ON THE UNIQUE EXISTENCE OF GLOBAL HOLOMORPHIC SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS WITH A PARAMETER IN A STEIN MANIFOLD

By

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(Received April 23, 1975)

1. Introduction.

Let X be a Stein manifold, $\mathbf{C}^n$ be the space of n complex variables $z_1, z_2, \dots, z_{n-1}$ and z_n , Ω be a Stein subdomain of the product manifold $\mathbf{C}^n \times X$ of $\mathbf{C}^n$ and X . Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions in Ω , m be a positive integer and $a_{jk}^i(z, x)$ be holomorphic functions in Ω for $i=1, 2, \dots, n$ and $j, k=1, 2, \dots, m$. We define homomorphisms T_i, S_j and S of $\mathcal{O}^m$ in $\mathcal{O}^m$ for $i, j=1, 2, \dots, n$ by putting

$$(1) \quad T_i u = \left(\frac{\partial u_1}{\partial z_i} + \sum_{k=1}^m a_{1k}^i(z, x) u_k, \frac{\partial u_2}{\partial z_i} + \sum_{k=1}^m a_{2k}^i u_k, \dots, \right. \\ \left. \frac{\partial u_m}{\partial z_i} + \sum_{k=1}^m a_{mk}^i(z, x) u_k \right)$$

$$(2) \quad S_1 u = T_n u, S_j u = T_{n-j+1}(S_{j-1} u) \quad (j=1, 2, \dots, n), S = S_n$$

for $u = (u_1, u_2, \dots, u_m) \in \mathcal{O}^m$.

Kajiwaru [1] obtained the necessary and sufficient condition that for any $v \in H^0(\Omega, \mathcal{O}^m)$ there exists $u \in H^0(\Omega, \mathcal{O}^m)$ satisfying the inhomogeneous equation $Su = v$ in case that $n=1$ and X consists of a single point. Suzuki [3] obtained the condition in case that $m=n=1$ and $Su = \partial u / \partial z$. Kajiwaru-Mori [2] obtained the condition in case that $n=1$. In the present paper we shall study the condition under the assumption that for any $i=1, 2, \dots, n$ there exists a point $(z_1, z_2, \dots, z_n, x)$ of Ω such that one of the connected components of $(\{(z_1, \dots, z_{i-1})\} \times \mathbf{C} \times \{(z_{i+1}, \dots, z_n, x)\}) \cap \Omega$ is not simply connected. In this case the existence of global holomorphic solution u of $Su = v$ is unique. The main methods are based on the above Kajiwaru-Mori [2].

2. The results of Kajiwara-Mori [2].

Let Ω be a Stein subdomain of the product manifold of $\mathbb{C} \times X$ of the complex plane $\mathbb{C}$ and a Stein manifold X and p be the projection of $\mathbb{C} \times X$ onto X . Let $a_{jk}(z, x)$ be holomorphic functions in Ω . We define a homomorphism T of $\mathcal{O}^m$ in $\mathcal{O}^m$ by putting

$$(3) \quad Tu = \left(\frac{\partial u_1}{\partial z} + \sum_{j=1}^m a_{1j}(z, x)u_j, \frac{\partial u_2}{\partial z} + \sum_{j=1}^m a_{2j}(z, x)u_j, \dots, \right. \\ \left. \frac{\partial u_m}{\partial z} + \sum_{j=1}^m a_{mj}(z, x)u_j \right)$$

for $u = (u_1, u_2, \dots, u_m) \in \mathcal{O}^m$. Let $\mathcal{A}$ be the kernel of T . The short exact sequence

$$(4) \quad 0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}^m \xrightarrow{T} \mathcal{O}^m \longrightarrow 0$$

of sheaves over Ω gives the long exact sequence

$$(5) \quad \dots \longrightarrow H^0(\Omega, \mathcal{O}^m) \longrightarrow H^0(\Omega, \mathcal{O}^m) \longrightarrow H^1(\Omega, \mathcal{A}) \longrightarrow H^1(\Omega, \mathcal{O}^m) \longrightarrow \dots$$

of cohomology groups. Since Ω is a Stein manifold, we have $H^1(\Omega, \mathcal{O}^m) = 0$. Hence we have

$$(6) \quad H^1(\Omega, \mathcal{A}) = H^0(\Omega, \mathcal{O}^m) / TH^0(\Omega, \mathcal{O}^m)$$

by (5). In other words, the necessary and sufficient condition that for any $v \in H^0(\Omega, \mathcal{O}^m)$ there exists $u \in H^0(\Omega, \mathcal{O}^m)$ satisfying the inhomogeneous equation

$$(7) \quad Tu = v$$

is that there holds

$$(8) \quad H^1(\Omega, \mathcal{A}) = 0.$$

For any point (z, x) of Ω we put

$$(9) \quad \Omega(z, x) = \text{the connected component of } (\mathbb{C} \times \{x\}) \cap \Omega \\ \text{containing } (z, x).$$

We say that two points (z, x) and (z', x') of Ω satisfy $(z, x) \sim (z', x')$ if and only if $x = x'$ and $\Omega(z, x) = \Omega(z', x')$. Then $\sim$ is an equivalence relation in Ω . Let $\tilde{\Omega}$ be the factor space of Ω by the equivalence relation $\sim$. Any point $\tilde{x}$ of $\tilde{\Omega}$

is an equivalence class in Ω . Let (z, x) be a representative of the class $\tilde{x}$. Then the subset $\Omega(z, x)$ of Ω coincides with the equivalence class $\tilde{x}$. Therefore we have

$$(10) \quad \tilde{\Omega} = \{\Omega(z, x); (z, x) \in \Omega\}.$$

We induce a factor topology of Ω in the factor space $\tilde{\Omega}$ of Ω . We put

$$(11) \quad \tilde{\Omega}_s = \text{the set of all simply connected } \Omega(z, x) \\ \text{for points } (z, x) \text{ of } \Omega$$

and

$$(12) \quad \tilde{\Omega}_d = \text{the set of all doubly connected } \Omega(z, x) \\ \text{for points } (z, x) \text{ of } \Omega.$$

THEOREM 1 (Kajiware-Mori [2]). *If $H^1(\Omega, \mathcal{A})=0$ then either $\tilde{\Omega}=\tilde{\Omega}_s$ or $\tilde{\Omega}=\tilde{\Omega}_d$. In case that $\tilde{\Omega}=\tilde{\Omega}_d$, the necessary and sufficient condition that $H^1(\Omega, \mathcal{A})=0$ is that $\tilde{\Omega}$ is a Hausdorff space and $H^0(\Omega(z, x), \mathcal{A})=0$ for any point (z, x) of Ω . In case that $\tilde{\Omega}=\tilde{\Omega}_s$, if $\tilde{\Omega}$ is a Hausdorff space and $\tilde{\Omega}$ is a Stein manifold for the natural complex structure in $\tilde{\Omega}$, we have $H^1(\Omega, \mathcal{A})=0$. In case that $\tilde{\Omega}=\tilde{\Omega}_s$, under the assumption (A) given in Kajiware-Mori [2] if $H^1(\Omega, \mathcal{A})=0$, $\tilde{\Omega}$ is a Hausdorff space and $\tilde{\Omega}$ is a Stein manifold for the natural complex structure in $\tilde{\Omega}$.*

3. The unique existence of a global holomorphic solution.

Let Ω be a Stein subdomain of the product manifold $\mathbb{C}^n \times X$ of $\mathbb{C}^n$ and a Stein manifold X . For any point $(z, x)=(z_1, z_2, \dots, z_n, x)$ of Ω , we put

$$(13) \quad \Omega_i(z, x) = \text{the connected component of } (\{(z_1, \dots, z_{i-1})\} \\ \times \mathbb{C} \times \{(z_{i+1}, \dots, z_n, x)\}) \cap \Omega \text{ containing } (z, x).$$

We induce an equivalence relations $\sim(i)$ in Ω for $i=1, 2, \dots, n$. We say that two points $(z, x)=(z_1, z_2, \dots, z_n, x)$ and $(z', x')=(z'_1, z'_2, \dots, z'_n, x')$ of Ω satisfy $(z, x) \sim(i) (z', x')$ if and only if $z_k = z'_k$ for $k=1, 2, \dots, i-1, i+1, \dots, n$, $x = x'$ and $\Omega_i(z, x) = \Omega_i(z', x')$. Let $\tilde{\Omega}_i$ be the factor space of Ω by the equivalence relations $\sim(i)$. Then we have

$$(14) \quad \tilde{\Omega}_i = \{\Omega_i(z, x); (z, x) \in \Omega\} \quad (i=1, 2, \dots, n).$$

In the factor spaces $\tilde{\Omega}_i$ of Ω we induce a factor topology of Ω .

Let $\mathcal{A}_{T_i}$, $\mathcal{A}_{S_j}$ and $\mathcal{A}_S$ be the kernels of T_i , S_j and S respectively. By the same method for (6), we have

$$(15) \quad H^1(\Omega, \mathcal{A}_{T_i}) = H^0(\Omega, \mathcal{O}^m) / T_i H^0(\Omega, \mathcal{O}^m)$$

for $i=1, 2, \dots, n$ and

$$(16) \quad H^1(\Omega, \mathcal{A}_{S_j}) = H^0(\Omega, \mathcal{O}^m) / S_j H^0(\Omega, \mathcal{O}^m)$$

for $j=1, 2, \dots, n$. By (16) the necessary and sufficient condition that for any $v \in H^0(\Omega, \mathcal{O}^m)$ there exists $u \in H^0(\Omega, \mathcal{O}^m)$ satisfying the inhomogeneous equation $Su=v$ is that there holds

$$(17) \quad H^1(\Omega, \mathcal{A}_S) = 0.$$

LEMMA 1. Assume that there exists a point (z, x) of Ω such that $\Omega_1(z, x)$ is not simply connected and that $H^1(\Omega, \mathcal{A}_S) = 0$. Then $H^1(\Omega, \mathcal{A}_{T_1}) = 0$, $\Omega_1(z, x)$ is doubly connected for any $(z, x) \in \Omega$, $H^0(\Omega_1(z, x), \mathcal{A}_{T_1}) = 0$ and the solution $u \in H^0(\Omega, \mathcal{O}^m)$ of $T_1 u = v$ is unique for any $v \in H^0(\Omega, \mathcal{O}^m)$.

PROOF. Let v be any element of $H^0(\Omega, \mathcal{O}^m)$. Since $H^1(\Omega, \mathcal{A}_S) = 0$, we have $H^0(\Omega, \mathcal{O}^m) = SH^0(\Omega, \mathcal{O}^m)$ by (16). Hence there exists an element u of $H^0(\Omega, \mathcal{O}^m)$ such that $Su = v$. Let $u' = S_{n-1}u$, then $T_1 u' = v$. Thus we have proved $H^0(\Omega, \mathcal{O}^m) = T_1 H^0(\Omega, \mathcal{O}^m)$. By (16) $H^1(\Omega, \mathcal{A}_{T_1}) = 0$. The last part of Lemma 1 is a consequence of Theorem 1.

LEMMA 2. Under the assumption of Lemma 1, if $H^1(\Omega, \mathcal{A}_S) = 0$, then we have $H^1(\Omega, \mathcal{A}_{S_{n-1}}) = 0$

PROOF. Let v be any element of $H^0(\Omega, \mathcal{O}^m)$. Since $T_1 v$ is an element of $H^0(\Omega, \mathcal{O}^m)$ and $H^0(\Omega, \mathcal{O}^m) = SH^0(\Omega, \mathcal{O}^m)$, there exists an element u of $H^0(\Omega, \mathcal{O}^m)$ such that $Su = T_1(S_{n-1}u) = T_1 v$. Hence we have $T_1(S_{n-1}u - v) = 0$. By Lemma 1, we have $S_{n-1}u = v$. Thus we have obtained $H^1(\Omega, \mathcal{A}_{S_{n-1}}) = 0$.

LEMMA 3. Assume that for any $i=1, 2, \dots, n$ there exists a point (z, x) of Ω such that $\Omega_i(z, x)$ is not simply connected and that $H^1(\Omega, \mathcal{A}_S) = 0$. Then $H^1(\Omega, \mathcal{A}_{T_i}) = 0$, $\Omega_i(z, x)$ is doubly connected, $\tilde{\Omega}_i$ is a Hausdorff space, $H^0(\Omega, \mathcal{A}_{T_i}) = 0$ and for any $v \in H^0(\Omega, \mathcal{O}^m)$ the solution $u \in H^0(\Omega, \mathcal{O}^m)$ of $T_i u = v$ is unique for $i=1, 2, \dots, n$.

PROOF. By Lemma 1 the statement is true for $i=1$. Assume that it is

true for $i=1, 2, \dots, k$. Then we have $H^1(\Omega, \mathcal{A}_{S_{n-k}})=0$ by Lemma 2. Since $T_{k+1}(S_{n-k-1}u)=S_{n-k}u$ for $u \in H^0(\Omega, \mathcal{O}^m)$, $H^1(\Omega, \mathcal{A}_{T_{k+1}})=0$. Hence for each $i=1, 2, \dots, n$ the statement is true.

LEMMA 4. If $H^1(\Omega, \mathcal{A}_{T_i})=0$ for each $i=1, 2, \dots, n$, then we have $H^1(\Omega, \mathcal{A}_S)=0$.

PROOF. Since $S_1=T_n$, $H^1(\Omega, \mathcal{A}_{S_1})=0$. Assume that $H^1(\Omega, \mathcal{A}_{S_j})=0$. Since $H^1(\Omega, \mathcal{A}_{T_{n-j}})=0$, for any $v \in H^0(\Omega, \mathcal{O}^m)$ there exists $u \in H^0(\Omega, \mathcal{O}^m)$ such that $T_{n-j}u=v$. By the assumption there exists $u' \in H^0(\Omega, \mathcal{O}^m)$ such that $S_ju'=u$. Therefore $S_{j+1}u'=T_{n-j}(S_ju')=T_{n-j}u=v$. Hence we have obtained $H^1(\Omega, \mathcal{A}_{S_{j+1}})=0$. By induction $H^0(\Omega, \mathcal{A}_S)=H^0(\Omega, \mathcal{A}_{S_n})=0$.

4. Statement of the theorem.

THEOREM 2. Let X be a Stein manifold and Ω be a Stein subdomain of $\mathbb{C}^n \times X$. For $i=1, 2, \dots, n$ and any point $(z, x)=(z_1, z_2, \dots, z_n, x)$ of Ω let $\Omega_i(z, x)$ be the connected component of $(\{(z_1, \dots, z_{i-1})\} \times \mathbb{C} \times \{(z_{i+1}, \dots, z_n, x)\}) \cap \Omega$ containing (z, x) . Let $\tilde{\Omega}_i$ be the set of all $\Omega_i(z, x)$ for $(z, x) \in \Omega$. In the factor set $\tilde{\Omega}_i$ of Ω we induce a factor topology of Ω . Let $\mathcal{O}$ be the sheaf of all germs of holomorphic functions in Ω , m be a positive integer and $a_{jk}^i(z, x)$ be holomorphic functions in Ω for $i=1, 2, \dots, n$ and $j, k=1, 2, \dots, m$. Let $\mathcal{A}_{T_i}$ be the kernel of the homomorphism T_i of $\mathcal{O}^m$ in $\mathcal{O}^m$ defined by

$$T_i u = \left(\frac{\partial u_1}{\partial z_i} + \sum_{k=1}^m a_{1k}^i(z, x) u_k, \frac{\partial u_2}{\partial z_i} + \sum_{k=1}^m a_{2k}^i(z, x) u_k, \dots, \frac{\partial u_m}{\partial z_i} + \sum_{k=1}^m a_{mk}^i(z, x) u_k \right)$$

for $u=(u_1, u_2, \dots, u_m) \in \mathcal{O}^m$. Let $\mathcal{A}_S$ be the kernel of the homomorphism of $\mathcal{O}^m$ in $\mathcal{O}^m$ defined by

$$Su = T_1(T_2(\dots(T_n u) \dots))$$

for $u \in \mathcal{O}^m$.

Assume that for any $i=1, 2, \dots, n$ there exists a point (z, x) of Ω such that $\Omega_i(z, x)$ is not simply connected. Then the necessary and sufficient condition that $H^1(\Omega, \mathcal{A}_S)=0$ is that $\Omega_i(z, x)$ is doubly connected for any point (z, x) of Ω , $H^0(\Omega_i(z, x), \mathcal{A}_{T_i})=0$ for any point (z, x) of Ω and $\tilde{\Omega}_i$ is a Hausdorff space for each $i=1, 2, \dots, n$.

References

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