{14} \circ \mu_{21}\}$. However we have shown $H(\sigma' \circ \sigma_{14} \circ \mu_{21}) = \eta_{13} \circ \sigma_{14} \circ \mu_{21}$ in (4.10). Hence $\{\sigma', \sigma_{14}^2, \eta_{28}\}_1 \subset \{\sigma' \circ \sigma_{14} \circ \mu_{21}\} + E\pi_{29}^6$. This completes the proof of (4.19). q.e.d.

Now consider the exact sequence $\pi_{33}^{17} \xrightarrow{A} \pi_{31}^8 \xrightarrow{E} \pi_{32}^9$, where $\pi_{33}^{17} = \{\eta_{17}^*\} \oplus \{\omega_{17}\} \oplus \{\sigma_{17} \circ \mu_{24}\} \approx Z_2 \oplus Z_2 \oplus Z_2$. We have already shown that

$$\Delta(\omega_{17}) = E\sigma' \circ \omega_{15},$$

$$\Delta(\sigma_{17} \circ \mu_{24}) = E\sigma' \circ \sigma_{15} \circ \mu_{22},$$

$$\Delta(\eta_{17}^*) \equiv \sigma_8 \circ \eta^{*'} \quad \text{mod } \sigma_8 \circ \mu_{22}, 4v_8 \circ \bar{\kappa}_{11}, E\sigma' \circ \omega_{15}, E\sigma' \circ \sigma_{15} \circ \mu_{22}.$$

Hence by (4.18) we have

$$(4.20) \quad E\pi_{30}^8 = \{E^2 \bar{\rho}'\} \oplus \{v_9 \circ \bar{\kappa}_{12}\} \oplus \{\phi_9\} \oplus \{\bar{\kappa}_9 \circ v_{29} - v_9 \circ \bar{\kappa}_{12}\} \\ \oplus \{\sigma_9^2 \circ \mu_{23}\} \oplus \{\sigma_9 \circ \omega_{16}\} \approx Z_8 \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

Consider next the exact sequence $\pi_{34}^{19} \xrightarrow{A} \pi_{32}^9 \xrightarrow{E} \pi_{33}^{19}$, where $\pi_{34}^{19} = \{\rho_{19}\} \oplus \{\bar{e}_{19}\} \approx Z_{32} \oplus Z_2$. By (4.15) and the statement just before (4.14), $\Delta\pi_{34}^{19} = \{\sigma_9^2 \circ \mu_{23}\}$. Thus we have

$$(4.12) \quad \pi_{33}^{19} \text{ has the following subgroup;}$$

$$\{E^3 \bar{\rho}'\} \oplus \{v_{10} \circ \bar{\kappa}_{13}\} \oplus \{\phi_{10}\} \oplus \{\bar{\kappa}_{10} \circ v_{30} - v_{10} \circ \bar{\kappa}_{13}\} \oplus \{\sigma_{10} \circ \omega_{17}\} \\ \approx Z_8 \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

The following is a key proposition to determine π_{32}^9 and π_{33}^{19} :

$$(4.22) \quad \sigma_{11} \circ \omega_{18} \equiv 0 \quad \text{mod } 8\bar{\rho}_{11}, 4v_{11} \circ \bar{\kappa}_{14}, \phi_{11},$$

$$\Delta(\sigma_{21}^2) \equiv \sigma_{10} \circ \omega_{17} \quad \text{mod } 4E^3 \bar{\rho}, 4v_{10} \circ \bar{\kappa}_{13}, \phi_{10}, \bar{\kappa}_{10} \circ v_{30} - v_{10} \circ \bar{\kappa}_{13}.$$

Proof. $\sigma_{11} \circ \omega_{18} \in \sigma_{11} \circ \{2\iota_{18}, \sigma_{18}^2, \eta_{32}\}_1 \subset \{2\sigma_{11}, \sigma_{18}^2, \eta_{32}\}_1 \equiv 0 \text{ mod } E^5 \pi_{29}^6$ by (5.19). The indeterminacy of $\{2\sigma_{11}, \sigma_{18}^2, \eta_{32}\}_1$ is $2\sigma_{11} \circ E\pi_{33}^{17} + \pi_{33}^{13} \circ \eta_{33} = \{4v_{11} \circ \bar{\kappa}_{14}\}$. Consider the exact sequence $\pi_{35}^{21} \xrightarrow{A} \pi_{33}^{19} \xrightarrow{E} \pi_{34}^{11}$, where $\pi_{35}^{21} = \{\sigma_{21}^2\} \oplus \{\kappa_{21}\} \approx Z_2 \oplus Z_2$. Since we have shown $\Delta(\kappa_{21}) = \bar{\kappa}_{10} \circ v_{30} - v_{10} \circ \bar{\kappa}_{13}$ in (4.16), we must have $\Delta(\sigma_{21}^2) \equiv \sigma_{10} \circ \omega_{17} \text{ mod } E^4 \pi_{29}^6, \bar{\kappa}_{10} \circ v_{30} - v_{10} \circ \bar{\kappa}_{13}$. Here

the indeterminacy must be $4E^3\bar{\rho}'$, $4v_{10}\circ\bar{\kappa}_{13}$, ϕ_{10} and $\bar{\kappa}_{10}\circ v_{30}-v_{10}\circ\bar{\kappa}_{13}$, since $\Delta(\sigma_{21}^2)$ is of order 2 and since $E^4\pi_{29}^6=\{2E^3\bar{\rho}'\}\oplus\{v_{10}\circ\bar{\kappa}_{13}\}\oplus\{\phi_{10}\}\approx Z_4\oplus Z_8\oplus Z_2$ by (4.21).

Therefore by (3.2) we also have $\sigma_{11}\circ\omega_{18}\equiv 0 \pmod{8\bar{\rho}_{11}}$, $4v_{11}\circ\bar{\kappa}_{14}$, ϕ_{11} .

In the above argument we have shown

$$(4.23) \quad \text{Ker}(\Delta: \pi_{35}^{21} \longrightarrow \pi_{33}^{19})=0.$$

$$(4.24) \quad \text{Ker}(E^2: \pi_{32}^9 \longrightarrow \pi_{34}^1) \subset \{\sigma_9^2\circ\mu_{23}, \sigma_9\circ\omega_{16}, 4E^2\bar{\rho}', \\ 4v_9\circ\bar{\kappa}_{12}, \bar{\kappa}_9\circ v_{29}-v_9\circ\bar{\kappa}_{12}\}.$$

We now determine π_{32}^9 . Consider the exact sequence

$$0 \longrightarrow E\pi_{31}^8 \longrightarrow \pi_{32}^9 \xrightarrow{H} \pi_{32}^{17} \xrightarrow{\Delta} \pi_{30}^8,$$

where $\pi_{32}^{17}=\{\rho_{17}\}\oplus\{\bar{\epsilon}_{17}\}\approx Z_{32}\oplus Z_2$ and $E\pi_{31}^8$ is given in (4.20). By (8.3), (8.5) of [MII], $\text{Ker}(\Delta: \pi_{32}^{17}\rightarrow\pi_{30}^8)=\{16\rho_{17}\}\approx Z_2$. By (3.2), (4.20) and (4.24), $\bar{\rho}_9$ is of order 16. Therefore we have

$$(4.25) \quad \pi_{32}^9=\{\bar{\rho}_9\}\oplus\{v_9\circ\bar{\kappa}_{12}\}\oplus\{\phi_9\}\oplus\{\bar{\kappa}_9\circ v_{29}-v_9\circ\bar{\kappa}_{12}\} \\ \oplus\{\sigma_9^2\circ\mu_{23}\}\oplus\{\sigma_9\circ\omega_{16}\}\approx Z_{16}\oplus Z_8\oplus Z_2\oplus Z_2\oplus Z_2.$$

The following two propositions will be used to compute π_{33}^{19} :

$$(4.26) \quad \Delta\sigma_{21}\in\{\sigma_{10}, \bar{v}_{17}+\varepsilon_{17}, 2\iota_{25}\}_4.$$

Proof. Observe that $\{\sigma_{10}, \bar{v}_{17}+\varepsilon_{17}, 2\iota_{25}\}_4=\{\sigma_{10}, \bar{v}_{17}+\varepsilon_{17}, 2\iota_{25}\}_1$. Then by Proposition 2.3 and (10.18) of [T], we have

$$H\{\sigma_{10}, \bar{v}_{17}+\varepsilon_{17}, 2\iota_{25}\}_4 \\ = -\Delta^{-1}(\sigma_9\circ(\bar{v}_{16}+\varepsilon_{16}))\circ 2\iota_{26} \\ \ni 2\sigma_{19},$$

while by Proposition 2.7 of [T], $H(\Delta\sigma_{21})=H\Delta(\iota_{21})\circ\sigma_{19}=2\sigma_{19}$. Note that $\text{Ker}(H: \pi_{28}^{10}\rightarrow\pi_{26}^9)=\{8\Delta\sigma_{21}\}\oplus\{\sigma_{10}\circ\mu_{17}\}$ is contained in the indeterminacy of $\{\sigma_{10}, \bar{v}_{17}+\varepsilon_{17}, 2\iota_{25}\}_4$. Hence we have $\Delta(\sigma_{21})\in\{\sigma_{10}, \bar{v}_{17}+\varepsilon_{17}, 2\iota_{25}\}_4$.

In π_{33}^{19} there appears the following new element which we did not define in §3.

$$(4.27) \quad \text{There exists an element } \psi_{10} \text{ of } \pi_{33}^{19} \text{ such that } H(\psi_{10})=\sigma_{19}^2, 2\psi_{10}=\Delta\sigma_{21}^2.$$

Proof. Choose an element ψ_{10} from $\{\sigma_{10}, \bar{v}_{17} + \varepsilon_{17}, \sigma_{25}\}_4$, then

$$\begin{aligned} H(\psi_{10}) &\in H\{\sigma_{10}, \bar{v}_{17} + \varepsilon_{17}, \sigma_{25}\}_1 \\ &= -\Delta^{-1}(\sigma_9 \circ (\bar{v}_{16} + \varepsilon_{16})) \circ \sigma_{26} \\ &= \sigma_{19}^2 \quad (\text{by (10.18) of [T]}) \end{aligned}$$

and

$$\begin{aligned} 2\psi_{10} &= \psi_{10} \circ 2\iota_{33} \in \{\sigma_{10}, \bar{v}_{17} + \varepsilon_{17}, \sigma_{25}\}_4 \circ 2\iota_{33} \\ &\subset \{\sigma_{10}, \bar{v}_{17} + \varepsilon_{17}, 2\sigma_{25}\}_4 \\ &\supset \{\sigma_{10}, \bar{v}_{17} + \varepsilon_{17}, 2\iota_{25}\}_4 \circ \sigma_{26} \\ &\ni \Delta(\sigma_{21}) \circ \sigma_{26} = \Delta\sigma_{21}^2 \quad (\text{by (4.26)}) \\ &\text{mod } \sigma_{10} \circ E^4 \pi_{28}^{13} + \pi_{26}^{10} \circ 2\sigma_{26} = 0. \end{aligned} \quad \text{q. e. d.}$$

Consider the exact sequence

$$\pi_{34}^{19} \xrightarrow{\Delta} \pi_{32}^9 \xrightarrow{E} \pi_{33}^{10} \xrightarrow{H} \pi_{33}^{19} \xrightarrow{\Delta} \pi_{31}^9.$$

As is stated just before (4.12), $\Delta\pi_{34}^{19} = \{\sigma_9^2 \circ \mu_{23}\}$. Hence $E\pi_{32}^9$ is determined by (4.25). By (4.22) and (4.27) we have $2\psi_{10} = \Delta\sigma_{21}^2 \equiv \sigma_{10} \circ \omega_{17} \text{ mod } 4E^3\bar{\rho}', 4v_{10} \circ \bar{\kappa}_{13}, \phi_{10}, \bar{\kappa}_{10} \circ v_{30} - v_{10} \circ \bar{\kappa}_{13}$, whence ψ_{10} is of order 4, while by pp. 317–8 of [MII] $\text{Ker}(\Delta: \pi_{33}^{19} \rightarrow \pi_{31}^9) = \{\sigma_{19}^2\} \approx Z_2$. Therefore we have

$$\begin{aligned} (4.28) \quad \pi_{33}^{19} &= \{\bar{\rho}_{10}\} \oplus \{v_{10} \circ \bar{\kappa}_{13}\} \oplus \{\phi_{10}\} \oplus \{\bar{\kappa}_{10} \circ v_{30} - v_{10} \circ \bar{\kappa}_{13}\} \oplus \{\psi_{10}\} \\ &\approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_4. \end{aligned}$$

We next show

$$\begin{aligned} (4.29) \quad \pi_{n+23}^n &= \{\bar{\rho}_n\} \oplus \{v_n \circ \bar{\kappa}_{n+3}\} \oplus \{\phi_n\} \oplus \{\psi_n\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \\ &\text{for } n = 11, 12, 13, 14. \end{aligned}$$

In fact this directly follows from the facts; ${}_2\pi_{12}^S = {}_2\pi_{13}^S = 0$; $\Delta(\zeta_{25})$ is of order 8 by p. 320 of [MII]; $\Delta(\zeta_{27}) = 0$ by (3.10); $\Delta(\eta_{27} \circ \mu_{28}) \neq 0$ by p. 321 of [MII].

Consider the exact sequences:

$$\begin{aligned} \pi_{39}^{29} &\xrightarrow{\Delta} \pi_{37}^{14} \xrightarrow{E} \pi_{38}^{15} \xrightarrow{H} \pi_{38}^{29} \xrightarrow{\Delta} \pi_{36}^{14}, \\ \pi_{40}^{31} &\xrightarrow{\Delta} \pi_{38}^{15} \xrightarrow{E} \pi_{39}^{16} \xrightarrow{H} \pi_{39}^{31}, \end{aligned}$$

where $\pi_{39}^{29} = \{\eta_{29} \circ \mu_{30}\} \approx Z_2$ and $\pi_{40}^{31} = \{v_{31}^3\} \oplus \{\mu_{31}\} \oplus \{\eta_{31} \circ \varepsilon_{32}\} \approx Z_2 \oplus Z_2 \oplus Z_2$. By (3.11) $\Delta(\eta_{29} \circ \mu_{30}) = \Delta H(\mu^{*'}) = 0$ and $\Delta(\mu_{31}) = \Delta H(\mu^*) = 0$. By Lemma 6.3 and (10.21) of [T] we have $\Delta(v_{31}^3) = \Delta(\eta_{31} \circ \bar{v}_{32}) = \Delta(\eta_{31}) \circ \bar{v}_{30} = 0$, and similarly $\Delta(\eta_{31} \circ \varepsilon_{32}) = 0$. Thus $\Delta\pi_{39}^{29} = \Delta\pi_{40}^{31} = 0$, whence $E: \pi_{37}^{14} \rightarrow \pi_{38}^{15}$ and $E: \pi_{38}^{15} \rightarrow \pi_{39}^{16}$ are monic. Consider the upper exact sequence, then we have

$$\text{Ker}(\Delta: \pi_{38}^{29} \longrightarrow \pi_{36}^{14}) = \{v_{29}^3\} \oplus \{\eta_{29} \circ \varepsilon_{30}\} \approx Z_2 \oplus Z_2.$$

By (3.4) $2E\bar{e}^{*'} = 2\Delta(\varepsilon_{33}) = \Delta(2\varepsilon_{33}) = 0$ and similarly $2E\bar{v}^{*'} = 0$. So we have $2\bar{e}^{*'} = 2\bar{v}^{*'} = 0$, since $E: \pi_{38}^{15} \rightarrow \pi_{39}^{16}$ is monic.

Therefore by (3.4) and (4.29) we have

$$(4.30) \quad \pi_{38}^{15} = \{\bar{\rho}_{15}\} \oplus \{v_{15} \circ \bar{\kappa}_{18}\} \oplus \{\phi_{15}\} \oplus \{\psi_{15}\} \oplus \{\bar{e}^{*'}\} \oplus \{\bar{v}^{*'}\} \\ \approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

Consider again the lower exact sequence. Recall $\pi_{39}^{31} = \{\bar{v}_{31}\} \oplus \{\varepsilon_{31}\} \approx Z_2 \oplus Z_2$. Therefore by (3.4) we easily have

$$(4.31) \quad \pi_{39}^{16} = \{\bar{\rho}_{16}\} \oplus \{v_{16} \circ \bar{\kappa}_{19}\} \oplus \{\phi_{16}\} \oplus \{\psi_{16}\} \oplus \{E\bar{e}^{*'}\} \oplus \{E\bar{v}^{*'}\} \\ \oplus \{\bar{e}_{16}^*\} \oplus \{\bar{v}_{16}^*\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

Next consider the exact sequence

$$\pi_{41}^{33} \xrightarrow{\Delta} \pi_{39}^{16} \xrightarrow{E} \pi_{40}^{17} \xrightarrow{H} \pi_{40}^{33} \xrightarrow{\Delta} \pi_{38}^{16},$$

where $\pi_{41}^{33} = \{\bar{v}_{33}\} \oplus \{\varepsilon_{33}\} \approx Z_2 \oplus Z_2$ and $\pi_{40}^{33} = \{\sigma_{33}\} \approx Z_{16}$. Then by p. 323 of [MII] $\pm \Delta(\sigma_{33}) \equiv 2\sigma_{16}^* - E\sigma^{*'} \pmod{\rho_{16} \circ \sigma_{31}}$, whence $\Delta(\sigma_{33})$ is of order 16 and $H: \pi_{40}^{17} \rightarrow \pi_{40}^{33}$ is trivial. Therefore by (3.4) and (4.31) we have

$$(4.32) \quad \pi_{40}^{17} = \{\bar{\rho}_{17}\} \oplus \{v_{17} \circ \bar{\kappa}_{20}\} \oplus \{\phi_{17}\} \oplus \{\psi_{17}\} \oplus \{\bar{e}_{17}^*\} \oplus \{\bar{v}_{17}^*\} \\ \approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

Consider the exact sequence

$$\pi_{42}^{35} \xrightarrow{\Delta} \pi_{40}^{17} \xrightarrow{E} \pi_{41}^{18} \xrightarrow{H} \pi_{41}^{35} \xrightarrow{\Delta} \pi_{39}^{17},$$

where $\pi_{42}^{35} = \{\sigma_{35}\} \approx Z_{16}$ and $\pi_{41}^{35} = \{v_{35}^2\} \approx Z_2$. Then $\Delta(\sigma_{35}) = \Delta(\iota_{35}) \circ \sigma_{33} \equiv \eta_{17}^* \circ \sigma_{33} \pmod{E^2\pi_{31}^{15} \circ \sigma_{33}} \subset E^2\pi_{38}^{15}$, while $\eta_{17}^* \circ \sigma_{33} = \bar{v}_{17}^* + \bar{e}_{17}^*$ by our definition (see the proof of (3.4)). So $\Delta(\sigma_{35}) \equiv v_{17}^* + \varepsilon_{17}^* \equiv 0 \pmod{E^2\pi_{38}^{15}}$. Now $\Delta(v_{35}^2) = \omega_{17} \circ v_{33} \neq 0$ by p. 323 of [MII], whence $H: \pi_{41}^{18} \rightarrow \pi_{41}^{35}$ is trivial. Therefore we have

$$(4.33) \quad \begin{aligned} \pi_{41}^{18} &= \{\bar{\rho}_{18}\} \oplus \{v_{18} \circ \bar{\kappa}_{21}\} \oplus \{\phi_{18}\} \oplus \{\psi_{18}\} \oplus \{\bar{\varepsilon}_{18}^*\} \\ &\approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

Also we have shown

$$(4.34) \quad \text{Ker}(\Delta: \pi_{42}^{35} \longrightarrow \pi_{40}^{17}) = \{2\sigma_{35}\} \approx Z_8.$$

$$(4.35) \quad \bar{v}_{18}^* \equiv \bar{\varepsilon}_{18}^* \pmod{E^3 \pi_{38}^{15}}.$$

We next show

$$(4.36) \quad \Delta v_{37}^2 \equiv \bar{\varepsilon}_{18}^* \pmod{E^3 \pi_{38}^{15}}.$$

Proof. Apply Proposition 11.11, ii) of [T] to the case $\alpha = v_{30}^2$, $n = 15$, $i = 38$. Then there exists an element β of π_{39}^{16} such that $\Delta v_{37}^2 = E^2 \beta$ and $H(\beta) \in \{\eta_{31}, 2\epsilon_{32}, v_{32}^2\}_2$. The Toda bracket $\{\eta_{31}, 2\epsilon_{32}, v_{32}^2\}_2$ contains ε_{31} by (6.1) of [T] and its indeterminacy is $\eta_{31} \circ E^2 \pi_{37}^{39} + \pi_{33}^{31} \circ v_{33}^2 = \{\eta_{31} \circ \sigma_{32}\}$, whence $H(\beta) = \varepsilon_{31}$ or $\bar{v}_{31}$ ($= \varepsilon_{31} + \eta_{31} \circ \sigma_{32}$). Since $\beta \equiv \bar{\varepsilon}_{16}^*$ or $\bar{v}_{16}^* \pmod{E \pi_{38}^{15}}$ by (3.4) we obtain $E^2 \beta \equiv \bar{\varepsilon}_{18}^* \pmod{E^3 \pi_{38}^{15}}$ by (4.35). Therefore we have $\Delta v_{37}^2 \equiv \bar{\varepsilon}_{18}^* \pmod{E^3 \pi_{38}^{15}}$.
q.e.d.

Recall ${}_2\pi_4^8 = {}_2\pi_5^8 = 0$. Then the following result immediately follows from (4.36) and the fact that $4\Delta(v_{41}) = 8\sigma_{20}^* \neq 0$ (Lemma 8.3 of [MII]).

$$(4.37) \quad \begin{aligned} \pi_{n+23}^n &= \{\bar{\rho}_n\} \oplus \{v_n \circ \bar{\kappa}_{n+3}\} \oplus \{\phi_n\} \oplus \{\psi_n\} \\ &\approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z_2 \quad \text{for } n = 19, 20, 21. \end{aligned}$$

Before determining π_{45}^{22} we note that

$$\Delta(\eta_{45}^2) = \Delta(\eta_{45}) \circ \eta_{44} = 2\sigma_{22}^* \circ \eta_{44} = 0,$$

$$H(\eta_{45}) = 2\sigma_{22}^* \neq 0, \quad (\text{by Lemma 8.3 of [MII]})$$

and that

$$\begin{aligned} \Delta(\eta_{47}) &= \Delta(\epsilon_{47}) \circ \eta_{45} \\ &= \sigma_{23}^* \circ \eta_{45} && (\text{by Lemma 8.3 of [MII]}) \\ &= \eta_{23} \circ \sigma_{24}^* && (\text{by Proposition 3.1 of [T]}) \\ &= 0. && (\text{by Lemma 8.3 of [MII]}) \end{aligned}$$

Since $H\Delta\iota_{49} = \pm 2\iota_{47}$ by Proposition 2.7 of [T], we have easily

$$\begin{aligned}
 (4.38) \quad & E: \pi_{45}^{22} \longrightarrow \pi_{46}^{23} \text{ is an isomorphism,} \\
 & E; \pi_{46}^{23} \longrightarrow \pi_{47}^{24} \text{ is a monomorphism,} \\
 & \pi_{47}^{24} = E\pi_{46}^{23} \oplus \{\Delta\iota_{49}\} \approx E\pi_{66}^{23} \oplus Z. \\
 & \text{and } E_\infty: \pi_{45}^{22} \rightarrow {}_2\pi_{23}^8 \text{ is an isomorphism.} \\
 (4.39) \quad & \psi_{22} \equiv 0 \pmod{8\bar{\rho}_{22}, 4v_{22} \circ \bar{\kappa}_{25}, \phi_{22}}.
 \end{aligned}$$

Proof. By virtue of (4.38) it is sufficient to show that $\psi \equiv 0 \pmod{8\bar{\rho}, 4v \circ \bar{\kappa}, \phi}$ in the stable range. Recall $\psi \in \langle \sigma, \bar{v} + \varepsilon, \sigma \rangle$ by definition and its indeterminacy is $\sigma \circ {}_2\pi_{16} = \{\sigma \circ \omega\}$. By (3.10) of [T] $\langle \sigma, \bar{v} + \varepsilon, \sigma \rangle$ and $\langle \bar{v} + \varepsilon, \sigma, 2\sigma \rangle$ have a common element. Then

$$\begin{aligned}
 \langle \bar{v} + \varepsilon, \sigma, 2\sigma \rangle &= \langle \sigma \circ \eta, \sigma, 2\sigma \rangle \subset \langle \sigma, \eta \circ \sigma \circ 2\iota, \sigma \rangle = \langle \sigma, 0, \sigma \rangle = \\
 &= \sigma \circ {}_2\pi_{16} = \{\sigma \circ \omega\}.
 \end{aligned}$$

Hence $\psi = 0$ or $\sigma \circ \omega$. Thus the statement follows from (4.22). q.e.d.

Consider the exact sequence

$$\pi_{46}^{43} \xrightarrow{\Delta} \pi_{44}^{21} \xrightarrow{E} \pi_{45}^{22} \xrightarrow{H} \pi_{45}^{43} \xrightarrow{\Delta} \pi_{43}^{21},$$

where $\pi_{46}^{43} = \{v_{43}\}$ and $\pi_{45}^{43} = \{\eta_{43}^2\}$. By (4.39) we must have $\Delta(v_{43}) \equiv \psi_{21} \pmod{8\bar{\rho}_{21}, 4v_{21} \circ \bar{\kappa}_{24}, \phi_{21}}$, while by Lemma 8.3 of [T] $\Delta(\eta_{43}^2) = 4\sigma_{21}^* \neq 0$.

Therefore by (4.38) we have the following results.

$$\begin{aligned}
 (4.40) \quad & \pi_{n+23}^n = \{\bar{\rho}_n\} \oplus \{v_n \circ \bar{\kappa}_{n+3}\} \oplus \{\phi_n\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \quad \text{for } n = 22, 23, \\
 & \pi_{47}^{24} = \{\bar{\rho}_{24}\} \oplus \{v_{24} \circ \bar{\kappa}_{27}\} \oplus \{\phi_{24}\} \oplus \{\Delta\iota_{49}\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \oplus Z, \\
 & \pi_{n+23}^n = \{\bar{\rho}_n\} \oplus \{v_n \circ \bar{\kappa}_{n+3}\} \oplus \{\phi_n\} \approx Z_{16} \oplus Z_8 \oplus Z_2 \quad \text{for } n \geq 25.
 \end{aligned}$$

§ 5. Calculation of the 24-stem.

We devote this section to computing π_{n+24}^n . It follows from (5.2) of [T] and (4.3)

$$(5.1) \quad \pi_{26}^2 = \{\eta_2 \circ \bar{\alpha}\} \approx Z_4.$$

Consider the exact sequence

$$\pi_{26}^2 \xrightarrow{E} \pi_{27}^3 \xrightarrow{H} \pi_{27}^5 \xrightarrow{A} \pi_{25}^2,$$

where $\pi_{27}^5 = \{\varepsilon_5 \circ \kappa_{13}\} \oplus \{v_5 \circ \bar{\zeta}_8\} \oplus \{v_5 \circ \bar{\sigma}_8\} \approx Z_2 \oplus Z_2 \oplus Z_2$. By (4.5) $E^2(\eta_2 \circ \bar{\alpha}) = \eta_4 \circ E^2 \bar{\alpha} = \eta_4 \circ 2v_5 \circ \bar{\kappa}_8 = 2\eta_4 \circ v_5 \circ \bar{\kappa}_8 = 0$. Since $E: \pi_{27}^3 \rightarrow \pi_{28}^4$ is monic, we have $E(\eta_2 \circ \bar{\alpha}) = 0$ and by (5.1) $E\pi_{26}^2 = 0$. By (4.2) $\Delta\pi_{27}^5 = 0$, whence $H: \pi_{27}^3 \rightarrow \pi_{27}^5$ is isomorphic. By Lemma 6.6 of [T], $H(\varepsilon' \circ \kappa_{13}) = H(\varepsilon') \circ \kappa_{13} = \varepsilon_5 \circ \kappa_{13}$. By Lemma 12.2 of [T] and (2.4), $H(\bar{\mu}_3 \circ \sigma_{20}) = H(\bar{\mu}_3) \circ \sigma_{20} = \rho^{IV} \circ \sigma_{20} = v_5 \circ \bar{\zeta}_8$. By (3.5), $H(\delta_3) \equiv v_5 \circ \bar{\sigma}_8 \pmod{v_5 \circ \bar{\zeta}_8}$. Therefore we have

$$(5.2) \quad \pi_{27}^3 = \{\delta_3\} \oplus \{\bar{\mu}_3 \circ \sigma_{20}\} \oplus \{\varepsilon' \circ \kappa_{13}\} \approx Z_2 \oplus Z_2 \oplus Z_2.$$

Then (5.6) of [T], Theorem A of [MII] and (5.2) imply

$$(5.3) \quad \begin{aligned} \pi_{28}^4 &= \{\delta_4\} \oplus \{\bar{\mu}_4 \circ \sigma_{21}\} \oplus \{E\varepsilon' \circ \kappa_{14}\} \oplus \{v_4 \circ \eta_7 \circ \bar{\kappa}_8\} \oplus \{v_4 \circ \sigma' \circ \kappa_{14}\} \\ &\approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

Now we show

$$(5.4) \quad \Delta\sigma_9^3 = 0.$$

Proof. Note that $\varepsilon' \circ \sigma_{13}^2 = 0$. For, by Lemma 6.6 and Lemma 10.7 of [T], $H(\varepsilon' \circ \sigma_{13}^2) = H(\varepsilon') \circ \sigma_{13}^2 = \varepsilon_5 \circ \sigma_{13}^2 = 0$, while $H: \pi_{27}^3 \rightarrow \pi_{27}^5$ is isomorphic; hence $\varepsilon' \circ \sigma_{13}^2 = 0$. Now by (7.16) of [T] and pp. 306–7 of [MII] we have $\Delta(\sigma_9^3) = \Delta(\sigma_9) \circ \sigma_{14}^2 = (x(v_4 \circ \sigma') \pm E\varepsilon') \circ \sigma_{14}^2 = 0$. q. e. d.

Consider the exact sequence

$$\pi_{30}^9 \xrightarrow{A} \pi_{28}^4 \xrightarrow{E} \pi_{29}^5 \xrightarrow{H} \pi_{29}^9 \xrightarrow{A} \pi_{27}^4,$$

where $\pi_{30}^9 = \{\eta_9 \circ \bar{\kappa}_{10}\} \oplus \{\sigma_9 \circ \kappa_{16}\} \oplus \{\sigma_9^3\} \approx Z_2 \oplus Z_2 \oplus Z_2$ and $\pi_{29}^9 = \{\bar{\kappa}_9\} \approx Z_8$. Then by (7.16) of [T], we have

$$E(v_4 \circ \sigma' \circ \kappa_{14}) = v_5 \circ E\sigma' \circ \kappa_{15} = 2(v_5 \circ \sigma_8) \circ \kappa_{15} = v_5 \circ \sigma_8 \circ 2\kappa_{15} = 0;$$

also by (7.10) of [T], $E(E\varepsilon' \circ \kappa_{14}) = E^2\varepsilon' \circ \kappa_{15} = \pm 2(v_5 \circ \sigma_8) \circ \kappa_{15} = v_5 \circ \sigma_8 \circ (\pm 2\kappa_{15}) = 0$. Hence $\text{Ker}(E: \pi_{28}^4 \rightarrow \pi_{29}^5)$ contains a subgroup $\{v_4 \circ \sigma' \circ \kappa_{14}\} \oplus \{E\varepsilon' \circ \kappa_{14}\} \approx Z_2 \oplus Z_2$. However since $\Delta(\sigma_9^3) = 0$ by (5.4), we must have $\Delta\pi_{30}^9 = \{v_4 \circ \sigma' \circ \kappa_{14}\} \oplus \{E\varepsilon' \circ \kappa_{14}\} \approx Z_2 \oplus Z_2$. Therefore $E\pi_{28}^4 = \{v_5 \circ \eta_8 \circ \bar{\kappa}_9\} \oplus \{\bar{\mu}_5 \circ \sigma_{22}\} \oplus \{\delta_5\} \approx Z_2 \oplus Z_2 \oplus Z_2$. Now by (7.9) of [T] and Lemma 15.4 of [MT], we have

$$H(v_5 \circ \sigma_8 \circ \kappa_{15}) = H(v_5 \circ \sigma_8) \circ \kappa_{15} = v_5^2 \circ \kappa_{15} = 4\bar{\kappa}_9.$$

Clearly $2(v_5 \circ \sigma_8 \circ \kappa_{15}) = v_5 \circ \sigma_8 \circ 2\kappa_{15} = 0$, while $\text{Ker}(\Delta: \pi_{29}^9 \rightarrow \pi_{27}^4) = \{4\bar{\kappa}_9\} \approx Z_2$. Therefore we have

$$(5.5) \quad \begin{aligned} \pi_{29}^5 &= \{\delta_5\} \oplus \{\bar{\mu}_5 \circ \sigma_{22}\} \oplus \{v_5 \circ \eta_8 \circ \bar{\kappa}_9\} \oplus \{v_5 \circ \sigma_8 \circ \kappa_{15}\} \\ &\approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

Consider the exact sequence

$$\pi_{31}^1 \xrightarrow{\Delta} \pi_{29}^5 \xrightarrow{E} \pi_{30}^6 \xrightarrow{H} \pi_{30}^1,$$

where $\pi_{31}^1 = \{\bar{\kappa}_{11}\} \oplus \{\theta' \circ \varepsilon_{23}\} \oplus \{\beta''\} \approx Z_8 \oplus Z_2 \oplus Z_2$, and $\pi_{30}^1 = \{\lambda' \circ \eta_{29}\} \oplus \{\xi' \circ \eta_{29}\} \oplus \{\bar{\sigma}_{11}\} \oplus \{\xi_{11}\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_8$. Then by (5.10) of [T] $\Delta(\bar{\kappa}_{11}) = \Delta(\iota_{11}) \circ \bar{\kappa}_9 = v_5 \circ \eta_8 \circ \bar{\kappa}_9$; by Lemma 12.11 of [T] $\Delta(\theta' \circ \varepsilon_{23}) = \Delta(\theta') \circ \varepsilon_{21} = \Delta(H\Delta E\theta) \circ \varepsilon_{21} = 0$; and by (3.7) $\Delta(\beta'') = \Delta H(\beta'') = 0$. Hence $E\pi_{29}^5 = \{v_6 \circ \sigma_9 \circ \kappa_{16}\} \oplus \{\bar{\mu}_6 \circ \sigma_{23}\} \oplus \{\delta_6\} \approx Z_2 \oplus Z_2 \oplus Z_2$. On the other hand by (3.3) we have

$$\begin{aligned} H\Delta(\lambda \circ \eta_{31}) &\equiv \lambda' \circ \eta_{29} \quad \text{mod } \eta_{11}^2 \circ \bar{\mu}_{13} = 4\bar{\xi}_{11}, \\ H\Delta(\xi_{13} \circ \eta_{31}) &\equiv \xi' \circ \eta_{29} \quad \text{mod } \eta_{11}^2 \circ \bar{\mu}_{13} = 4\bar{\xi}_{11}. \end{aligned}$$

Hence by (3.8), $\bar{\sigma}'_6 - \bar{\sigma}_6 \in E\pi_{29}^5 \cup \{4\bar{\xi}'_6, \Delta(\xi_{13} \circ \eta_{31})\}$. Therefore $2(\bar{\sigma}'_6 - \bar{\sigma}_6) = 0$. But since $2\bar{\sigma}'_6 = 0$ by (3.8), we have $2\bar{\sigma}_6 = 0$. Thus we have

$$(5.6) \quad \begin{aligned} \pi_{30}^6 &= \{\delta_6\} \oplus \{\bar{\mu}_6 \circ \sigma_{23}\} \oplus \{v_6 \circ \sigma_9 \circ \kappa_{16}\} \oplus \{\bar{\xi}'_6\} \oplus \{\bar{\sigma}'_6\} \\ &\oplus \{\Delta(\lambda \circ \eta_{31})\} \oplus \{\Delta(\xi_{13} \circ \eta_{31})\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

$$(5.7) \quad \begin{aligned} \Delta(\bar{\sigma}_{13}) &= 0, \\ \Delta(\bar{\xi}_{13}) &\equiv 2\bar{\xi}'_6 \quad \text{mod } 4\bar{\xi}'_6. \end{aligned}$$

Proof. The first one follows from

$$\begin{aligned} \Delta(\bar{\sigma}_{13}) &= \Delta(\iota_{13}) \circ \bar{\sigma}_{11} \in \{v_6, \eta_9, 2\iota_{10}\}_3 \circ \bar{\sigma}_{10} \quad (\text{by Lemma 5.10 of [T]}) \\ &\subset \{v_6, \eta_9, 2\bar{\sigma}_{10}\}_3 \\ &= v_6 \circ E^3 \pi_{27}^6 \\ &= v_6 \circ E^3 \{\eta_6 \circ \bar{\kappa}_7\} \quad (\text{by Theorem A of [MII]}) \\ &= 0. \quad (\text{by (5.9) of [T]}) \end{aligned}$$

The second is obtained as follows. By (3.8) we have $\bar{\zeta}'_6 - x\bar{\zeta}''_6 \in E\pi_{2,9}^5$ for some odd integer x . Since $2(\bar{\zeta}'_6 - x\bar{\zeta}''_6) \in 2E\pi_{2,9}^5 = 0$, $2\bar{\zeta}'_6 \equiv 2\bar{\zeta}''_6 \pmod{4\bar{\zeta}'_6}$. Hence by (3.8) $\Delta(\bar{\zeta}'_6) \equiv 2\bar{\zeta}'_6 \pmod{4\bar{\zeta}'_6}$. q.e.d.

Now consider the exact sequence

$$\pi_{3,2}^{1,3} \xrightarrow{\Delta} \pi_{3,0}^6 \xrightarrow{E} \pi_{3,1}^7 \xrightarrow{H} \pi_{3,1}^{1,3} \xrightarrow{\Delta} \pi_{2,9}^6,$$

where $\pi_{3,2}^{1,3} = \{\xi_{13} \circ \eta_{3,1}\} \oplus \{E\omega'\} \oplus \{\bar{\sigma}_{13}\} \oplus \{\bar{\zeta}_{13}\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_8$. Since $EA(\lambda \circ \eta_{3,1}) = EA(\xi_{13} \circ \eta_{3,1}) = 0$ by (5.6) and (5.7), we have

$$\begin{aligned} E\pi_{3,0}^6 &= \{\bar{\sigma}'_7\} \oplus \{\bar{\zeta}'_7\} \oplus \{v_7 \circ \sigma_{10} \circ \kappa_{17}\} \oplus \{\bar{\mu}_7 \circ \sigma_{24}\} \oplus \{\delta_7\} \\ &\approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

On the other hand, as is shown in the proof of (4.9), $\text{Ker}(\Delta: \pi_{3,1}^{1,3} \rightarrow \pi_{2,9}^6) = \{4\xi_{13} + 4y\lambda\} \oplus \{\eta_{13} \circ \bar{\mu}_{14}\} \approx Z_2 \oplus Z_2$, where $t=0$ or 1 . By Lemma 5.14 of [T] $H(\sigma' \circ \bar{\mu}_{14}) = H(\sigma') \circ \bar{\mu}_{14} = \eta_{13} \circ \bar{\mu}_{14}$, $2(\sigma' \circ \bar{\mu}_{14}) = \sigma' \circ 2\bar{\mu}_{14} = 0$ and $2(\sigma' \circ \omega_{14} \circ \eta_{30}) = \sigma' \circ \omega_{14} \circ 2\eta_{30} = 0$. Therefore we have

$$\begin{aligned} (5.8) \quad \pi_{3,1}^7 &= \{\delta_7\} \oplus \{\bar{\mu}_7 \circ \sigma_{24}\} \oplus \{v_7 \circ \sigma_{10} \circ \kappa_{17}\} \oplus \{\bar{\zeta}'_7\} \oplus \{\bar{\sigma}'_7\} \oplus \{\sigma' \circ \bar{\mu}_{14}\} \oplus \{\sigma' \circ \omega_{14} \circ \eta_{30}\} \\ &\approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

Then (5.15) and Theorem 12.17 of [T], and (5.8) imply

$$\begin{aligned} (5.9) \quad \pi_{3,2}^8 &= \{\delta_8\} \oplus \{\bar{\mu}_8 \circ \sigma_{25}\} \oplus \{v_8 \circ \sigma_{11} \circ \kappa_{18}\} \oplus \{\bar{\zeta}'_8\} \oplus \{\bar{\sigma}'_8\} \oplus \{E\sigma' \circ \bar{\mu}_{15}\} \\ &\quad \oplus \{E\sigma' \circ \omega_{15} \circ \eta_{31}\} \oplus \{\sigma_8 \circ \bar{\mu}_{15}\} \oplus \{\sigma_8^2 \circ \eta_{22} \circ \mu_{23}\} \oplus \{\sigma_8 \circ v_{15} \circ \kappa_{18}\} \oplus \{\sigma_8 \circ \varepsilon_{15}^*\} \\ &\quad \oplus \{\sigma_8 \circ \eta^{*'} \circ \eta_{31}\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \\ &\quad \oplus Z_2. \end{aligned}$$

$$(5.10) \quad \bar{\zeta}'_7 = \sigma' \circ \varepsilon_{14} \circ \mu_{22}.$$

Proof. By definition

$$\begin{aligned} \bar{\zeta}'_7 &\in -\{E\zeta', 8\iota_{23}, 2\sigma_{23}\}_1 \\ &= -\{\sigma' \circ \eta_{14} \circ \varepsilon_{15}, 8\iota_{23}, 2\sigma_{23}\}_1 \quad (\text{by (12.4) of [T]}) \\ &\subset \{\sigma' \circ \varepsilon_{14}, \eta_{22} \circ 8\iota_{23}, 2\sigma_{23}\}_1 \\ &= \sigma' \circ \varepsilon_{14} \circ E\pi_{3,0}^{2,1} + \pi_{2,4}^7 \circ 2\sigma_{24} \end{aligned}$$

$$= \{\sigma' \circ \varepsilon_{14} \circ \mu_{22}\}.$$

Since $\bar{\zeta}'_7 \neq 0$, we have $\bar{\zeta}'_7 = \sigma' \circ \varepsilon_{14} \circ \mu_{22}$. q. e. d.

Recall from [T]

$$\begin{aligned} \pi_{34}^{17} &= \{\eta_{17}^* \circ \eta_{33}\} \oplus \{\varepsilon_{17}^*\} \oplus \{\sigma_{17} \circ \eta_{24} \circ \mu_{25}\} \oplus \{v_{17} \circ \kappa_{20}\} \oplus \{\bar{\mu}_{17}\} \\ &\approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

(5.11) For the homomorphism $\Delta: \pi_{34}^{17} \rightarrow \pi_{32}^8$, we have

$$\begin{aligned} \Delta(\eta_{17}^* \circ \eta_{33}) &\equiv \sigma_8 \circ \eta^{*'} \circ \eta_{31} \pmod{\sigma_8^2 \circ \eta_{22} \circ \mu_{23}, \sigma' \circ \omega_{15} \circ \eta_{31}, \bar{\zeta}'_8}, \\ \Delta(\varepsilon_{17}^*) &= E\sigma' \circ \omega_{15} \circ \eta_{31}, \\ \Delta(\sigma_{17} \circ \eta_{24} \circ \mu_{25}) &= \bar{\zeta}'_8, \\ \Delta(v_{17} \circ \kappa_{20}) &= v_8 \circ \sigma_{11} \circ \kappa_{18}, \\ \Delta(\bar{\mu}_{17}) &= E\sigma' \circ \bar{\mu}_{15}. \end{aligned}$$

Proof. Firstly by the statement just before (4.13) and by (2.1) and (5.10) we have $\Delta(\eta_{17}^* \circ \eta_{33}) = \Delta(\eta_{17}^* \circ \eta_{31}) \equiv \sigma_8 \circ \eta^{*'} \circ \eta_{31} \pmod{\sigma_8^2 \circ \eta_{22} \circ \mu_{23}, E\sigma' \circ \omega_{15} \circ \eta_{31}, \bar{\zeta}'_8}$. By (5.16) and Lemma 12.15 of [T] $\Delta(\varepsilon_{17}^*) = \Delta(\iota_{17})\varepsilon_{15}^* = (2\sigma_8 - E\sigma') \circ \varepsilon_{15}^* = E\sigma' \circ \varepsilon_{15}^* = E\sigma' \circ \omega_{15} \circ \eta_{31}$. Similarly $\Delta(\bar{\mu}_{17}) = E\sigma' \circ \bar{\mu}_{15}$ and by (7.19) of [T] $\Delta(v_{17} \circ \kappa_{20}) = E\sigma' \circ v_{15} \circ \kappa_{18} = v_8 \circ \sigma_{11} \circ \kappa_{18}$. Finally by Lemma 6.4 of [T], (2.1) and (5.10) $\Delta(\sigma_{17} \circ \eta_{24} \circ \mu_{25}) = E\sigma' \circ \sigma_{15} \circ \eta_{22} \circ \mu_{23} = E\sigma' \circ (\varepsilon_{15} + \bar{v}_{15}) \circ \mu_{23} = E\sigma' \circ \varepsilon_{15} \circ \mu_{23} = \bar{\zeta}'_8$. q. e. d.

Consider the exact sequence

$$\pi_{34}^{17} \xrightarrow{\Delta} \pi_{32}^8 \xrightarrow{E} \pi_{33}^9 \xrightarrow{H} \pi_{33}^{17} \xrightarrow{\Delta} \pi_{31}^8.$$

As is shown in the previous section $\Delta: \pi_{33}^{17} \rightarrow \pi_{31}^8$ is monic, whence $H: \pi_{33}^9 \rightarrow \pi_{33}^{17}$ is trivial and $\pi_{33}^9 = E\pi_{32}^8$. Therefore we have

$$\begin{aligned} (5.12) \quad \pi_{33}^9 &= \{\delta_9\} \oplus \{\bar{\mu}_9 \circ \sigma_{26}\} \oplus \{\bar{\sigma}_9\} \oplus \{\sigma_9 \circ \bar{\mu}_{16}\} \oplus \{\sigma_9^2 \circ \eta_{23} \circ \mu_{24}\} \oplus \{\sigma_9 \circ v_{16} \circ \kappa_{19}\} \\ &\oplus \{\sigma_9 \circ \varepsilon_{16}^*\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

$$\begin{aligned} (5.13) \quad \Delta(\sigma_{19} \circ \mu_{26}) &= \sigma_9^2 \circ \eta_{23} \circ \mu_{24}, \\ \Delta(\omega_{19}) &\equiv \sigma_9 \circ \varepsilon_{16}^* \pmod{E^3 \pi_{30}^6}. \end{aligned}$$

Proof. By Lemma 6.4 and (10.18) of [T] we have $\Delta(\sigma_{19} \circ \mu_{26}) = \Delta(\sigma_{19}) \circ \mu_{24} =$

$(\sigma_9 \circ \bar{v}_{16} + \sigma_9 \circ \varepsilon_{16}) \circ \mu_{24} = \sigma_9 \circ (\bar{v}_{16} + \varepsilon_{16}) \circ \mu_{24} = \sigma_9^2 \circ \eta_{23} \circ \mu_{24}$. By (7.1) and Lemma 12.15 of [T], we have $\Delta(\omega_{19}) = \Delta(\iota_{19}) \circ \omega_{17} = (\sigma_9 \circ \eta_{16} + \bar{v}_9 + \varepsilon_9) \circ \omega_{17} = \sigma_9 \circ \eta_{16} \circ \omega_{17} + \bar{v}_9 \circ \omega_{17} + \varepsilon_9 \circ \omega_{17} \equiv \sigma_9 \circ \varepsilon_{16}^* \pmod{E^3 \pi_{30}^6}$. q.e.d.

Consider the exact sequence

$$\pi_{35}^{19} \xrightarrow{\Delta} \pi_{33}^9 \xrightarrow{E} \pi_{34}^{10} \xrightarrow{H} \pi_{34}^{19} \xrightarrow{\Delta} \pi_{32}^9,$$

where $\pi_{35}^{19} = \{\omega_{19}\} \oplus \{\sigma_{19} \circ \mu_{26}\} \approx Z_2 \oplus Z_2$ and $\pi_{34}^{19} = \{\rho_{19}\} \oplus \{\bar{\varepsilon}_{19}\} \approx Z_{32} \oplus Z_2$. Then (5.12) and (5.13) imply

$$(5.14) \quad E\pi_{33}^9 = \{\sigma_{10} \circ v_{17} \circ \kappa_{20}\} \oplus \{\sigma_{10} \circ \bar{\mu}_{17}\} \oplus \{\bar{\sigma}'_{10}\} \oplus \{\bar{\mu}_{10} \circ \sigma_{27}\} \oplus \{\delta_{10}\} \\ \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

On the other hand by (4.15) $\text{Ker}(\Delta: \pi_{34}^{19} \rightarrow \pi_{32}^9) = \{2\rho_{19}\} \oplus \{\bar{\varepsilon}_{19}\} \approx Z_{16} \oplus Z_2$. By (3.9) $H(\bar{\varepsilon}_{10}) = \bar{\varepsilon}_{19}$ and $2\bar{\varepsilon}_{10} = \sigma_{10} \circ v_{17} \circ \kappa_{20}$. So $\bar{\varepsilon}_{10}$ is of order 4. By Proposition 2.7 of [T] $H\Delta\rho_{21} = \pm 2\rho_{19}$. Apply Corollary 3.3 of [T] to the case $\alpha = \bar{\mu}_3$, $\beta = \sigma_8$, then

$$16\Delta\rho_{21} = \bar{\mu}_{10} \circ \sigma_{27} + \sigma_{10} \circ \bar{\mu}_{17},$$

which is non zero, whence $\Delta\rho_{21}$ is of order 32. Therefore we have

$$(5.15) \quad \pi_{34}^{10} = \{\delta_{10}\} \oplus \{\bar{\mu}_{10} \circ \sigma_{27}\} \oplus \{\bar{\sigma}'_{10}\} \oplus \{\bar{\varepsilon}_{10}\} \oplus \{\Delta\rho_{21}\} \\ \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_4 \oplus Z_{32}.$$

Consider the exact sequence

$$\pi_{36}^{21} \xrightarrow{\Delta} \pi_{34}^{10} \xrightarrow{E} \pi_{35}^{11} \xrightarrow{H} \pi_{35}^{21} \xrightarrow{\Delta} \pi_{33}^{10},$$

where $\pi_{36}^{21} = \{\rho_{21}\} \oplus \{\bar{\varepsilon}_{21}\}$. By (3.12) $\Delta(\bar{\varepsilon}_{21}) = \Delta H(\hat{\varepsilon}_{11}) = 0$. As is shown in the previous section $\Delta: \pi_{35}^{21} \rightarrow \pi_{33}^{10}$ is monic, whence $H: \pi_{35}^{11} \rightarrow \pi_{35}^{21}$ is trivial and $\pi_{35}^{11} = E\pi_{34}^{10}$. Therefore we have

$$(5.16) \quad \pi_{35}^{11} = \{\delta_{11}\} \oplus \{\bar{\mu}_{11} \circ \sigma_{28}\} \oplus \{\bar{\sigma}'_{11}\} \oplus \{\bar{\varepsilon}_{11}\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_4.$$

Next we show

$$(5.17) \quad \pi_{n+24}^n = \{\delta_n\} \oplus \{\bar{\mu}_n \circ \sigma_{n+17}\} \oplus \{\bar{\sigma}'_n\} \oplus \{\bar{\varepsilon}_n\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \\ \text{for } n = 12, 13.$$

Proof. Consider the exact sequences

$$\begin{aligned}\pi_{37}^{23} &\xrightarrow{A} \pi_{35}^{11} \xrightarrow{E} \pi_{36}^{12} \xrightarrow{H} \pi_{36}^{23} = 0, \\ 0 = \pi_{38}^{25} &\xrightarrow{A} \pi_{36}^{12} \xrightarrow{E} \pi_{37}^{13} \xrightarrow{H} \pi_{37}^{25} = 0,\end{aligned}$$

where $\pi_{37}^{23} = \{\sigma_{23}^2\} \oplus \{\kappa_{23}\} \approx Z_2 \oplus Z_2$. By Lemma 12.14 of [T] $\Delta(\sigma_{23}^2) = \Delta(\sigma_{23}) \circ \sigma_{28} = \Delta H(\xi_{12}) \circ \sigma_{28} = 0$. By (7.21) of [T] and (3.9) $\Delta(\kappa_{23}) = \Delta(\iota_{23}) \circ \kappa_{21} = \sigma_{11} \circ \nu_{18} \circ \kappa_{21} = 2\bar{e}_{11}$. Hence (5.17) follows immediately. q.e.d.

Now consider the exact sequence

$$0 = \pi_{39}^{27} \xrightarrow{A} \pi_{37}^{13} \xrightarrow{E} \pi_{38}^{14} \xrightarrow{H} \pi_{38}^{27} \xrightarrow{A} \pi_{36}^{13},$$

where $\pi_{38}^{27} = \{\zeta_{27}\} \approx Z_8$. By (3.10) $x\zeta^* - \zeta^{*'} \in E\pi_{37}^{13}$ for some odd integer x , moreover $2(x\zeta^* - \zeta^{*'}) = 2E\pi_{37}^{13} = 0$. Hence $8\zeta^* = 8\zeta^{*'} = 0$. Therefore we have

$$\begin{aligned}(5.18) \quad \pi_{38}^{14} &= \{\delta_{14}\} \oplus \{\bar{\mu}_{14} \circ \sigma_{31}\} \oplus \{\bar{\sigma}'_{14}\} \oplus \{\bar{e}_{14}\} \oplus \{\zeta^*\} \\ &\approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_8.\end{aligned}$$

We prove

$$\begin{aligned}(5.19) \quad 2E\zeta^* &= 0, \\ \Delta(\zeta_{29}) &= \pm 2\zeta^*.\end{aligned}$$

Proof. Note that $2x\zeta^* = 2\zeta^{*'}$ for some odd integer x . Then $2xE\zeta^* = 2\iota_{15} \circ E\zeta^{*'} \in -2\iota_{15} \circ \{\omega_{15}, 8\iota_{31}, 2\sigma_{31}\} = \{2\iota_{15}, \omega_{15}, 8\iota_{31}\} \circ 2\sigma_{32} = 0$, since $\{2\iota_{15}, \omega_{51}, 8\iota_{31}\} \subset \pi_{32}(S^{15}) = \pi_{32}^{15} = Z_2 \oplus Z_2 \oplus Z_2$. Hence $2\zeta^*$ is contained in $\text{Im}(\Delta: \pi_{40}^{29} \rightarrow \pi_{38}^{14})$, where $\pi_{40}^{29} = \{\zeta_{29}\} \approx Z_8$. Since $H(\zeta^*) = \zeta_{27}$ by (3.10), and since $H(\Delta\zeta_{29}) = \pm 2\zeta_{27}$ by Proposition 2.7 of [T], we must have $\Delta\zeta_{29} = \pm 2\zeta^*$. q.e.d.

Consider the exact sequences

$$\pi_{40}^{29} \xrightarrow{A} \pi_{38}^{14} \xrightarrow{E} \pi_{39}^{15} \xrightarrow{H} \pi_{39}^{29},$$

and

$$\pi_{41}^{31} \xrightarrow{A} \pi_{39}^{15} \xrightarrow{E} \pi_{40}^{16} \xrightarrow{H} \pi_{40}^{31},$$

where

$$\pi_{40}^{29} = \{\zeta_{29}\} \approx Z_8, \quad \pi_{39}^{29} = \{\eta_{29} \circ \mu_{30}\} \approx Z_2,$$

and

$$\pi_{41}^3 = \{\eta_{31} \circ \mu_{32}\} \approx Z_2.$$

By (3.11) we have $H(\mu^{*'}) = \eta_{29} \circ \mu_{30}$. To settle π_{39}^1 we show $2\mu^{*'} = 0$. In the second exact sequence $\Delta(\eta_{31} \circ \mu_{32}) = \Delta(\eta_{31}) \circ \mu_{30} = 0$ by (10.21) of [T] and so $E: \pi_{39}^1 \rightarrow \pi_{40}^0$ is monic. Hence we have $2\mu^{*'} = 0$, since $2E\mu^{*'} = \Delta(2\mu_{33}) = 0$ by (3.11). Therefore we have

$$(5.20) \quad \begin{aligned} \pi_{39}^1 &= \{\delta_{15}\} \oplus \{\bar{\mu}_{15} \circ \sigma_{32}\} \oplus \{\bar{\sigma}'_{15}\} \oplus \{\bar{\varepsilon}_{15}\} \oplus \{E\zeta^*\} \oplus \{\mu^{*'}\} \\ &\approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

Consider again the second exact sequence. Recall

$$\pi_{40}^1 = \{v_{31}^3\} \oplus \{\mu_{31}\} \oplus \{\eta_{31} \circ \varepsilon_{32}\} \approx Z_2 \oplus Z_2 \oplus Z_2.$$

By Lemma 6.3 and Lemma 12.14 of [T], $H(\eta_{16}^* \circ \bar{v}_{32}) = H(\eta_{16}^*) \circ \bar{v}_{32} = \eta_{31} \circ \bar{v}_{32} = v_{31}^3$. Similarly $H(\eta_{16}^* \circ \varepsilon_{32}) = \eta_{31} \circ \varepsilon_{32}$. By (3.11) $H(\mu^*) = \mu_{31}$ and $2\mu^* = 0$. Hence we have obviously

$$(5.21) \quad \begin{aligned} \pi_{40}^1 &= \{\delta_{16}\} \oplus \{\bar{\mu}_{16} \circ \sigma_{33}\} \oplus \{\bar{\sigma}'_{16}\} \oplus \{\bar{\varepsilon}_{16}\} \oplus \{E^2\zeta^*\} \oplus \{E\mu^{*'}\} \\ &\quad \oplus \{\mu_{16}^*\} \oplus \{\eta_{16}^* \circ \varepsilon_{32}\} \oplus \{\eta_{16}^* \circ \bar{v}_{32}\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \\ &\quad \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

Before determining π_{n+24}^n for $n \geq 17$, we prove some results on π_{n+24}^n for $n \geq 20$ to simplify the argument.

Apply Proposition 11.10, ii) of [T] to the case $\alpha = \iota_{46}$, $n = 23$, $i = 47$, then we have

$$(5.22) \quad \text{There exists an element } \tilde{\eta} \text{ of } \pi_{48}^{24} \text{ such that } H(\tilde{\eta}) = \eta_{47} \text{ and } \Delta(\iota_{51}) = E\tilde{\eta}.$$

Consider the exact sequence

$$\pi_{49}^{47} \xrightarrow{\Delta} \pi_{47}^{23} \xrightarrow{H} \pi_{48}^{24}, \quad \text{where} \quad \pi_{49}^{47} = \{\eta_{47}^2\} \approx Z_2.$$

Then $\Delta(\eta_{47}^2) = \Delta(\eta_{47}) \circ \eta_{46} = \Delta(H\tilde{\eta}) \circ \eta_{46} = 0$ by (5.22). Hence we have

$$(5.23) \quad E: \pi_{47}^{23} \longrightarrow \pi_{48}^{24} \text{ is monic.}$$

Next we apply Proposition 11.16 of [T] to the case $\alpha = \tilde{\eta} \in \pi_{48}^{24}$, $\beta = \eta_{44} \in \pi_{43}^{44}$, $n = 22$, $i = 46$, then we have

$$(5.24) \quad \text{There exists an element } \tilde{\eta}' \text{ of } \pi_{43}^{23} \text{ such that } \Delta(\eta_{49}) \pm 2\tilde{\eta} = E\tilde{\eta}' \text{ and } H(\tilde{\eta}') = \eta_{45}^2.$$

We also show

$$(5.25) \quad \begin{aligned} 2\tilde{\eta}' &= 2\tilde{\eta} = 0, \\ \Delta(\eta_{49}) &= E\tilde{\eta}'. \end{aligned}$$

Proof. Firstly we have $2E\tilde{\eta} = \Delta(2\iota_{51}) = \pm \Delta(H\Delta\iota_{53}) = 0$ by Proposition 2.7 of [T] and (5.22). Then $E^2\tilde{\eta}' = E(\Delta(\eta_{49}) \pm 2\tilde{\eta}) = 0$ by (5.24). Since $E: \pi_4^{23} \rightarrow \pi_8^{24}$ is monic by (5.23), $E\tilde{\eta}' \neq 0$. Hence $E\tilde{\eta}' \in \text{Im}(\Delta: \pi_5^{49} \rightarrow \pi_8^{24})$ and so $\Delta(\eta_{49}) = E\tilde{\eta}'$. However $\Delta(\eta_{49}) \pm 2\tilde{\eta} = E\tilde{\eta}'$ as is shown in (5.24), whence $2\tilde{\eta} = 0$. Moreover $2E\tilde{\eta}' = \Delta(2\eta_{49}) = 0$, whence $2\tilde{\eta}' = 0$ by (5.23). q. e. d.

$$(5.26) \quad \begin{aligned} H(\Delta v_{45}) &= \pm 2v_{43}, \\ 4\Delta v_{45} &= 0. \end{aligned}$$

Proof. The first one is immediate from Proposition 2.7 of [T]. By (5.5) of [T] and (5.24),

$$4\Delta v_{45} = \Delta(4v_{45}) = \Delta(\eta_{45}^3) = \Delta(\eta_{45}^2) \circ \eta_{45} = \Delta(H\tilde{\eta}') \circ \eta_{45} = 0.$$

q. e. d.

Now the following results are immediate:

$$(5.27) \quad \begin{aligned} \pi_4^{21} &= E\pi_{44}^{20}, \\ \pi_4^{22} &= E^2\pi_{44}^{20} \oplus \{\Delta v_{45}\} \approx E^2\pi_{44}^{22} \oplus Z_4, \\ \pi_4^{23} &= E^3\pi_{44}^{20} \oplus \{\tilde{\eta}'\} \approx E^3\pi_{44}^{20} \oplus Z_2, \\ \pi_4^{24} &= E^4\pi_{44}^{20} \oplus \{E\tilde{\eta}'\} \oplus \{\tilde{\eta}\} \approx E^4\pi_{44}^{20} \oplus Z_2 \oplus Z_2, \\ \pi_4^{25} &= E^5\pi_{44}^{20} \oplus \{E\tilde{\eta}\} \approx E^5\pi_{44}^{20} \oplus Z_2, \\ \pi_{n+24}^n &= E^{n-20}\pi_{44}^{20}, \\ \text{and also } E^n: \pi_{44}^{20} &\rightarrow \pi_{n+44}^{n+20} \text{ is monic for } n \geq 1. \end{aligned}$$

It remains to settle π_{n+24}^n for $n = 17, 18, 19, 20$.

$$(5.28) \quad \begin{aligned} \pi_{41}^{17} &= E^2\pi_{39}^{15} \oplus \{\eta_{17}^* \circ \bar{v}_{33}\} \oplus \{\eta_{17}^* \circ e_{33}\} \oplus \{\mu_{17}^*\} \\ &\approx E^2\pi_{36}^{15} \oplus Z_2 \oplus Z_2 \oplus Z_2. \end{aligned}$$

Proof. Since $E\pi_{49}^{30} = Z_2 \oplus Z_2 \oplus Z_2$, we have by Proposition 2.7 of [T]

$H\Delta\pi_{42}^{33} = H\Delta E^3\pi_{39}^{39} = H(\Delta\iota_{33})^\circ E\pi_{39}^{39} = \pm 2E\pi_{49}^{39} = 0$. Hence $\Delta\pi_{42}^{33} \subset E\pi_{39}^{39}$. Consider the exact sequence

$$\pi_{42}^{33} \xrightarrow{\Delta} \pi_{40}^{16} \xrightarrow{E} \pi_{41}^{17} \xrightarrow{H} \pi_{41}^{33} \xrightarrow{\Delta} \pi_{39}^{16}.$$

As is shown in the previous section $\Delta: \pi_{41}^{33} \rightarrow \pi_{39}^{16}$ is monic and so $H: \pi_{41}^{17} \rightarrow \pi_{41}^{33}$ is trivial. Hence the statement follows since $\pi_{41}^{17} = E\pi_{40}^{16}$. (Here note that $E^2\mu^{*'} = 0$ in $E^2\pi_{39}^{16}$ by (3.11)). q.e.d.

$$(5.29) \quad \begin{aligned} \Delta\bar{v}_{35} &\equiv \eta_{17}^* \circ \bar{v}_{33} \pmod{E^2\pi_{39}^{16}}, \\ \Delta\epsilon_{35} &\equiv \eta_{17}^* \circ \epsilon_{33} \pmod{E^2\pi_{39}^{16}}. \end{aligned}$$

Proof. Apply Proposition 11.10, ii) of [T] to the case $\alpha = \bar{v}_{30}$ (resp. ϵ_{30}), $n=15$, $i=39$, then we may take $\beta = \eta_{16}^* \circ \bar{v}_{32}$ (resp. $\eta_{16}^* \circ \epsilon_{32}$) $\pmod{E\pi_{39}^{16}}$ and we have $\Delta(\bar{v}_{35}) = \eta_{17}^* \circ \bar{v}_{33}$ (resp. $\Delta(\epsilon_{35}) = \eta_{17}^* \circ \epsilon_{33}$) $\pmod{E^2\pi_{39}^{16}}$. q.e.d.

Therefore we have

$$(5.30) \quad E\pi_{41}^{17} = E^3\pi_{39}^{16} \oplus Z_2 \approx E^3\pi_{39}^{16} \oplus \{\mu_{18}^*\}.$$

Next we show

$$(5.31) \quad 8\Delta\sigma_{37} \equiv \mu_{18}^* \pmod{E^3\pi_{39}^{16}}.$$

Proof. Apply Proposition 3.4 of [T] to the case $\alpha = \sigma_8$, $\beta = 2\sigma_8$, $\gamma = \mu_3$, then it follows that $\{\sigma_{19}, 2\sigma_{26}, \mu_{33}\}$ and $\{\mu_{19}, 2\sigma_{28}, \sigma_{25}\}$ have a common element. Observe that $\{\mu_{15}, 2\sigma_{24}, \sigma_{31}\}_1$ is well-defined. Then we can easily check that $\{\mu_{19}, 2\sigma_{28}, \sigma_{35}\} \subset E^4\pi_{39}^{16}$. The indeterminacy of $\{\sigma_{19}, 2\sigma_{26}, \mu_{33}\}$ is also contained in $E^4\pi_{39}^{16}$. Thus $\mu_{19}^* \in \{\sigma_{19}, 2\sigma_{26}, \mu_{33}\} \subset E^4\pi_{39}^{16}$. Therefore by (5.30) $\mu_{18}^* + c$ is in $\text{Im}(\Delta: \pi_{44}^{37} \rightarrow \pi_{42}^{18})$, where c is some element of $E^4\pi_{39}^{16}$. However since $H\Delta\sigma_{37} = \pm 2\sigma_{35}$ by Proposition 2.7 of [T], we must have $8\Delta\sigma_{37} \equiv \mu_{18}^* \pmod{E^3\pi_{39}^{16}}$. q.e.d.

$$(5.32) \quad \begin{aligned} v_{13} \circ \omega_{16} &= 0, \\ \bar{v}_{11} \circ \omega_{19} &= 0, \\ \omega_{19} \circ \bar{v}_{35} &\in E^{10}\pi_{33}^9. \end{aligned}$$

Proof. Since $v_{11} \circ \omega_{14} \equiv \lambda' \circ \eta_{29} \pmod{\xi' \circ \eta_{29}}$, we have $v_{13} \circ \omega_{16} = 0$ by Lemma 12.19 of [T].

$$\begin{aligned}
 \bar{v}_{11} \circ \omega_{19} &\in \{v_{11}, \eta_{14}, v_{15}\}_2 \circ \omega_{19} \subset \{v_{11}, \eta_{14}, v_{15} \circ \omega_{18}\}_2 \\
 &= v_{11} \circ E^2 \pi_{33}^1 = 0. \\
 \omega_{19} \circ \bar{v}_{35} &\in \{\sigma_{19}, 2\sigma_{26}, \eta_{33}\}_1 \circ \bar{v}_{35} \subset \{\sigma_{19}, 2\sigma_{26} \circ \bar{v}_{33}, \eta_{41}\}_1 \\
 &= \sigma_{19} \circ E \pi_{42}^5 + \pi_{42}^9 \circ \eta_{42}.
 \end{aligned}$$

Observe that $E \pi_{42}^5 = E^{10} \pi_{33}^6$. Now $\pi_{42}^9 = \{\bar{\rho}_{19}\} \oplus \{v_{19} \circ \bar{\kappa}_{22}\} \oplus \{\phi_{19}\} \oplus \{\psi_{19}\}$, then $\bar{\rho}_{19} \circ \eta_{42}$, $v_{19} \circ \bar{\kappa}_{22} \circ \eta_{42}$ and $\phi_{19} \circ \eta_{42}$ belong to $E^{10} \pi_{33}^9$. And by Proposition 3.1 of [T] $\psi_{19} \circ \eta_{42} = \eta_{19} \circ \psi_{20} \in E^{10} \pi_{33}^9$. Therefore we have $\omega_{19} \circ \bar{v}_{35} \in E^{10} \pi_{33}^9$.
q.e.d.

$$(5.33) \quad \tilde{\varepsilon} \equiv 0 \pmod{E^\infty \pi_{33}^9}.$$

Proof. $\tilde{\varepsilon} \in -\langle \sigma, \bar{\varepsilon}, \eta \rangle = \langle \eta, \bar{\varepsilon}, \sigma \rangle \pmod{\sigma \circ_2 \pi_{17}^5 + \eta \circ_2 \pi_{23}^5} \subset E^\infty \pi_{33}^9$. However $\{\eta_9, \bar{\varepsilon}_{10}, \sigma_{25}\}_1 \subset \pi_{33}^9$ is well defined, whence (5.33) follows.
q.e.d.

$$(5.34) \quad \Delta v_{39}^2 = \omega_{19} \circ \bar{v}_{35} \in E^{10} \pi_{33}^9.$$

Proof. Apply Corollary 3.3 of [T] to the case $\alpha = \omega_{14}$, $\beta = \bar{v}_6$, then from (5.32) this follows immediately.
q.e.d.

Now consider the following diagram

$$\begin{array}{ccccccc}
 \pi_{44}^{20} & \xleftarrow{E} & \pi_{43}^{19} & \xleftarrow{E} & \pi_{42}^{18} & \xleftarrow{E} & \pi_{41}^{17} & \xleftarrow{E} & \pi_{40}^{16} & \xleftarrow{E} & \pi_{39}^{15} \\
 & & \Delta \uparrow & & \Delta \uparrow & & \Delta \uparrow & & \Delta \uparrow & & \\
 & & \pi_{45}^{39} & & \pi_{44}^{37} & & \pi_{43}^{35} & & \pi_{42}^{33} & &
 \end{array}$$

where $\pi_{n+22}^{2n+1} \xrightarrow{\Delta} \pi_{n+24}^n \xrightarrow{E} \pi_{n+25}^{n+1}$ ($16 \leq n \leq 19$) is exact. Recall and notice (3.10), (3.11) and the argument mentioned above. Then the only possibility for the relation (5.33) to hold is that v_{33}^3 projects to $\tilde{\varepsilon}_{16} \pmod{E^7 \pi_{33}^9}$ through Δ i.e., $\Delta(v_{33}^3) \equiv \tilde{\varepsilon}_{16} \pmod{E^7 \pi_{33}^9}$. Now we have settled π_{41}^{17} , that is,

$$\begin{aligned}
 (5.35) \quad \pi_{41}^{17} &= \{\delta_{17}\} \oplus \{\bar{\mu}_{17} \circ \sigma_{34}\} \oplus \{\bar{\sigma}'_{17}\} \oplus \{\mu_{17}^*\} \oplus \{\eta_{17}^* \circ \varepsilon_{33}\} \oplus \{\eta_{17}^* \circ \bar{v}_{33}\} \\
 &\approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.
 \end{aligned}$$

By (5.29), (5.30) and (5.31) we have

$$(5.36) \quad \pi_{42}^{18} = \{\delta_{18}\} \oplus \{\bar{\mu}_{18} \circ \sigma_{35}\} \oplus \{\bar{\sigma}'_{18}\} \oplus \{\Delta \sigma_{37}\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_{16}.$$

So by (4.36) we have

$$(5.37) \quad \pi_{43}^1 = \{\delta_{19}\} \oplus \{\bar{\mu}_{19} \circ \sigma_{36}\} \oplus \{\bar{\sigma}'_{19}\} \approx Z_2 \oplus Z_2 \oplus Z_2.$$

We next show

$$(5.38) \quad \sigma'_{20} \equiv 0 \pmod{\sigma_{20} \circ \bar{\mu}_{27}, \delta_{20}}.$$

Proof. By virtue of (5.27), it is enough to show $\bar{\sigma}' \equiv 0 \pmod{\sigma \circ \bar{\mu}, \delta}$ in stable range. Since $\delta \in \langle \varepsilon, \eta \circ \sigma, \sigma \rangle$ and $\bar{\sigma}' \in \langle \bar{v}, \eta \circ \sigma, \sigma \rangle$, we have $\delta + \bar{\sigma}' \in \langle \varepsilon, \eta \circ \sigma, \sigma \rangle + \langle \bar{v}, \eta \circ \sigma, \sigma \rangle \subset \langle \varepsilon + \bar{v}, \eta \circ \sigma, \sigma \rangle = \langle \eta \circ \sigma, \eta \circ \sigma, \sigma \rangle \subset \eta \circ \langle \sigma, \eta \circ \sigma, \sigma \rangle$. Now $\langle \sigma, \eta \circ \sigma, \sigma \rangle$ and $\langle \eta \circ \sigma, 2\sigma, \sigma \rangle$ have a common element by (3.10) of [T], while $\langle \eta \circ \sigma, 2\sigma, \sigma \rangle \subset \eta \circ \langle \sigma, 2\sigma, \sigma \rangle = 0$ by p. 324 of [MII]. Then $\langle \eta \circ \sigma, 2\sigma, \sigma \rangle = \eta \circ \sigma \circ_2 \pi_{15}^S + \rho \circ_2 \pi_{16}^S = \{\sigma \circ \omega\}$ by [15.3], [15.4] of [MT], Lemma 8.2 of [MII], and (2.3). Hence $\eta \circ \langle \sigma, \eta \circ \sigma, \sigma \rangle$ contains $x\eta \circ \sigma \circ \omega$ for some integer x . However $\eta \circ \sigma \circ \omega \equiv 0 \pmod{\eta \circ \phi}$ by (4.22). Thus $\delta + \bar{\sigma}' \equiv 0 \pmod{\eta \circ \phi}$. The indeterminacy is $\varepsilon \circ_2 \pi_{16}^S + \bar{v} \circ_2 \pi_{16}^S + \sigma \circ_2 \pi_{17}^S \subset \{\eta \circ \phi, \bar{\mu} \circ \sigma\}$. For ${}_2\pi_{16}^S = \{\omega\} \oplus \{\mu \circ \sigma\}$, ${}_2\pi_{17}^S = \{\varepsilon^*\} \oplus \{\sigma \circ \mu \circ \eta\} \oplus \{v \circ \kappa\} \oplus \{\bar{\mu}\}$, by Lemma 6.4 of [T] and (4.22) and (5.32) $\varepsilon \circ \omega = (\eta \circ \sigma + \bar{v}) \circ \omega = \eta \circ \sigma \circ \omega \subset \{\eta \circ \phi\}$, by Lemma 10.7 of [T] and (5.32) $\varepsilon \circ \mu \circ \sigma = \bar{v} \circ \omega = \bar{v} \circ \mu \circ \sigma = 0$, by Lemma 12.15 of [T] $\sigma \circ \varepsilon^* = \sigma \circ \omega \circ \eta \subset \{\eta \circ \phi\}$, by (2.3) $\sigma^2 \circ \mu \circ \eta = 0$, by (7.20) of [T] $\sigma \circ v \circ \kappa = 0$. Now $\eta \circ \phi = E^\infty(\eta_4 \circ \phi_5) \subset E^\infty \pi_{28}^4 = \{\bar{\mu} \circ \sigma, \delta\}$. Hence we have $\bar{\sigma}' \equiv 0 \pmod{\sigma \circ \bar{\mu}, \delta}$. q.e.d.

Therefore the only possibility for the relation (5.38) to hold is

$$(5.39) \quad \Delta(v_{39}^2) \equiv \bar{\sigma}'_{19} \pmod{\sigma_{19} \circ \bar{\mu}_{19}, \delta_{19}}.$$

We now settle π_{n+24}^n for $n \geq 20$ by (5.27), which is given as follows.

$$(5.40) \quad \begin{aligned} \pi_{n+24}^n &= \{\delta_n\} \oplus \{\bar{\mu}_n \circ \sigma_{n+17}\} \approx Z_2 \oplus Z_2 \quad \text{for } n=20, 21, \\ \pi_{46}^2 &= \{\delta_{22}\} \oplus \{\bar{\mu}_{22} \circ \sigma_{39}\} \oplus \{\Delta v_{45}\} \approx Z_2 \oplus Z_2 \oplus Z_4, \\ \pi_{47}^3 &= \{\delta_{23}\} \oplus \{\bar{\mu}_{23} \circ \sigma_{40}\} \oplus \{\tilde{\eta}'\} \approx Z_2 \oplus Z_2 \oplus Z_2, \\ \pi_{48}^4 &= \{\delta_{24}\} \oplus \{\bar{\mu}_{24} \circ \sigma_{41}\} \oplus \{E\tilde{\eta}'\} \oplus \{\tilde{\eta}\} \approx Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2, \\ \pi_{49}^5 &= \{\delta_{25}\} \oplus \{\bar{\mu}_{25} \circ \sigma_{42}\} \oplus \{E\tilde{\eta}\} \approx Z_2 \oplus Z_2 \oplus Z_2, \\ \pi_{n+24}^n &= \{\delta_n\} \oplus \{\bar{\mu}_n \circ \sigma_{n+17}\} \approx Z_2 \oplus Z_2 \quad \text{for } n \geq 26. \end{aligned}$$

This completes the proof of Theorem 1.1.

§ 6. Some remarks on ${}_2\pi_{23}^S$ and ${}_2\pi_{24}^S$

In this section we add some remarks on the product structure of the stable homotopy groups ${}_2\pi_{23}^S$ and ${}_2\pi_{24}^S$, and prove Theorem 1.2.

$$(6.1) \quad \langle \varepsilon, \sigma, \bar{v} \rangle = \{\varepsilon \circ \omega\}.$$

Proof. Since $\{\varepsilon_5, \sigma_{13}, v_{20}\}_1$ is well defined in π_{24}^S by (7.20) and Lemma 10.7 of [T], and since $E^\infty \pi_{24}^S = \{\bar{\varepsilon}\} \approx Z_8$, we have $\langle \varepsilon, \sigma, v \rangle \ni y\bar{\varepsilon}$ for some integer y . However $\langle \varepsilon, \sigma, v \rangle$ is a coset of $\varepsilon \circ {}_2\pi_{11}^S + v \circ {}_2\pi_{16}^S = 0$ by Theorem 14.1 of [T] and (5.32), and so $\langle \varepsilon, \sigma, v \rangle = y\bar{\varepsilon}$. Then $y \equiv 0 \pmod{4}$, since $2 < \varepsilon, \sigma, v \rangle < 2\varepsilon$, $\sigma, v \rangle = v \circ {}_2\pi_{16}^S = 0$. Now by (3.7) of [T] $\langle \varepsilon, \sigma, \langle v, \eta, v \rangle \rangle + \langle \varepsilon, \langle \sigma, v, \eta \rangle, v \rangle + \langle \varepsilon, \sigma, v \rangle, \eta, v \rangle \ni 0$, while $\bar{v} = \langle v, \eta, v \rangle$, $\langle \sigma, v, \eta \rangle \subset {}_2\pi_{12}^S = 0$, and $\langle \varepsilon, \sigma, v \rangle, \eta, v \rangle = \langle 4y\bar{\varepsilon}, \eta, v \rangle \subset \langle 2y\bar{\varepsilon}, 2\eta, v \rangle = 2y\bar{\varepsilon} \circ {}_2\pi_{21}^S + v \circ {}_2\pi_{21}^S = 0$. Hence $\langle \varepsilon, \sigma, \bar{v} \rangle + \{\varepsilon \circ \omega\} \ni 0$. Thus $\langle \varepsilon, \sigma, \bar{v} \rangle = \{\varepsilon \circ \omega\}$ since this is a coset of $\varepsilon \circ {}_2\pi_{16}^S + \bar{v} \circ {}_2\pi_{16}^S = \{\varepsilon \circ \omega\}$. q.e.d.

$$(6.2) \quad \delta \equiv \varepsilon \circ \omega \pmod{\sigma \circ \bar{\mu}}.$$

Proof. $\delta \in \langle \varepsilon, \sigma \circ \eta, \sigma \rangle \subset \langle \varepsilon, \sigma, \eta \circ \sigma \rangle = \langle \varepsilon, \sigma, \varepsilon + \bar{v} \rangle \subset \langle \varepsilon, \sigma, \varepsilon \rangle + \langle \varepsilon, \sigma, \bar{v} \rangle$. Then by (3.10) of [T], $\langle \varepsilon, \sigma, \varepsilon \rangle$ and $\sigma \circ {}_2\pi_{17}^S$ have a common element, where $\sigma \circ {}_2\pi_{17}^S = \{\varepsilon \circ \omega, \sigma \circ \bar{\mu}\}$, since ${}_2\pi_{17}^S = \{\varepsilon^*\} \oplus \{\bar{\mu}\} \oplus \{\sigma \circ \eta \circ \mu\} \oplus \{v \circ \kappa\}$, and since by (7.20) of [T] $\sigma \circ v \circ \kappa = 0$, by (2.3) $\sigma^2 \circ \mu \circ \eta = 0$, by Lemmas 6.4 and 12.15 of [T] and (5.32) $\varepsilon^* \circ \sigma = \omega \circ \eta \circ \sigma = \omega \circ (\varepsilon + \bar{v}) = \omega \circ \varepsilon$. Since $\langle \varepsilon, \sigma, \varepsilon \rangle$ is a coset of $\varepsilon \circ {}_2\pi_{16}^S = \{\varepsilon \circ \omega\}$, $\langle \varepsilon, \sigma, \varepsilon \rangle \subset \{\varepsilon \circ \omega, \sigma \circ \bar{\mu}\}$. Therefore since $\langle \varepsilon, \sigma, \bar{v} \rangle = \{\varepsilon \circ \omega\}$ by (6.1), $\langle \varepsilon, \sigma, \varepsilon \rangle + \langle \varepsilon, \sigma, \bar{v} \rangle \subset \{\varepsilon \circ \omega, \sigma \circ \bar{\mu}\}$, whence for some integers x, y , $x\varepsilon \circ \omega + y\sigma \circ \bar{\mu} \in \langle \varepsilon, \sigma \circ \eta, \sigma \rangle \pmod{\{\varepsilon \circ \omega, \sigma \circ \bar{\mu}\}}$. Therefore $\delta \in \{\varepsilon \circ \omega, \sigma \circ \bar{\mu}\}$. But since as is calculated in § 5 ${}_2\pi_{24}^S = \{\delta\} \oplus \{\sigma \circ \bar{\mu}\} \approx Z_2 \oplus Z_2$, we must have $\delta \equiv \varepsilon \circ \omega \pmod{\sigma \circ \bar{\mu}}$. q.e.d.

$$(6.3) \quad \begin{aligned} \eta \circ \sigma \circ \omega &= \varepsilon \circ \omega = \eta \circ \phi, \\ \phi &\equiv \sigma \circ \omega \pmod{4v \circ \bar{\kappa}, 8\bar{\rho}}. \end{aligned}$$

Although J. Mukai has a direct proof from the unstable view point, our proof depends on the whole computations of 23 and 24-stems.

Proof. By (5.32) $\eta \circ \sigma \circ \omega = (\bar{v} + \varepsilon) \circ \omega = \varepsilon \circ \omega$. By (4.22) $\sigma \circ \omega = x\phi + 4y\phi \circ v \circ \bar{\kappa} + 8z\bar{\rho}$ for some integers x, y, z . Multiply by η and we have $\varepsilon \circ \omega = \eta \circ \sigma \circ \omega = x\eta \circ \phi$.

Since $\varepsilon \circ \omega \neq 0$ by (6.2) we have $x=1$. Hence $\sigma \circ \omega = \phi + 4yv\bar{\kappa} + 8z\bar{\rho}$ and this completes the proof of (6.3). q.e.d.

$$(6.4) \quad \langle \sigma^2, 2\iota, \varepsilon \rangle \ni \phi \pmod{4v\bar{\kappa}, 8\bar{\rho}}.$$

Proof. Since $2\langle \sigma^2, 2\iota, \varepsilon \rangle \subset \langle \sigma^2, 2\iota, 2\varepsilon \rangle = \sigma^2 \circ_2 \pi_2^S = 0$, we have $\langle \sigma^2, 2\iota, \varepsilon \rangle \in x\phi + 4yv\bar{\kappa} + 8z\bar{\rho}$. Then $\omega \circ \varepsilon = x\eta \circ \phi$, since by (4.19) $\eta \circ \langle \sigma^2, 2\iota, \varepsilon \rangle = \langle \eta, \sigma^2, 2\iota \rangle \circ \varepsilon \ni \omega \circ \varepsilon \pmod{\eta \circ (\sigma^2 \circ_2 \pi_2^S + \varepsilon \circ_2 \pi_1^S)} = 0$. Hence we must have $x=1$ and $\langle \sigma^2, 2\iota, \varepsilon \rangle \ni \phi \pmod{4v\bar{\kappa}, 8\bar{\rho}}$. q.e.d.

Summing up (6.1)–(6.4) we get a proof of Theorem 1.2.

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