INTERPOLATION, II⁽¹⁾

By

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1. Introduction. It is well known that there exists a trigonometrical polynomial at most of n -th order taking the given values at $2n+1$ points in $[0, 2\pi]$. Recently LOZINSKI⁽²⁾ has considered an interpolating sum of order n $L_n(x)$ which takes the values $F_n(x_\nu)$ at $x_\nu = 2\nu\pi/(2n+1)$, $\nu=0, 1, \dots, 2n$, where

$$(1.1) \quad F_n(x) = \frac{1}{2\delta_n} \int_{x-\delta_n}^{x+\delta_n} f(u) du, \quad \delta_n = \frac{\pi}{2n+1}$$

and $f(x)$ is an integrable function in $(0, 2\pi)$, and he discussed the mean convergence of $L_n(x)$ to $f(x)$ and proved that $L_n(x)$ behaves like the partial sum of the Fourier series of $f(x)$ so far as the mean convergence concerns. In the present paper a convergence property will be discussed.

Let $f(x)$ be integrable and let $S_n(x)$ be the partial sum of the Fourier series of $f(x)$, and further let $t_n(x)$ be the n -th partial sum of the Fourier series of $F_n(x)$ in (1.1). First in §2 we shall discuss the relation between $s_n(x)$ and $t_n(x)$.

2. A relation between $s_n(x)$ and $t_n(x)$.

Theorem 1. If

$$(2.1) \quad \int_0^t |f(x+u) - s| du = o(t),$$

then

$$\lim_{n \rightarrow \infty} \left(\frac{\pi}{2} t_n(x) - s_n(x) \right) = \left(\frac{\pi}{2} - 1 \right) s.$$

(1) G. MARUYAMA and T. KAWATA, *Interpolation*, I, Mem. Fac. Sc. Kyūsyū Imp. Univ., Ser. A, 2 (1942), 205-215.

(2) S. M. LOZINSKI, *Über Interpolation*, Rec. Math. Moscou, 8 (1940), 57-67.

Proof. As is well known, we have

$$(2.2) \quad s_n(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x+t) \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin \frac{1}{2}t} dt,$$

$$(2.3) \quad t_n(x) = \frac{1}{2\pi} \int_0^{2\pi} F_n(x+t) \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin \frac{1}{2}t} dt.$$

Thus we have

$$(2.4) \quad \begin{aligned} & \frac{\pi}{2} t_n(x) - s_n(x) - \left(\frac{\pi}{2} - 1\right)s = \frac{\pi}{2}(t_n(x) - s) - (s_n(x) - s) \\ &= \frac{1}{4} \int_0^{2\pi} \{F_n(x+t) - s\} D_n(t) dt - \frac{1}{2\pi} \int_0^{2\pi} \{f(x+t) - s\} D_n(t) dt \\ &= t_n^{(1)}(x) - s_n^{(1)}(x), \end{aligned}$$

where

$$D_n(t) = \sin\left(n + \frac{1}{2}\right)t / \sin \frac{1}{2}t.$$

Putting $f(x+t) - s = \varphi(t)$, we have

$$(2.5) \quad \begin{aligned} t_n^{(1)}(x) &= \frac{1}{4} \int_0^{2\pi} D_n(t) dt \cdot \frac{1}{2\delta_n} \int_{t-\delta_n}^{t+\delta_n} \varphi(u) du \\ &= \frac{1}{8\delta_n} \int_{-\delta_n}^{\delta_n} \varphi(u) du \int_0^{u+\delta_n} D_n(t) dt + \frac{1}{8\delta_n} \int_{\delta_n}^{2\delta_n} \varphi(u) du \int_{u-\delta_n}^{u+\delta_n} D_n(t) dt \\ &\quad + \frac{1}{8\delta_n} \int_{2\delta_n}^{2\pi-\delta_n} \varphi(u) du \int_{u-\delta_n}^{u+\delta_n} D_n(t) dt + \frac{1}{8\delta_n} \int_{2\pi-\delta_n}^{2\pi} \varphi(u) du \int_{u-\delta_n}^{2\pi} D_n(t) dt \\ &= K_1 + K_2 + K_3 + K_4, \end{aligned}$$

say. Since $D_n(t) = O(t)$ for every t and $\delta_n = O(n)$, we have, by (2.1),

$$(2.6) \quad |K_1| = O\left(n^2 \int_{-\delta_n}^{\delta_n} |\varphi(u)| du \int_0^{u+\delta_n} dt\right) = O\left(n \int_{-\delta_n}^{\delta_n} |\varphi(u)| du\right) = o(1).$$

Similarly we obtain

$$(2.7) \quad |K_2| = o(1), \quad |K_4| = o(1).$$

Hence putting (2.6) and (2.7) into (2.5) we get

$$(2.8) \quad t_n^{(1)}(x) = K_3 + o(1).$$

Now

$$(2.9) \quad \begin{aligned} s_n^{(1)}(x) &= \frac{1}{2\pi} \int_0^{2\delta_n} \varphi(t) D_n(t) dt + \frac{1}{2\pi} \int_{2\delta_n}^{2\pi-\delta_n} \varphi(t) D_n(t) dt + \frac{1}{2\pi} \int_{2\pi-\delta_n}^{2\pi} \varphi(t) D_n(t) dt \\ &= K_5 + K_6 + K_7, \end{aligned}$$

say. Since $D_n(t) = O(1/t)$, we have

$$\begin{aligned} |K_5| &= O\left(n \int_0^{\delta_n} |\varphi(t)| dt\right) = o(1), \\ |K_7| &= O\left(\int_{2\pi-\delta_n}^{2\pi} \frac{|\varphi(t)|}{t} dt\right) = O\left(\int_{2\pi-\delta_n}^{2\pi} |\varphi(t)| dt\right) = o(1), \end{aligned}$$

and

$$(2.10) \quad s_n^{(1)}(x) = K_6 + o(1).$$

By (2.8), (2.10) and (2.4)

$$(2.11) \quad \begin{aligned} \frac{\pi}{2} t_n(x) - s_n(x) - \left(\frac{\pi}{2} - 1\right)s &= K_3 - K_6 + o(1) \\ &= \frac{1}{2\pi} \int_{2\delta_n}^{2\pi-\delta_n} \varphi(u) du \left(\frac{\pi}{4\delta_n} \int_{u-\delta_n}^{u+\delta_n} D_n(t) dt - D_n(u) \right) + o(1). \end{aligned}$$

Now by integration by parts we have

$$(2.12) \quad \begin{aligned} \frac{\pi}{4\delta_n} \int_{u-\delta_n}^{u+\delta_n} D_n(t) dt &= \frac{\pi}{4\delta_n} \left[-\frac{\cos\left(n + \frac{1}{2}\right)t}{\left(n + \frac{1}{2}\right) \sin \frac{t}{2}} \right]_{u-\delta_n}^{u+\delta_n} \\ &\quad - \frac{\pi}{8\delta_n} \int_{u-\delta_n}^{u+\delta_n} \frac{\cos\left(n + \frac{1}{2}\right)t \cos \frac{t}{2}}{\left(n + \frac{1}{2}\right) \sin^2 \frac{t}{2}} dt = K_8 + K_9, \end{aligned}$$

say. We have

$$|K_9| = O\left(\frac{1}{\delta_n} \int_{u-\delta_n}^{u+\delta_n} \frac{dt}{nt^2}\right) = O\left(\frac{1}{n(u-\delta_n)^2}\right).$$

Thus

$$\begin{aligned} \frac{1}{2\pi} \int_{2\delta_n}^{2\pi-\delta_n} \varphi(u) K_9 du &= O\left(\frac{1}{n} \int_{2\delta_n}^{2\pi-\delta_n} \frac{|\varphi(u)|}{u^2} du\right) \\ &= O\left(\frac{1}{n} \left[\frac{\Phi(u)}{u^2} \right]_{2\delta_n}^{2\pi-\delta_n} + \frac{1}{n} \int_{2\delta_n}^{2\pi-\delta_n} \frac{|\Phi(u)|}{u^3} du\right) = o(1), \end{aligned}$$

where we put

$$\Phi(u) = \int_0^u |\varphi(t)| dt.$$

Hence we get by (2.11)

$$(2.13) \quad \begin{aligned} & \frac{\pi}{2} t_n(x) - s_n(x) - \left(\frac{\pi}{2} - 1\right) s \\ &= \frac{1}{2\pi} \int_{2\delta_n}^{2\pi - \delta_n} \varphi(u) [K_8 - D_n(u)] du + o(1). \end{aligned}$$

Taking into account

$$\begin{aligned} K_8 &= \frac{1}{2} \left[\frac{\cos\left(n + \frac{1}{2}\right) \left(u - \frac{\pi}{2n+1}\right)}{\sin \frac{u - \delta_n}{2}} - \frac{\cos\left(n + \frac{1}{2}\right) \left(u + \frac{\pi}{2n+1}\right)}{\sin \frac{u + \delta_n}{2}} \right] \\ &= \frac{1}{2} \sin\left(n + \frac{1}{2}\right) u \cdot \left(\frac{1}{\sin(u - \delta_n)/2} + \frac{1}{\sin(u + \delta_n)/2} \right), \end{aligned}$$

we can estimate the left-hand member of (2.13) as follows:

$$\begin{aligned} & \frac{\pi}{2} t_n(x) - s_n(x) - \left(\frac{\pi}{2} - 1\right) s \\ &= \frac{1}{2\pi} \int_{2\delta_n}^{2\pi - \delta_n} \varphi(u) \sin\left(n + \frac{1}{2}\right) u \\ & \quad \times \left\{ \frac{1}{2} \left(\frac{1}{\sin(u - \delta_n)/2} + \frac{1}{\sin(u + \delta_n)/2} \right) - \frac{1}{\sin u/2} \right\} du \\ &= \frac{1}{2\pi} \int_{2\delta_n}^{2\pi - \delta_n} \varphi(u) \sin\left(n + \frac{1}{2}\right) u \cdot \left(\frac{1}{u - \delta_n} + \frac{1}{u + \delta_n} - \frac{2}{u} \right) du + o(1) \\ &= O \left(\int_{2\delta_n}^{2\pi - \delta_n} |\varphi(u)| \frac{\delta_n^2}{u(u^2 - \delta_n^2)} du \right) = O \left(\frac{1}{n^2} \int_{2\delta_n}^{2\pi} \frac{|\varphi(u)|}{u^3} du \right) \\ &= O \left(\frac{1}{n^2} \left[\frac{\Phi(u)}{u^3} \right]_{2\delta_n}^{2\pi} + 3 \frac{1}{n^2} \int_{2\delta_n}^{2\pi} \frac{\Phi(u)}{u^4} du \right) \\ &= o(1) + o \left(\frac{1}{n^2} \int_{2\delta_n}^{\pi} \frac{du}{u^3} \right) = o(1). \end{aligned} \quad \text{Q. E. D.}$$

By Theorem 1, we have immediately the following theorems.

Theorem 2. If the relation (2.1) holds, then the convergency of $s_n(x)$ implies that of $t_n(x)$ and vice versa.

Theorem 3. If

$$(2.14) \quad \int_0^t |f(x+u) - s|^p du = o(t), \quad p > 1,$$

then

$$(2.15) \quad \sum_{n=1}^N |t_n(x) - s|^k = o(N)$$

for $k > 0$.

Proof. We have

$$\begin{aligned} \sum_{n=1}^N |t_n(x) - s|^k &= \sum_{n=1}^N \left| t_n(x) - \frac{2}{\pi} s_n(x) - \frac{\pi-2}{\pi} s + \frac{2}{\pi} s_n(x) - \frac{2}{\pi} s \right|^k \\ &\leq A_k \sum_{n=1}^N \left| t_n(x) - \frac{2}{\pi} s_n(x) - \frac{\pi-2}{\pi} s \right|^k + A_k \sum_{n=1}^N |s_n(x) - s|^k, \end{aligned}$$

which is, by Theorem 1 (the condition (2.1) is a consequence of (2.14)),

$$= o(N) + O\left(\sum_{n=1}^N |s_n(x) - s|^k\right),$$

of which the latter term is $O(N)$ by the known theorem concerning the strong summability of Fourier series.

Theorem 4. If

$$f(x) \in L_p \quad (p \geq 1),$$

then

$$\int_0^{2\pi} \left| \frac{\pi}{2} t_n(x) - s_n(x) \right|^p dx \leq A_p \int_0^{2\pi} |f(x)|^p dx,$$

where A_p does not depend on $f(x)$.

Proof. In the proof of Theorem 1 we can easily see, putting $s=0$,

$$\begin{aligned} &\left| \frac{\pi}{2} t_n(x) - s_n(x) \right| \\ &\leq A \left(n \int_{-2\delta_n}^{2\delta_n} |f(x+t)| dt + \frac{1}{n^2} \int_{2\delta_n}^{2\pi-\delta_n} |f(x+t)| \frac{dt}{t^3} + \frac{1}{n} \int_{2\delta_n}^{2\pi-\delta_n} |f(x+t)| \frac{dt}{t^2} \right). \end{aligned}$$

Thus for $p > 1$, ($1/p + 1/q = 1$), we obtain

$$\int_0^{2\pi} \left| \frac{\pi}{2} t_n(x) - s_n(x) \right|^p dx$$

$$\begin{aligned}
&\leq A_p \left(\int_0^{2\pi} n^p dx \int_{-2\delta_n}^{2\delta_n} |f(x+t)|^p dt (4\delta_n)^{\frac{p}{q}} \right) \\
&+ A_p \left(n^{-2p} \int_0^{2\pi} dx \left(\int_{2\delta_n}^{2\pi} \frac{|f(x+u)|}{u^3} du \right)^p \right) \\
&+ A_p \left(n^{-p} \int_0^{2\pi} dx \left(\int_{2\delta_n}^{2\pi} \frac{|f(x+u)|}{u^2} du \right)^p \right) \\
&\leq A_p n \int_{-2\delta_n}^{2\delta_n} dt \int_0^{2\pi} |f(x+t)|^p dx \\
&+ A_p n^{-2p} \int_0^{2\pi} dx \left(\int_{2\delta_n}^{2\pi} \frac{|f(x+u)|}{u^{\frac{3}{p}}} \cdot \frac{1}{u^{\frac{3}{q}}} du \right)^p \\
&+ A_p n^{-p} \int_0^{2\pi} dx \left(\int_{2\delta_n}^{2\pi} \frac{|f(x+u)|}{u^{\frac{2}{p}}} \cdot \frac{1}{u^{\frac{2}{q}}} du \right)^p \\
&\leq A_p \int_0^{2\pi} |f(x)|^p dx + A_p n^{-2p} \int_0^{2\pi} |f(x)|^p dx \left(\int_{2\delta_n}^{2\pi} \frac{du}{u^3} \right)^{1+\frac{p}{q}} \\
&+ A_p n^{-p} \int_0^{2\pi} |f(x)|^p dx \left(\int_{2\delta_n}^{2\pi} \frac{du}{u^2} \right)^{1+\frac{p}{q}} \leq A_p \int_0^{2\pi} |f(x)|^p dx.
\end{aligned}$$

The case $p=1$ will more easily be proved.

From the above theorem and the known fact concerning mean convergence of Fourier series, we have the

Theorem 5.

- (i) If $f(x) \in L_p$, $p > 1$, then $\int_0^{2\pi} |t_n(x) - f(x)|^p dx \rightarrow 0$;
- (ii) if $|f(x)| \log^+ |f(x)| \in L$, then $\int_0^{2\pi} |t_n(x) - f(x)| dx \rightarrow 0$;
- (iii) if $f(x) \in L$, then $\int_0^{2\pi} |t_n(x) - f(x)|^r dx \rightarrow 0$ ($0 < r < 1$).

3. In this section we discuss the convergence property of LOZINSKI's interpolating sum $L_n(x)$.

Theorem 6. If $f(x) \in L$ and $f(x)$ satisfies the LEBESGUE condition

$$(3.1) \quad \int_0^t |\varphi(u)| d\omega(t),$$

$$(3.2) \quad \int_{\eta}^{2\pi} \frac{|\varphi(u) - \varphi(u+\eta)|}{u} du = o(1) \quad \text{as } \eta \rightarrow 0,$$

then $L_n(x)$ converges to s , where $\varphi(u) = f(x+u) - s$.

Proof. As is well known we can write $L_n(x)$ in the form

$$(3.3) \quad L_n(x) = \frac{1}{2\pi} \int_0^{2\pi} F_n(t) D_n(x-t) d\varphi_n(t),$$

where $F_n(t)$ is the one in (1.1) and

$$(3.4) \quad \varphi_n(t) = \frac{2\pi}{2n+1} \quad \text{for } k \frac{2\pi}{2n+1} \leq t < (k+1) \frac{2\pi}{2n+1}, \\ k=0, 1, \dots, 2n-1.$$

Putting $\psi_n(t) = \varphi_n(t+x)$, we have

$$(3.5) \quad L_n(x) - s = \frac{1}{2\pi} \int_{t-\delta_n}^{2\pi} \frac{1}{2\delta_n} \int_{t-\delta_n}^{t+\delta_n} (f(u) - s) du D_n(x-t) d\varphi_n(t) \\ = \frac{1}{4\pi\delta_n} \int_{-\delta_n}^{\delta_n} \varphi(u) du \int_0^{u+\delta_n} D_n(t) d\psi_n(t) \\ + \frac{1}{4\pi\delta_n} \int_{\delta_n}^{2\delta_n} \varphi(u) du \int_{u-\delta_n}^{u+\delta_n} D_n(t) d\psi_n(t) + \frac{1}{4\pi\delta_n} \int_{2\delta_n}^{2\pi-\delta_n} \varphi(u) du \int_{u-\delta_n}^{u+\delta_n} D_n(t) d\psi_n(t) \\ + \frac{1}{4\pi\delta_n} \int_{2\pi-\delta_n}^{2\pi+\delta_n} \varphi(u) du \int_{u-\delta_n}^{2\pi} D_n(t) d\psi_n(t) \\ = K_{10} + K_{11} + K_{12} + K_{13},$$

say. Since $D_n(t) = O(n)$ for all t , if $0 < u < \delta_n$, we have

$$\left| \int_0^{u+\delta_n} D_n(t) d\psi_n(t) \right| \leq O\left(n \int_0^{u+\delta_n} d\psi_n(t)\right) = O(1), \\ \left| \int_{u-\delta_n}^{u+\delta_n} D_n(t) d\psi_n(t) \right| = O(1).$$

Hence by (3.1) we have

$$(3.6) \quad |K_{10}| = o(1), \quad |K_{11}| = o(1).$$

Similarly we get

$$(3.7) \quad |K_{13}| \leq O\left(n \int_{2\pi-\delta_n}^{2\pi+\delta_n} |\varphi(u)| du \int_{u-\delta_n}^{2\pi} \frac{d\psi_n(t)}{t}\right) = o(1).$$

For K_{12} , we obtain the estimate

$$\begin{aligned}
 K_{12} &= \frac{1}{4\pi\delta_n} \int_{2\delta_n}^{2\pi-\delta_n} \varphi(u) du \int_{u-\delta_n}^{u+\delta_n} D_n(t) d\psi_n(t) \\
 &= \frac{1}{4\pi\delta_n} \int_0^{2\pi-3\delta_n} \varphi(u+2\delta_n) du \int_{u-\delta_n}^{u+\delta_n} D_n(t+2\delta_n) d\psi_n(t) \\
 &= \frac{1}{8\pi\delta_n} \left(\int_{2\delta_n}^{2\pi-\delta_n} \varphi(u) du \int_{u-\delta_n}^{u+\delta_n} D_n(t) d\psi_n(t) \right. \\
 &\quad \left. + \int_0^{2\pi-3\delta_n} \varphi(u+2\delta_n) du \int_{u-\delta_n}^{u+\delta_n} D_n(t+2\delta_n) d\psi_n(t) \right) \\
 &= \frac{1}{8\pi\delta_n} \left(\int_{2\delta_n}^{2\pi-3\delta_n} \varphi(u) du \int_{u-\delta_n}^{u+\delta_n} D_n(t) d\psi_n(t) + \int_{2\pi-3\delta_n}^{2\pi-\delta_n} \varphi(u) du \int_{u-\delta_n}^{u+\delta_n} D_n(t) d\psi_n(t) \right. \\
 &\quad \left. + \int_0^{2\delta_n} \varphi(u+2\delta_n) du \int_{u-\delta_n}^{u+\delta_n} D_n(t+2\delta_n) d\psi_n(t) \right. \\
 &\quad \left. + \int_{2\delta_n}^{2\pi-3\delta_n} \varphi(u+2\delta_n) du \int_{u-\delta_n}^{u+\delta_n} D_n(t+2\delta_n) d\psi_n(t) \right) \\
 (3.8) \quad &= K_{14} + K_{15} + K_{16} + K_{17},
 \end{aligned}$$

say. We have as in (3.6)

$$\begin{aligned}
 (3.9) \quad &|K_{15}| = o(1), \quad |K_{16}| = o(1), \\
 K_{14} + K_{17} &= \frac{1}{8\pi\delta_n} \int_{2\delta_n}^{2\pi-3\delta_n} \{\varphi(u) - \varphi(u+2\delta_n)\} du \int_{u-\delta_n}^{u+\delta_n} D_n(t) d\psi_n(t) \\
 &\quad + \frac{1}{8\pi\delta_n} \int_{2\delta_n}^{2\pi-3\delta_n} \varphi(u+2\delta_n) du \int_{u-\delta_n}^{u+\delta_n} (D_n(t) + D_n(t+2\delta_n)) d\psi_n(t) \\
 (3.10) \quad &= K_{18} + K_{19},
 \end{aligned}$$

say. Since

$$\begin{aligned}
 (3.11) \quad D_n(t) + D_n(t+2\delta_n) &= \sin\left(n + \frac{1}{2}\right)t \left(\frac{1}{\sin t/2} - \frac{1}{\sin(t+2\delta_n)/2} \right) \\
 &= O\left(\frac{1}{nt^2}\right) \quad \text{for } t > 2\delta_n,
 \end{aligned}$$

we have by (3.1)

$$\begin{aligned}
K_{19} &= O\left(\int_{2\delta_n}^{2\pi-3\delta_n} |\varphi(u+2\delta_n)| du \int_{u-\delta_n}^{u+\delta_n} \frac{d\phi_n(t)}{t^2}\right) \\
&= O\left(\frac{1}{n} \int_{2\delta_n}^{2\pi} \frac{|\varphi(u)|}{u^2} du\right) = o(1).
\end{aligned}$$

Combining above estimates we get

$$(3.12) \quad L_n(x) - s = K_{18} + o(1).$$

Now since

$$D_n(t) = O(1/t),$$

we have

$$\begin{aligned}
K_{18} &= O\left(n \int_{2\delta_n}^{2\pi} |\varphi(u) - \varphi(u+2\delta_n)| du \int_{u-\delta_n}^{u+\delta_n} \frac{d\phi_n(t)}{t}\right) \\
&= O\left(\int_{2\delta_n}^{2\pi} \frac{|\varphi(u) - \varphi(u+2\delta_n)|}{u} du\right)
\end{aligned}$$

which is $o(1)$ by (3.2). Thus the proof is complete.