A CERTAIN OPERATOR ALGEBRA IN AN INDEFINITE INNER PRODUCT SPACE

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§1. Introduction.

Recently, M. Tomita in [8] provided the concept of a Lorentz algebra, which is a Banach $*$ -algebra on a Hilbert space with a particular involution. On the other hand, about fifteen years ago, R. S. Phillips [5] studied the connection with algebraic commutative Lorentz algebras and their particular invariant subspaces in an indefinite inner product space.

In the present paper, we will study a Lorentz algebra with respect to an indefinite metric and obtain the necessary and sufficient conditions in order that a Lorentz algebra may be a C^* -algebra and also the sufficient condition for $*$ -semisimplicity of its algebra.

For general terms not defined in this paper, a reference is made to Rickart [7].

§2. Definition of Lorentz algebras and its involution.

We shall discuss Banach algebras with two involutions, therefore, we denote a Banach algebra $\mathfrak{A}$ with involution $*$ by $(\mathfrak{A}, *)$. And in this paper we assume that any Banach algebra has a unit.

Let $(\mathfrak{A}_0, *)$ be a C^* -algebra and $(\mathfrak{A}, \varphi)$ be a Banach subalgebra of $\mathfrak{A}_0$ which is a Banach $*$ -algebra under a certain involution φ and which contains the unit of $\mathfrak{A}_0$.

DEFINITION 2.1. $(\mathfrak{A}, \varphi)$ is said to be a *Lorentz algebra* with respect to $(\mathfrak{A}_0, *)$ if there exists a hermitian unitary element J in $(\mathfrak{A}_0, *)$ satisfying

$$A^\varphi = JA^*J$$

for all A in $\mathfrak{A}$. In particular, when we emphasize that J is associated with involution φ , $(\mathfrak{A}, \varphi)$ is denoted by $(\mathfrak{A}, \varphi; J)$.

Before giving some examples, we recall some concept of indefinite inner product space. Let $\mathcal{H}$ be a Hilbert space with a usual inner product (x, y) and its norm $\|x\| = (x, x)^{\frac{1}{2}}$ and $[x, y]_{\varphi}$ be a sesquilinear form on $\mathcal{H}$.

DEFINITION 2.2. $[x, y]_{\varphi}$ is said to be a *Minkowsky form* if it satisfies

$$[x, y]_{\varphi} = \overline{[y, x]_{\varphi}}$$

for all x, y in $\mathcal{H}$, and

$$\|x\| = \sup \{ |[x, y]_{\varphi}| : \|y\| \leq 1 \}$$

for all x in $\mathcal{H}$.

Furthermore, $[x, y]_{\varphi}$ is said to be *indefinite* if $[x, x]_{\varphi}$ (x in $\mathcal{H}$) takes value of opposite sign, and to be *definite*, otherwise. And $\mathcal{H}$ is said to be a *J-space* if it has the above Minkowsky form.

We can easily prove that a sesquilinear form $[x, y]_{\varphi}$ is a Minkowsky form if and only if there exists uniquely a hermitian unitary operator J on $\mathcal{H}$ satisfying

$$[x, y]_{\varphi} = (Jx, y)$$

For all x, y in $\mathcal{H}$.

LEMMA 2.3. The above Minkowsky form $[x, y]_{\varphi}$ is definite if and only if $J=1$ or -1 .

Proof. By symmetry, we can assume that $[x, x]_{\varphi} \geq 0$ for any x in $\mathcal{H}$. Then from the positiveness of J , we have

$$J = (J^2)^{\frac{1}{2}} = 1.$$

EXAMPLE 2.4. Let $\mathcal{H}$ be the above J -space and $\mathfrak{A}_0$ be the set of all bounded operators on $\mathcal{H}$. And let $\mathfrak{A}$ be any fixed Banach subalgebra of $\mathfrak{A}_0$ with involution φ defined by the adjoint operation with respect to the above Minkowsky form. Then $(\mathfrak{A}, \varphi)$ becomes a Lorentz algebra. In fact, from the above facts, we have

$$\begin{aligned} [Ax, y]_{\varphi} &= (JAx, y) = (x, A^*Jy) \\ &= (Jx, JA^*Jy) = [x, JA^*Jy]_{\varphi} \end{aligned}$$

for all x, y in $\mathcal{H}$, therefore we have $A^{\varphi} = JA^*J$ for all A in $\mathfrak{A}$. In this case, we

simply call this algebra the Lorentz algebra on $\mathcal{H}$. Thus this algebra has close connections with the structure of the underlying J -space. We shall consider its connection in section 3.

EXAMPLE 2.5. We consider a C^* -algebra

$$\mathfrak{A}_\circ = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathfrak{B} \right\} \quad \text{on } \mathcal{H} \oplus \mathcal{H}$$

where $\mathfrak{B}$ is a C^* -algebra on $\mathcal{H}$ with involution $*$, and we put

$$\mathfrak{A} = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} : A, B \in \mathfrak{B} \right\},$$

furthermore we define its involution as follows;

$$X^\varphi = JX^*J$$

for all X in $\mathfrak{A}$, where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{A}_\circ$.

Then we know that $(\mathfrak{A}, \varphi)$ is a Lorentz algebra with respect to $(\mathfrak{A}_\circ, *)$ and can easily prove from semisimplicity of $\mathfrak{B}$ that $\mathfrak{A}$ is semisimple if and only if

$$\mathfrak{A} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in \mathfrak{B} \right\}.$$

In particular, under the above example, let δ be a hermitian derivation of C^* -algebra $\mathfrak{B}$, that is,

$$\delta(A)^* = \delta(A^*)$$

for any A in $\mathfrak{B}$. And we put

$$\mathfrak{A} = \left\{ \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix} : A \in \mathfrak{B} \right\}$$

Then it is not difficult to show that $(\mathfrak{A}, \varphi)$ is a hermitian semisimple (that is, $*$ -semisimple) Lorentz algebra with respect to $(\mathfrak{A}_\circ, *)$.

Now we discuss some involution properties of Lorentz algebras. We denote the set of all unitary elements in a Banach algebra $(\mathfrak{A}, \Psi)$ by $U(\mathfrak{A}, \Psi)$.

PROPOSITION 2.6. (M. TOMITA) *Under the conditions of Definition 2.1, we assume that a Lorentz algebra $(\mathfrak{A}, \varphi; J)$ is closed under involution $*$ of $(\mathfrak{A}_\circ, *)$. Suppose that A belongs to $U(\mathfrak{A}, \varphi)$, then we have a unique decomposition as follows;*

$$A = BC$$

where B in $U(\mathfrak{A}, \varphi) \cap (\mathfrak{A}_0, *)^+$,

C in $U(\mathfrak{A}, \varphi) \cap U(\mathfrak{A}_0, *)$, (the positive cone of $(\mathfrak{A}_0, *)$ is denoted by $(\mathfrak{A}_0, *)^+$.)

Proof. Let A be a unitary element of $(\mathfrak{A}, \varphi)$. Then we have

$$(AA^*)^\varphi = JAJ \cdot JA^*J = (A^*)^{-1}A^{-1} = (AA^*)^{-1},$$

that is,

$$AA^* \text{ in } U(\mathfrak{A}, \varphi) \cap (\mathfrak{A}_0, *)^+.$$

Since $(\mathfrak{A}, \varphi)$ is a C^* -subalgebra with involution $*$ of $\mathfrak{A}_0$ by our assumption and we use a general spectral theory, we have

$$(AA^*)^{\frac{1}{2}} \in \mathfrak{A}, \text{ and } J(AA^*)^{\frac{1}{2}}J = (AA^*)^{-\frac{1}{2}}.$$

Put

$$B = (AA^*)^{\frac{1}{2}}, \text{ and } C = B^{-1}A,$$

then we easily have

$$B, B^{-1} \in U(\mathfrak{A}, \varphi) \cap (\mathfrak{A}_0, *)^+, \quad C \in U(\mathfrak{A}, \varphi).$$

Therefore, we obtain

$$C^*C = A^*B^{-1}B^{-1}A = A^*(AA^*)^{-1}A = A^*(A^*)^{-1}A^{-1}A = 1,$$

similarly, $CC^* = 1$.

Thus we get the decomposition of Proposition 2.6, on the other hand, it is not difficult to prove the uniqueness of its decomposition. This completes the proof.

Next we ask under what conditions a Lorentz algebra becomes a C^* -algebra.

PROPOSITION 2.7. *Let $(\mathfrak{A}_0, *)$ be a C^* -algebra and $(\mathfrak{A}, \varphi; J)$ be a Lorentz algebra with respect to $(\mathfrak{A}_0, *)$. Then the following conditions are mutually equivalent.*

- 1° $(\mathfrak{A}, \varphi)$ is a C^* -algebra.
- 2° For every A in $\mathfrak{A}$, we have $A^\varphi = A^*$.
- 3° Every element of $\mathfrak{A}$ commutes with J .

4° $\mathfrak{A}$ is closed under involution $*$ of $\mathfrak{A}_0$ and $(\mathfrak{A}, \varphi)$ is hermitean.

5° $\mathfrak{A}$ is closed under involution $*$ of $\mathfrak{A}_0$ and

$$U(\mathfrak{A}, \varphi) \cap (\mathfrak{A}_0, *)^+ \subset \{X \in \mathfrak{A}: \|X\| = 1\}.$$

Proof.

1° $\Rightarrow$ 2° Considering the embedding map of $(\mathfrak{A}, \varphi)$ into $(\mathfrak{A}_0, *)$, this implication is obvious by [2, Lemma 8].

2° $\Rightarrow$ 3° Let A be any element of $\mathfrak{A}$. Then from our assumption we have

$$A^* = JA^*J, \text{ namely, } JA = AJ.$$

3° $\Rightarrow$ 4° and 4° $\Rightarrow$ 5° They are obvious.

5° $\Rightarrow$ 1° Let A be any unitary element of $(\mathfrak{A}, \varphi)$.

By Proposition 2.6, A is decomposed as follows;

$$A = BC,$$

where $B \in U(\mathfrak{A}, \varphi) \cap (\mathfrak{A}_0, *)^+$, $C \in U(\mathfrak{A}, \varphi) \cap U(\mathfrak{A}_0, *)$.

Then, by our assumption, we have

$$\|A\| = \|BC\| = \|B\| = 1,$$

that is to say,

$$U(\mathfrak{A}, \varphi) \subset \{X \in \mathfrak{A}: \|X\| = 1\}.$$

This fact implies the condition 1° from the result of T. W. PALMER [4]. This completes the proof.

COROLLARY 2.8. *Suppose that a Lorentz algebra $(\mathfrak{A}, \varphi; J)$ is weakly dense in $(\mathfrak{A}_0, *) = \mathfrak{B}(\mathcal{H})$. Then for $(\mathfrak{A}, \varphi; J)$ to be a C^* -algebra it is necessary and sufficient that J -space $\mathcal{H}$ defined by J is definite.*

Proof. The commutator $\mathfrak{A}'$ of $\mathfrak{A}$ is the scalar algebra and J belongs to $\mathfrak{A}'$. Then we have the Corollary.

§3. Lorentz algebras and invariant subspaces.

Let $\mathcal{H}$ be a J -space with a Minkowsky form $[x, y]_\varphi$. Then, from the section 2, there exists a hermitian unitary operator J on $\mathcal{H}$ associated with its Minkowsky form such that

$$[x, y]_{\varphi} = (Jx, y)$$

for any $x, y \in \mathcal{H}$. We recall some of the basic definitions (see, [1, 3]). A closed subspace $\mathfrak{M}$ is said to be *J-nonnegative* (*J-positive*) if

$$[x, x]_{\varphi} \geq 0 \quad ([x, x]_{\varphi} > 0)$$

for arbitrary $x \in \mathfrak{M}$ (for arbitrary $x \in \mathfrak{M}, x \neq 0$) respectively. $\mathfrak{M}$ is said to be *J-uniformly positive* if there exists a constant $\gamma > 0$ such that

$$[x, x]_{\varphi} \geq \gamma(x, x)$$

for all $x \in \mathfrak{M}$. Similarly, the concepts of *J-nonpositive*, *J-negative* and *J-uniformly negative* closed subspaces are introduced analogously. Furthermore, a closed subspace is said to be *J-strongly definite* (*J-uniformly definite*) if it is *J-positive* or *J-negative* (*J-uniformly positive* or *J-uniformly negative*) respectively, and a *J-nonnegative* closed subspace is said to be *maximal* if it is not a proper part of any other *J-nonnegative* closed subspace of $\mathcal{H}$.

Let $\mathfrak{M}$ be a closed subspace. The totality of $x \in \mathcal{H}$ such that $[x, y]_{\varphi} = 0$ for all $y \in \mathfrak{M}$ will be called the *J-orthogonal complement* of $\mathfrak{M}$ and be denoted by $\mathfrak{M}^{\perp}$. Then we easily see the next lemma.

LEMMA 3.1. Suppose that $\mathfrak{M}$ is invariant for a bounded linear operator T on a *J-space* $\mathcal{H}$. Then $\mathfrak{M}^{\perp}$ is invariant for $T^{\varphi} = JT^*J$.

Let $(\mathfrak{A}, \varphi, J)$ be a Lorentz algebra on the above *J-space*. We easily see that $J = p_+ - p_-$, where p_+ and p_- are orthogonal complementary projections on $\mathcal{H}$. Put $p_+\mathcal{H} = \mathcal{H}_+$ and $p_-\mathcal{H} = \mathcal{H}_-$. Then we have $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (direct sum).

THEOREM 3.2. Let $(\mathfrak{A}, \varphi, J)$ be a Lorentz algebra on the above *J-space* $\mathcal{H}$. Then it is a *C*-algebra* if and only if $\mathcal{H}_+$ and $\mathcal{H}_-$ are its invariant subspaces.

Proof. By Proposition 2.7, we can easily prove that $(\mathfrak{A}, \varphi, J)$ is a *C*-algebra* if and only if

$$\mathfrak{A} \cong \left\{ \begin{pmatrix} p_+ x p_+ & 0 \\ 0 & p_- x p_- \end{pmatrix} : x \in \mathfrak{A} \right\} \quad \text{w.r.t. } \mathcal{H}_+, \mathcal{H}_-.$$

Therefore this theorem follows.

REMARK. In Theorem 3.2, $\mathcal{H}_+(\mathcal{H}_-)$ is a maximal *J-uniformly positive*

(maximal J -uniformly negative) respectively. The general case that a Lorentz algebra has a maximal J -uniformly invariant subspace will be treated in Theorem 3.5.

LEMMA 3.3 *Suppose that there exists a J -strongly definite closed subspace $\mathfrak{M}$ which is invariant under $\mathfrak{A}$. Then $*$ -radical R^* of $(\mathfrak{A}, \varphi)$ is contained in the set such that each element of the set vanishes on $\mathfrak{M}$.*

Proof. By symmetry, we have only to prove the assertion for the case of $\mathfrak{M}$ to be J -positive. For arbitrary fixed $\zeta \in \mathfrak{M}$, we define the linear functional on $\mathfrak{A}$ as follows;

$$f_{\zeta}(X) = [X\zeta, \zeta]_{\varphi}$$

for all X in $\mathfrak{A}$. Then f_{ζ} is a positive linear functional on $(\mathfrak{A}, \varphi)$, because of our assumption that $\mathfrak{M}$ is $\mathfrak{A}$ -invariant and J -positive. Let X be an element of R^* . Then, since for any positive linear functional f we have

$$f(X^{\varphi}X) = 0,$$

we have

$$[X\zeta, X\zeta]_{\varphi} = 0$$

for any $\zeta \in \mathfrak{M}$. On the other hand, $\mathfrak{M}$ is J -positive and so we obtain $X\zeta = 0$ for all ζ in $\mathfrak{M}$. This lemma is proved.

THEOREM 3.4. *Suppose that there exist two J -strongly definite closed subspaces such that both of them are invariant under $\mathfrak{A}$ and $\mathcal{H}$ is the closed linear span of them. Then $(\mathfrak{A}, \varphi; J)$ becomes $*$ -semisimple. In particular, if there exists a maximal J -strongly definite closed subspace which is invariant under $\mathfrak{A}$, then $(\mathfrak{A}, \varphi)$ is $*$ -semisimple.*

Proof. The first assertion is clear from Lemma 3.3. Let $\mathfrak{N}$ be a maximal J -strongly definite $\mathfrak{A}$ -invariant. We can assume that $\mathfrak{N}$ is J -positive by symmetry. Then, from the results of YU. P. GINZBURG and I. S. IOKIVIDOV [1], $\mathfrak{N}^{[\perp]}$ is J -negative, and from the J -positiveness of $\mathfrak{N}$ it is not difficult to show that $\mathfrak{N} \oplus \mathfrak{N}^{[\perp]}$ (direct sum) is dense in $\mathcal{H}$. On the other hand, $\mathfrak{N}^{[\perp]}$ is also invariant under $\mathfrak{A}$ by Lemma 3.1. Hence it follows from the first assertion that $(\mathfrak{A}, \varphi)$ is $*$ -semisimple. The theorem is proved.

THEOREM 3.5. *Suppose that there exist J -uniformly positive closed subspace*

and J -uniformly negative closed subspace such that both of them are invariant under $\mathfrak{A}$ and $\mathcal{H}$ is the direct sum of them. Then $(\mathfrak{A}, \varphi; J)$ becomes a C^* -equivalent algebra. In particular, if there exists a maximal J -uniformly definite closed subspace which is invariant under $\mathfrak{A}$, then $(\mathfrak{A}, \varphi; J)$ is a C^* -equivalent algebra.

Proof. From the proof of [3, Theorem 7], it is easily seen that there exists a constant C such that

$$\|X\| \leq C$$

for any X in $U(\mathfrak{A}, \varphi; J)$. Hence the first assertion follows from [6, Theorem 8.4]. Next let $\mathfrak{M}$ be a maximal J -uniformly positive (we can assume by symmetry) which is invariant under $\mathfrak{A}$. Then by [1], J -orthogonal complement is a maximal J -negative $\mathfrak{A}$ -invariant closed subspace, and we have

$$\mathcal{H} = \mathfrak{M} [+] \mathfrak{M}^{[\perp]} \quad (J\text{-orthogonal, direct sum})$$

Hence the second assertion follows from the analogous argument of the previous theorem. We have proved the theorem.

REMARK AND PROBLEM. Is the converse of Theorem 3.5 true?, that is to say, “for a C^* -equivalent Lorentz algebra on a J -space $\mathcal{H}$, does there exist its maximal J -uniformly positive invariant subspace?”. In the commutative case this answer is affirmative by [5, Theorem 6.1].

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