A CONSTRUCTION OF CLOSED CATEGORIES RELATED TO k -SPACES

By

Yasuo KAWAHARA*) and Tatsuji KUDO

(Received Oct. 27, 1975)

The purpose of this note is to construct a cartesian closed category $\mathcal{SC}$ starting from an initial data $\{\mathcal{A}, \mathcal{C}, \mathcal{S}, k\}$ consisting of a category $\mathcal{A}$, a subcategory $\mathcal{C}$ of $\mathcal{A}$ with $Ob(\mathcal{C}) = Ob(\mathcal{A})$, a full subcategory $\mathcal{S}$ of $\mathcal{C}$, and a functor $k: \mathcal{SC}^{op} \times \mathcal{SC} \rightarrow \mathcal{A}$, where $\mathcal{SC}$ is a subcategory of $\mathcal{A}$ consisting of all $\mathcal{S}$ -maps, i.e., morphisms $f: X \rightarrow Y$ of $\mathcal{A}$ such that $f\alpha \in \mathcal{C}$ for all morphisms $\alpha: A \rightarrow X$ with $A \in \mathcal{S}$ and $\alpha \in \mathcal{C}$. In the special case where $\mathcal{A}$ is the category of topological spaces and (set) maps and $\mathcal{C}$ the category of topological spaces and continuous maps, the category $\mathcal{SC}$ is the category of topological spaces and k -continuous maps in the sense of R. M. Vogt [3]. It is clear that $\mathcal{A} \supset \mathcal{SC} \supset \mathcal{C} \supset \mathcal{S}$.

The initial data $\{\mathcal{A}, \mathcal{C}, \mathcal{S}, k\}$ must subject to several axioms which will be given separately in suitable places.

AXIOM (A.1) $\mathcal{A}$ has finite products.

AXIOM (A.2) If $(p, q)h = (f, g)$, $h \in \mathcal{C}$ if and only if $f \in \mathcal{C}$ and $g \in \mathcal{C}$, where $p: X \times Y \rightarrow X$, $q: X \times Y \rightarrow Y$ are projections and $f: Z \rightarrow X$, $g: Z \rightarrow Y$, $h: Z \rightarrow X \times Y$ are morphisms of $\mathcal{A}$.

PROPOSITION 1. In the same situation as in (A.2), $h \in \mathcal{SC}$ if and only if $f \in \mathcal{SC}$ and $g \in \mathcal{SC}$. In particular, if $f \in \mathcal{SC}$ and $g \in \mathcal{SC}$, then $h = \langle f, g \rangle \in \mathcal{SC}$.

PROPOSITION 2. If $f \in \mathcal{SC}$ and $g \in \mathcal{SC}$, then $f \times g \in \mathcal{SC}$, where $f: X \rightarrow Z$, $g: Y \rightarrow W$, and $f \times g: X \times Y \rightarrow Z \times W$ is given by $f \times g = \langle fp, gq \rangle$, p and q being projections.

*) Yamaguchi University.

COROLLARY 3. *Products in $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{SC}$ coincide.*

Proofs of these propositions are obvious.

AXIOM (B). *Functors $\mathbf{L}, \mathbf{M}, \mathbf{N}: \mathcal{SC}^{op} \times \mathcal{SC}^{op} \times \mathcal{SC} \longrightarrow \mathcal{Set}$ being defined by $\mathbf{L}(X, Y, Z) = \mathcal{SC}(X \times Y, Z)$, $\mathbf{M}(X, Y, Z) = \mathcal{A}(X, k(Y, Z))$, $\mathbf{N}(X, Y, Z) = \mathcal{A}(X \times Y, Z)$, there are given two natural transformations $\phi: \mathbf{L} \longrightarrow \mathbf{M}$, $\psi: \mathbf{M} \longrightarrow \mathbf{N}$ such that $\psi(X, Y, Z)\phi(X, Y, Z): \mathbf{L}(X, Y, Z) \longrightarrow \mathbf{N}(X, Y, Z)$ is an inclusion map $\iota(X, Y, Z): \mathcal{SC}(X \times Y, Z) \longrightarrow \mathcal{A}(X \times Y, Z)$, and $\phi(X, Y, Z)\iota(X, Y, Z)^{-1}\psi(X, Y, Z): \psi(X, Y, Z)^{-1}[\mathbf{L}(X, Y, Z)] \longrightarrow \mathbf{M}(X, Y, Z)$ is an inclusion map $\psi(X, Y, Z)^{-1}[\mathcal{SC}(X \times Y, Z)] \longrightarrow \mathcal{A}(X, k(Y, Z))$.*

We put

- (1) $\tilde{f} = \phi(X, Y, Z)(f): X \longrightarrow k(Y, Z)$ for $\mathcal{SC} \ni f: X \times Y \longrightarrow Z$,
- (2) $\tilde{h} = \psi(X, Y, Z)(h): X \times Y \longrightarrow Z$ for $\mathcal{A} \ni h: X \longrightarrow k(Y, Z)$,
- (3) $y^* = k(y, Z): k(Y, Z) \longrightarrow k(Y', Z)$ for $\mathcal{SC} \ni y: Y' \longrightarrow Y$,
- (4) $z_* = k(Y, z): k(Y, Z) \longrightarrow k(Y, Z')$ for $\mathcal{SC} \ni z: Z \longrightarrow Z'$.

Then (B) amounts to assume the following formulas:

- (B.1) $\tilde{f}x = \overline{f(x \times Y)}$, (B.2) $y^*\tilde{f} = \overline{f(X \times y)}$, (B.3) $z_*\tilde{f} = \overline{zf}$,
- (B.4) $\widetilde{hx} = \tilde{h}(x \times Y)$, (B.5) $\widetilde{y^*h} = \tilde{h}(X \times y)$, (B.6) $\widetilde{z_*h} = z\tilde{h}$,
- (B.7) $\tilde{f} = f$, (B.8) $\tilde{h} = h$ provided that $\tilde{h} \in \mathcal{SC}$.

Following J. Dugundji [1] we make the following Definition 4 (and Definition 10).

DEFINITION 4. *An object Y of $\mathcal{A}$ is said to be ($\mathcal{SC}$ -) admissible if $h \in \mathcal{C}$ ($h \in \mathcal{SC}$) implies $\tilde{h} \in \mathcal{C}$ ($\tilde{h} \in \mathcal{SC}$) for all morphisms h of $\mathcal{A}$ of the form $h: X \longrightarrow k(Y, Z)$.*

If $E: k(Y, Z) \longrightarrow k(Y, Z)$ is the identity morphism, we call the morphism

$$(5) \quad \varepsilon_Y = \tilde{E}: k(Y, Z) \times Y \longrightarrow Z$$

the evaluation map associated with Y .

PROPOSITION 5. *An object Y is ($\mathcal{SC}$ -) admissible if and only if $\varepsilon_Y \in \mathcal{C}$ ($\varepsilon_Y \in \mathcal{SC}$).*

PROOF. Only if part follows from the definition. If part follows from $\tilde{h} = \widetilde{Eh} = \tilde{E}(h \times Y) = \varepsilon_Y(h \times Y)$, where $E = k(Y, Z)$.

COROLLARY 6. *If Y is admissible, then it is also $\mathcal{S}$ -admissible.*

AXIOM (C). *Any object of $\mathcal{S}$ is admissible.*

The following two axioms are stated relative to some fixed full subcategory $\mathcal{U}$ of $\mathcal{C}$ such that $\mathcal{K} \supset \mathcal{U} \supset \mathcal{S}$, where $\mathcal{K}$ is the full subcategory of $\mathcal{C}$ consisting of all k -objects K , i.e., objects K such that all morphisms f of $\mathcal{S}\mathcal{C}$ of the form $f: K \longrightarrow Z$ belong to $\mathcal{C}$. The inclusion relation $\mathcal{K} \supset \mathcal{S}$ is obvious.

AXIOM (D). *A morphism h of $\mathcal{A}$ of the form $h: X \longrightarrow k(Y, Z)$ belongs to $\mathcal{C}$ if and only if u^*h belongs to $\mathcal{C}$ for all $u: U \longrightarrow Y$ with $U \in \mathcal{U}$ and $u \in \mathcal{C}$.*

PROPOSITION 7. *If $u: U \longrightarrow Y$ with $U \in \mathcal{U}$ belongs to $\mathcal{C}$, then $u^*: k(Y, Z) \longrightarrow k(U, Z)$ belongs to $\mathcal{C}$ for all Z .*

AXIOM (E). *If $U \in \mathcal{U}$, then $\mathcal{C} \ni z: Z \longrightarrow Z'$ implies $\mathcal{C} \ni z_*: k(U, Y) \longrightarrow k(U, Z')$.*

PROPOSITION 8. *If $\mathcal{C} \ni z: Z \longrightarrow Z'$, then $\mathcal{C} \ni z_*: k(Y, Z) \longrightarrow k(Y, Z')$ for all Y .*

PROOF. It suffices to prove by (D) that the composite $k(Y, Z) \xrightarrow{z_*} k(Y, Z') \xrightarrow{u^*} k(U, Z')$ belongs to $\mathcal{C}$ for $\mathcal{C} \ni u: U \longrightarrow Y$ with $U \in \mathcal{U}$. But this composite is equal to the composite $k(Y, Z) \xrightarrow{u^*} k(U, Z) \xrightarrow{z_*} k(U, Z')$, which is continuous by Proposition 7 and (E).

PROPOSITION 9. *If $\mathcal{S}\mathcal{C} \ni y: Y' \longrightarrow Y$, then $\mathcal{C} \ni y^*: k(Y, Z) \longrightarrow k(Y', Z)$ for all Z .*

PROOF. For any $\mathcal{C} \ni u: U \longrightarrow Y'$, the composite $k(Y, Z) \xrightarrow{y^*} k(Y', Z) \xrightarrow{u^*} k(U, Z)$ is equal to $(yu)^*: k(Y, Z) \longrightarrow k(U, Z)$, and the latter is continuous by Proposition 7, because $U \in \mathcal{U} \subset \mathcal{K}$ and hence $yu \in \mathcal{C}$.

DEFINITION 10. *An object Y of $\mathcal{A}$ is said to be ($\mathcal{S}$ -) proper if $f \in \mathcal{C} (f \in \mathcal{S}\mathcal{C})$ implies $\bar{f} \in \mathcal{C} (\bar{f} \in \mathcal{S}\mathcal{C})$ for all morphisms f of $\mathcal{A}$ of the form $f: X \times Y \longrightarrow Z$.*

If $I: X \times Y \longrightarrow X \times Y$ is the identity morphism, we call the morphism

$$(6) \quad \eta_Y = \bar{I}: X \longrightarrow k(Y, X \times Y)$$

the coevaluation map associated with Y .

PROPOSITION 11. Y is proper if and only if $\eta_Y \in \mathcal{C}$.

PROOF. Only if part follows from the definition. If part follows from $\bar{f} = \overline{fI} = f_*\bar{I} = f_*\eta_Y$ and Proposition 8.

AXIOM (F). Any object of $\mathcal{U}$ is proper.

PROPOSITION 12. Any object of $\mathcal{A}$ is proper.

PROOF. To prove that $\eta_Y = \bar{I} \in \mathcal{C}$, where $I = X \times Y$ and $\eta_Y: X \longrightarrow k(Y, I)$, it suffices to see $u^*\bar{I} \in \mathcal{C}$ for $u: U \longrightarrow Y, u \in \mathcal{C}, U \in \mathcal{U}$. But $u^*\bar{I} = \overline{I(X \times u)} = \overline{X \times u}$ and $X \times U \xrightarrow{X \times u} X \times Y = I$. Therefore $u^*\bar{I} = \overline{X \times u} \in \mathcal{C}$ by (F) as desired.

COROLLARY 13. If $\mathcal{C} \ni f: X \times Y \longrightarrow Z$, then $\mathcal{C} \ni \bar{f}: X \longrightarrow k(Y, Z)$.

AXIOM (G). If $A, B \in \mathcal{S}$, then $A \times B \in \mathcal{X}$.

PROPOSITION 14. If $K \in \mathcal{X}$ and $B \in \mathcal{S}$, then $K \times B \in \mathcal{X}$.

PROOF. Let $\mathcal{S}\mathcal{C} \ni f: K \times B \longrightarrow Z$. We have to prove $f \in \mathcal{C}$. But, since $f = \tilde{f} = \varepsilon_B(\tilde{f} \times B)$, under the existence of the Axiom (C), it suffices to prove that $\tilde{f} \in \mathcal{C}$. Now $\tilde{f}: K \longrightarrow k(B, Z)$ has $K \in \mathcal{X}$ as its domain, and hence $\tilde{f} \in \mathcal{C}$ follows from $\tilde{f} \in \mathcal{S}\mathcal{C}$. Therefore, we will prove $\tilde{f} \in \mathcal{S}\mathcal{C}$, or equivalently, $\mathcal{C} \ni \tilde{f}\alpha: A \longrightarrow k(B, Z)$ for $\mathcal{C} \ni \alpha: A \longrightarrow K, A \in \mathcal{S}$. But, since $\tilde{f}\alpha = \overline{f(\alpha \times B)}$ and $f(\alpha \times B): A \times B \longrightarrow Z$, the facts $f(\alpha \times B) \in \mathcal{S}\mathcal{C}$, $A \times B \in \mathcal{X}$ ((G)) and (F) ensure the validity of $\tilde{f}\alpha \in \mathcal{C}$.

PROPOSITION 15. Any object of $\mathcal{A}$ is $\mathcal{S}$ -proper.

PROOF. We will show $\mathcal{S}\mathcal{C} \ni \bar{f}: X \longrightarrow k(Y, Z)$ for a given $\mathcal{S}\mathcal{C} \ni f: X \times Y \longrightarrow Z$. For this it suffices to see $u^*\bar{f}\alpha \in \mathcal{C}$ for $\alpha: A \longrightarrow X, A \in \mathcal{S}, \alpha \in \mathcal{C}$, and $u: U \longrightarrow Y, U \in \mathcal{U}, u \in \mathcal{C}$. Now $u^*\bar{f}\alpha = u^*(\overline{f(\alpha \times Y)}) = \overline{f(\alpha \times Y)(A \times u)} = \overline{f(\alpha \times u)}$, $f(\alpha \times u): A \times U \longrightarrow Z$, and $A \times U \in \mathcal{X}$ (Proposition 14) first imply $f(\alpha \times u) \in \mathcal{C}$ and then imply (by virtue of (F)) $u^*\bar{f}\alpha = \overline{f(\alpha \times u)} \in \mathcal{C}$ as desired.

COROLLARY 16. If $\mathcal{S}\mathcal{C} \ni f: X \times Y \longrightarrow Z$, then $\mathcal{S}\mathcal{C} \ni \bar{f}: X \longrightarrow k(Y, Z)$.

PROPOSITION 17. Any object of $\mathcal{A}$ is $\mathcal{S}$ -admissible.

PROOF. We will show $\mathcal{S}\mathcal{C} \ni \varepsilon_Y = \tilde{E}: k(Y, Z) \times Y \longrightarrow Z$, where $E = k(Y, Z)$.

To do this, we take $\langle \alpha, \beta \rangle : A \longrightarrow k(Y, Z) \times Y$, $A \in \mathcal{S}$, $\alpha \in \mathcal{C}$, $\beta \in \mathcal{C}$, and want to see $\varepsilon_Y \langle \alpha, \beta \rangle \in \mathcal{C}$. But, since $\varepsilon_Y \langle \alpha, \beta \rangle = \tilde{E}(E \times \beta) \langle \alpha, A \rangle = (\widetilde{\beta^* E}) \langle \alpha, A \rangle = \widetilde{\beta^*} \langle \alpha, A \rangle = (\widetilde{E_0 \beta^*}) \langle \alpha, A \rangle = \tilde{E}_0(\beta^* \times A) \langle \alpha, A \rangle = \varepsilon_A(\beta^* \times A) \langle \alpha, A \rangle$ with $E_0 = k(A, Z)$, $\varepsilon_A \in \mathcal{C}((\mathbf{C}))$, $(\beta^* \times A) \in \mathcal{C}$ (Proposition 7), and $\langle \alpha, A \rangle \in \mathcal{C}$, we have $\varepsilon_Y \langle \alpha, \beta \rangle = \varepsilon_A(\beta^* \times A) \langle \alpha, A \rangle \in \mathcal{C}$ as desired.

COROLLARY 18. *If $\mathcal{S}\mathcal{C} \ni h : X \longrightarrow k(Y, Z)$, then $\mathcal{S}\mathcal{C} \ni \tilde{h} : X \times Y \longrightarrow Z$.*

PROPOSITION 19. *If $\mathcal{S}\mathcal{C} \ni z : Z \longrightarrow Z'$, then $\mathcal{S}\mathcal{C} \ni z_* : k(Y, Z) \longrightarrow k(Y, Z')$.*

PROOF. Putting $E = k(Y, Z')$, we have $z_* = z_* E = z_* \tilde{E}$ ((B.8) and the above Corollary) $= z_* \varepsilon_Y = \overline{z \varepsilon_Y} \in \mathcal{S}\mathcal{C}$ (Proposition 17 and Corollary 16).

PROPOSITION 20. *$k : \mathcal{S}\mathcal{C}^{op} \times \mathcal{S}\mathcal{C} \longrightarrow \mathcal{A}$ is essentially a functor $k : \mathcal{S}\mathcal{C}^{op} \times \mathcal{S}\mathcal{C} \longrightarrow \mathcal{S}\mathcal{C}$.*

PROOF. y^* and z_* both belong to $\mathcal{S}\mathcal{C}$ by Propositions 9 and 19.

PROPOSITION 21. *Natural transformations ϕ and ψ induce natural transformations*

$$\phi : \mathbf{L} \longrightarrow \mathbf{P} \text{ and } \psi : \mathbf{P} \longrightarrow \mathbf{L}$$

which are inverses to each other, where $\mathbf{L} : \mathcal{S}\mathcal{C}^{op} \times \mathcal{S}\mathcal{C}^{op} \times \mathcal{S}\mathcal{C} \longrightarrow \mathcal{S}et$ is one given in (B), and $\mathbf{P} : \mathcal{S}\mathcal{C}^{op} \times \mathcal{S}\mathcal{C}^{op} \times \mathcal{S}\mathcal{C} \longrightarrow \mathcal{S}et$ is given by $\mathbf{P}(X, Y, Z) = \mathcal{S}\mathcal{C}(X, k(Y, Z))$.

PROOF. We have only to define ϕ, ψ by

$$(7) \quad \mathcal{S}\mathcal{C}(X \times Y, Z) \ni f \longmapsto_{\phi} \tilde{f} \in \mathcal{S}\mathcal{C}(X, k(Y, Z))$$

$$\text{and } \mathcal{S}\mathcal{C}(X, k(Y, Z)) \ni h \longmapsto_{\psi} \tilde{h} \in \mathcal{S}\mathcal{C}(X \times Y, Z).$$

THEOREM 1. *There exists a natural isomorphism in $\mathcal{S}\mathcal{C}$*

$$(8) \quad \Phi : k(X \times Y, Z) \longrightarrow k(X, k(Y, Z))$$

with an inverse natural isomorphism in $\mathcal{S}\mathcal{C}$

$$(9) \quad \Psi : k(X, k(Y, Z)) \longrightarrow k(X \times Y, Z).$$

PROOF. We first define a natural isomorphism $\hat{\phi} : \mathcal{S}\mathcal{C}(W, k(X \times Y, Z)) \longrightarrow \mathcal{S}\mathcal{C}(W, k(X, k(Y, Z)))$ to be the following composite of natural isomorphisms

$$\begin{aligned} \mathcal{S}\mathcal{C}(W, k(X \times Y, Z)) &\xrightarrow{\psi_{X \times Y}} \mathcal{S}\mathcal{C}(W \times (X \times Y), Z) \xrightarrow{\mu} \mathcal{S}\mathcal{C}((W \times X) \times Y, Z) \\ &\xrightarrow{\phi_Y} \mathcal{S}\mathcal{C}(W \times X, k(Y, Z)) \xrightarrow{\phi_X} \mathcal{S}\mathcal{C}(W, k(X, k(Y, Z))), \end{aligned}$$

where ϕ_Y, ψ_Y are shortenings of $\phi(X, Y, Z)$, $\psi(X, Y, Z)$ respectively, and μ denotes the natural isomorphism from the functor $(X, Y, Z) \longrightarrow (X \times Y) \times Z$ to the functor $(X, Y, Z) \longrightarrow X \times (Y \times Z)$. Now it is well known that from the existence of a natural isomorphism of the form $\hat{\phi}: \mathcal{S}\mathcal{C}(W, k(X \times Y, Z)) \longrightarrow \mathcal{S}\mathcal{C}(W, k(X, k(Y, Z)))$ follows a natural isomorphism of the form $\Phi: k(X \times Y, Z) \longrightarrow k(X, k(Y, Z))$, and hence the proof is complete.

REMARK. One can directly verify the last fact by making use of (B.1)~(B.6), where we put $\Phi = \hat{\phi}$ [the identity morphism of $W = k(X \times Y, Z)$] and $\Psi = \hat{\phi}^{-1}$ [the identity morphism of $W = k(X, k(Y, Z))$].

THEOREM 2. *There exists a natural isomorphism in $\mathcal{S}\mathcal{C}$*

$$(10) \quad \theta: k(X, Y \times Z) \longrightarrow k(X, Y) \times k(X, Z).$$

PROOF. We have only to use the following composite of natural isomorphisms

$$\begin{aligned} \mathcal{S}\mathcal{C}(W, k(X, Y \times Z)) &\longrightarrow \mathcal{S}\mathcal{C}(W \times X, Y \times Z) \longrightarrow \mathcal{S}\mathcal{C}(W \times X, Y) \\ &\times \mathcal{S}\mathcal{C}(W \times X, Z) \longrightarrow \mathcal{S}\mathcal{C}(W, k(X, Y)) \times \mathcal{S}\mathcal{C}(W, k(X, Z)) \longrightarrow \\ &\mathcal{S}\mathcal{C}(W, k(X, Y) \times k(X, Z)), \end{aligned}$$

where the second and the last natural isomorphisms come from Proposition 1.

THEOREM 3. *If H is admissible, then $K \times H \in \mathcal{K}$ for any $K \in \mathcal{K}$.*

PROOF. Assume $\mathcal{S}\mathcal{C} \ni f: K \times H \longrightarrow Z$. Then $\mathcal{S}\mathcal{C} \ni \tilde{f}: K \longrightarrow k(H, Z)$ by Corollary 16, and hence $\mathcal{C} \ni \tilde{f}$ because $K \in \mathcal{K}$. Now, putting $E = k(H, Z)$, we have $f = \tilde{f} = E\tilde{f} = \tilde{E}(H \times \tilde{f}) = \varepsilon_H(H \times \tilde{f}) \in \mathcal{C}$, which completes the proof.

THEOREM 4. *If $\mathcal{A}$ has a point and any trivial morphism in $\mathcal{A}$ belongs to $\mathcal{C}$, then any admissible object H belongs to $\mathcal{K}$.*

PROOF. Under our assumption any point $*$ belongs to $\mathcal{K}$, therefore $H \simeq * \times H \in \mathcal{K}$ by the preceding theorem.

THEOREM 5. *Let $\{\mathcal{A}, \mathcal{C}, \mathcal{S}, k_1\}$ be another initial data satisfying all*

the axioms. Then k and k_1 , when considered as functors $\mathcal{S}\mathcal{C}^{op} \times \mathcal{S}\mathcal{C} \longrightarrow \mathcal{S}\mathcal{C}$ (Proposition 20), are naturally isomorphic. In particular, for any $Y, Z \in \mathcal{A}$, we have

$$(11) \quad k(Y, Z) \simeq k_1(Y, Z) \quad \text{in } \mathcal{S}\mathcal{C}.$$

PROOF. Using Proposition 21, we see that there is a natural isomorphism $\mathcal{S}\mathcal{C}(X, k(Y, Z)) \simeq \mathcal{S}\mathcal{C}(X, k_1(Y, Z))$. From this we can see that there exists (as in the proof of Theorem 1) a natural isomorphism between two functors k and k_1 .

EXAMPLE 1. Let $\mathcal{A}$ be the category of topological spaces and (set) maps, $\mathcal{C}$ the category of topological spaces and continuous maps, and $\mathcal{S}$ some full subcategory of $\mathcal{C}$. In $\mathcal{C}$ we consider the function space functor $F: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$, each $F(Y, Z)$ being provided with the compact-open topology. Let $\mathcal{S}$ be at first arbitrary. Then $\mathcal{S}\mathcal{C}$ is the category of topological spaces and k -continuous maps in the sense of R. M. Vogt, and $\mathcal{X}$ the category of k -spaces in the same sense. Here (A.1), (A.2) already hold. Now let $k(Y, Z)$ be the set of all k -continuous maps. It defines at first a functor $k: \mathcal{S}\mathcal{C}^{op} \times \mathcal{S}\mathcal{C} \longrightarrow \mathcal{S}et$. Let us fix a full subcategory $\mathcal{U}$ of $\mathcal{C}$ such that $\mathcal{X} \supset \mathcal{U} \supset \mathcal{S}$, and let us put $k(Y, Z) = F(Y, Z)$ in case $Y \in \mathcal{U}$. (Note that for $Y \in \mathcal{X}$ and hence for $Y \in \mathcal{U}$ the set $k(Y, Z)$ is identical with the underlying set of $F(Y, Z)$.) Let Y be arbitrary. In this case we can introduce in $k(Y, Z)$ a unique topology by the condition (D). At this moment k has changed into a functor $k: \mathcal{S}\mathcal{C}^{op} \times \mathcal{S}\mathcal{C} \longrightarrow \mathcal{A}$, and we obtain an initial data $\{\mathcal{A}, \mathcal{C}, \mathcal{S}, k\}$.

If we consider the following expressions

$$(12) \quad \tilde{f}(x)(y) = f(x, y), \quad \tilde{h}(x, y) = h(x)(y),$$

we can easily see that the axiom (B) is satisfied if we define ϕ and ψ by

$$\mathcal{S}\mathcal{C}(X \times Y, Z) \ni f \mapsto_{\phi} \tilde{f} \ni \mathcal{A}(X, k(Y, Z)),$$

and

$$\mathcal{A}(X, k(Y, Z)) \ni h \mapsto_{\psi} \tilde{h} \ni \mathcal{A}(X \times Y, Z).$$

As for (C), (E), (F), these are conditions on $F(-, -)$ and not on (proper) $k(-, -)$. (E), (F) are known properties of $F(-, -)$ and (C) is satisfied if $\mathcal{S}$ is a full subcategory of the category of locally compact Hausdorff spaces. Finally, (G) is satisfied by various families of topological spaces, p.e., the family of compact spaces. After all, in this example, essential assumptions are only (C) and

(G), and under these few restrictions the exponential laws

$$(13) \quad k(X \times Y, Z) \simeq k(X, k(Y, Z)) \quad \text{and} \quad k(X, Y \times Z) \simeq k(X, Y) \times k(X, Z)$$

freely hold (this means these laws are valid for arbitrary topological spaces X , Y and Z), and these are reduced into the ordinary form as soon as some of the spaces ($X \times Y$, X , Y in the first formula, and X in the second) turn out to be from $\mathcal{U}$. Here $\simeq$ in (13) is not an ordinary homeomorphism, but only an isomorphism in $\mathcal{S}\mathcal{C}$. But it does not matter very much, because, for many choices of $\mathcal{S}$, homotopy groups and singular (co-)homology groups are invariant under isomorphisms in $\mathcal{S}\mathcal{C}$, as pointed out by Vogt [3].

EXAMPLE 2. In the preceding example let us specialize $\mathcal{S}$ to be $\mathcal{S} = \{Z^+\}$, where $Z^+ = \{0, 1, 1/2, 1/3, \dots\}$ is a subspace of the real line. Then, since continuous maps $Z^+ \longrightarrow X$ are equivalent to convergent sequences in X , $\mathcal{S}\mathcal{C}$ is just the category of topological spaces and sequentially continuous maps, and $\mathcal{K}$ is the category of sequential spaces. We then specialize $\mathcal{U}$ in two ways: $\mathcal{U} = \mathcal{K}$ and $\mathcal{U}_{cs} = \mathcal{S}$. Let $\{\mathcal{A}, \mathcal{C}, \mathcal{S}, k\}$ and $\{\mathcal{A}, \mathcal{C}, \mathcal{S}, k_{cs}\}$ be the corresponding initial datas. According to the general rule we have

$$(14) \quad k(X, Y) = F(X, Y) \quad \text{for} \quad X \in \mathcal{U} = \mathcal{K}, \quad \text{and} \quad k_{cs}(Z^+, Y) = F(Z^+, Y).$$

For general X , the topology of $k_{cs}(X, Y)$ which satisfies (D) is the coarsest topology which makes all the mappings of the form

$$(15) \quad \alpha^*: k_{cs}(X, Y) \longrightarrow k_{cs}(Z^+, Y) = F(Z^+, Y) (\mathcal{C} \ni \alpha: Z^+ \longrightarrow X)$$

continuous, and hence generated by all subsets of $k_{cs}(X, Y)$ of the form

$$(16) \quad (\alpha, V) = \{f \in k_{cs}(X, Y); f(\alpha(Z^+)) \subset V\},$$

with $\mathcal{C} \ni \alpha: Z^+ \longrightarrow X$ and with V open in Y . This topology is called the *cs*-open topology, and similarly for the subspace $F_{cs}(X, Y)$ consisting of all continuous functions $X \longrightarrow Y$, which is identical with $k_{cs}(X, Y)$ when $X \in \mathcal{K}$. Now we can apply Theorem 5 to compare $F(X, Y)$ and $F_{cs}(X, Y)$. Since $k(X, Y)$ and $k_{cs}(X, Y)$ are isomorphic in $\mathcal{S}\mathcal{C}$, $F(X, Y)$ and $F_{cs}(X, Y)$ are isomorphic in $\mathcal{S}\mathcal{C}$ in case where $X \in \mathcal{K}$; in particular, for metric X , Y , uniform convergence $f_n \longrightarrow f$ on all convergent sequences on X implies one on all compact subsets of X .

References

- [1] J. DUGUNDJI, *Topology*, Allyn and Bacon, Inc., Boston, 1965.
- [2] N. E. STEENROD, *A convenient category of topological spaces*, Michigan Math. J., **14** (1967), 133–150.
- [3] R. M. VOGT, *Convenient categories of topological spaces for homotopy theory*, Arch. Math., **22** (1971), 545–555.