NUMERICAL METHODS FOR INTEGRO-DIFFERENCE EQUATIONS

By

Muhammad Anwar CHAUDHRY

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This paper contains a brief study of numerical methods for solving linear integro-difference equations. A number of convergent approximation schemes are discussed and a numerical example is given.

1. Introduction

Recently, the study of a parking problem by M. Lal and P. Gillard [1] has led to an integro-difference equation. In this paper we consider the following linear integro-difference equation

$$(1.1) \quad \begin{cases} u(x) = f(x) + \int_0^x k(x, t)u(t-\tau)dt, & x > 0 \\ u(x) = \phi(x), & -\tau \leq x \leq 0 \end{cases}$$

and suggest methods for its numerical solution. It is assumed that in (1.1) the following conditions are satisfied

- (i) $f(x)$, $\phi(x)$ and $k(x, t)$ are continuous on I , I' and S , respectively, where $I = [0, a]$, $I' = [-\tau, 0]$, τ is a positive constant, and $S = \{(x, t) | 0 \leq t \leq x \leq a\}$,
- (ii) $f(0) = \phi(0)$.

Then the unique solution $u(x)$, continuous on I , of (1.1), that is, continuous extension of $\phi(x)$ on I , can be found by the method of steps. However, we note that even if the functions k , f and ϕ are sufficiently smooth on their respective domains, $u(x)$, generally speaking, is not differentiable at the point $x=0$. Because differentiating (1.1) at $x=0$, we get

$$(1.2) \quad \begin{cases} u'_+(0) = f'(0) + k(0, 0)\phi(-\tau) \\ u'_-(0) = \phi'(0), \end{cases}$$

where $u'_+(0)$ and $u'_-(0)$ denote the right hand and left hand derivatives at $x=0$, respectively, and they are generally not equal. If the condition

$$\phi'(0) = f'(0) + k(0, 0)\phi(-\tau)$$

is satisfied, then it is possible to guarantee the existence as well as continuity of $u'(x)$ at $x=0$. We also note that

$$u'(x) = f'(x) + k(x, x)u(x-\tau) + \int_0^x \frac{\partial k}{\partial x}(x, t)u(t-\tau)dt$$

is continuous everywhere on I except perhaps at the point $x=0$, but at the point $x=\tau$, $u''(x)$ may not exist due to the presence of $u'(x-\tau)$ in it. Continuing, we conclude that $u^k(t)$, generally, does not exist at $x=(k-1)\tau$, $k=1, 2, \dots$, while all the derivatives of lower order exist and are continuous. The above fact, however, will not affect our analysis in any way.

The explicit determination of the solution of (1.1) by the method of steps requires the evaluation of successive integrals, therefore, the procedure will become tedious especially when τ is much smaller than the length of the interval on which solution is sought. Hence, the numerical methods are most desirable. We will derive numerical schemes by approximating the integral in (1.1) by certain simple quadrature formulas.

2. Notations and Definitions

To solve (1.1) on an interval $[0, a]$, we divide it into smaller intervals of width h , where h will be defined later in each case separately. The i th point of subdivision is denoted by x_i , such that, $x_i = ih$, $i=0, 1, 2, \dots, N$, N being the integral part of a/h . The approximate solution will be defined at these meshpoints and denoted by U_i . We also state the following definitions which are given in P. Linz [3].

DEFINITION 1. Let $U_0(h), U_1(h), \dots$ denote the approximation obtained by a given method using stepsize h . Then the method is said to be convergent if and only if

$$\max_{0 \leq i \leq N} |U_i(h) - u(x_i)| \longrightarrow 0$$

as $h \rightarrow 0$.

DEFINITION 2. A method is said to be of order p if p is the largest number for which there exists a finite constant C such that

$$\max_{0 \leq i \leq N} |U_i(h) - u(x_i)| \leq Ch^p$$

for all $0 < h \leq a$.

3. The Rectangular Method

The approximation in this case is obtained by replacing the kernel $k(x, t)$ by $k(x, x_i)$ in each interval $x_i \leq t \leq x_{i+1}$. Discretizing the equation

$$(3.1) \quad u(x) = f(x) + \int_0^x k(x, t)u(t-\tau)dt,$$

we get

$$(3.2) \quad u(x_n) = f(x_n) + \int_0^{x_n} k(x_n, t)u(t-\tau)dt \quad n=1, 2, \dots, N.$$

Let h be chosen such that $\tau = mh$, m being a fixed positive integer. Applying the rectangular quadrature formula to (3.2), we have

$$(3.3) \quad u(x_n) = f(x_n) + h \sum_{i=0}^{n-1} k(x_n, x_i)u(x_i-\tau) + \delta_n \quad n=1, 2, \dots, N,$$

where δ_n is the discretization error. Denoting $(x_i - \tau)$, which equals $(i-m)h$, by x_{i-m} , $i=0, 1, 2, \dots, N$, we obtain

$$(3.4) \quad u(x_n) = f(x_n) + h \sum_{i=0}^{n-1} k(x_n, x_i)u(x_{i-m}) + \delta_n.$$

Neglecting δ_n , we get the approximation scheme

$$(3.5) \quad U_n = f(x_n) + h \sum_{i=0}^{n-1} k(x_n, x_i)U_{i-m} \quad n=1, 2, \dots, N,$$

where

$$U_{i-m} = u(x_{i-m}) = \phi(x_{i-m}) \quad i=0, 1, 2, \dots, m.$$

Now we suppose that the following conditions are satisfied

- (i) $f(x)$, $\phi(x)$ and $k(x, t)$ are continuously differentiable on their respective domains,
- (ii) $u'_+(0) = f'(0) + k(0, 0)\phi(-\tau)$ is finite.

THEOREM 1. *Under the above assumptions, the approximation method (3.5) is convergent with order one.*

Proof. Let $\varepsilon_n = u(x_n) - U_n$, $n = -m, -m+1, \dots, 0, 1, 2, \dots, N$. Subtraction of (3.5) from (3.4) yields

$$(3.6) \quad \varepsilon_n = h \sum_{i=0}^{n-1} k(x_n, x_i) \varepsilon_{i-m} + \delta_n \quad n = 1, 2, \dots, N$$

where

$$(3.7) \quad \delta_n = \sum_{i=0}^{n-1} \left[\int_{x_i}^{x_{i+1}} k(x_n, t) u(t-\tau) dt - h k(x_n, x_i) u(x_{i-m}) \right].$$

Due to our assumptions, $u(x)$ is continuously differentiable on $I'' = [-\tau, 0) \cup (0, a]$ and $u'(x)$ is bounded there. Expansion of the integrand in (3.7) by means of Taylor's theorem around the point x_i gives

$$(3.8) \quad \delta_n = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{\partial}{\partial t} [k(x_n, t) u(t-\tau)] (t-x_i) dt, \quad x_i < c_i < x_{i+1}.$$

From (3.8) we conclude that there exists a positive constant M such that

$$|\delta_n| \leq Mh.$$

Let $B = \max_S |k(x, t)|$, then (3.6) gives

$$(3.9) \quad |\varepsilon_n| \leq Bh \sum_{i=0}^{n-1} \varepsilon_{i-m} + Mh \quad n = 1, 2, \dots, N.$$

From (3.9) it follows by induction that $\varepsilon_n = O(h)$, which proves the theorem.

4. The Trapezoidal Method

Let h be the same as in the previous case. Discretizing (3.1) and applying the trapezoidal quadrature formula, we have

$$(4.1) \quad u(x_n) = f(x_n) + h \left[\frac{1}{2} k(x_n, x_0) u(x_0 - \tau) + \sum_{i=1}^{n-1} k(x_n, x_i) u(x_i - \tau) \right. \\ \left. + \frac{1}{2} k(x_n, x_n) u(x_n - \tau) \right] + \delta_n \quad n = 1, 2, \dots, N$$

where δ_n is the discretization error. Neglecting δ_n , we get the approximation scheme

$$(4.2) \quad U_n = f(x_n) + h \left[\frac{1}{2} k(x_n, x_0) U_{-m} + \sum_{i=1}^{n-1} k(x_n, x_i) U_{i-m} \right. \\ \left. + \frac{1}{2} k(x_n, x_n) U_{n-m} \right] \quad n = 1, 2, \dots, N.$$

In the subsequent analysis we will suppose that the following conditions are satisfied

- (i) $f(x)$, $\phi(x)$ and $k(x, t)$ are twice continuously differentiable on their respective domains,
- (ii) $\phi'(0) = f'(0) + k(0, 0)\phi(-\tau)$,
- (iii) $f(0) = \phi(0)$,
- (iv) $u''_+(0)$ is finite.

Then $u(x)$ is twice continuously differentiable on $I'' = [-\tau, 0) \cup (0, a]$ and $u''(x)$ is bounded there. We will now show that $\delta_n = O(h^2)$. Since

$$\delta_n = \left[\int_0^{x_{\frac{1}{2}}} k(x_n, t) u(t - \tau) dt - \frac{h}{2} k(x_n, x_0) u(x_{-m}) \right] \\ + \sum_{i=1}^{n-1} \left[\int_{x_{i-\frac{1}{2}}}^{x_i} k(x_n, t) u(t - \tau) dt - \frac{h}{2} k(x_n, x_i) u(x_{i-m}) \right] \\ + \sum_{i=1}^{n-1} \left[\int_{x_i}^{x_{i+\frac{1}{2}}} k(x_n, t) u(t - \tau) dt - \frac{h}{2} k(x_n, x_i) u(x_{i-m}) \right] \\ + \left[\int_{x_{n-\frac{1}{2}}}^{x_n} k(x_n, t) u(t - \tau) dt - \frac{h}{2} k(x_n, x_n) u(x_{n-m}) \right],$$

therefore, Taylor expansion of integrands in the first and fourth terms around the points $t = x_{1/2}$ and $t = x_n$, respectively, obviously gives that both terms are $O(h^2)$. Expanding the integrands in the second and third terms by means of

Taylor's theorem around the point $t = x_i$, we have

$$\begin{aligned}
 (4.3) \quad \delta_n = & \sum_{i=1}^{n-1} \left[\int_{x_{i-1/2}}^{x_i} \left[k(x_n, x_i) u(x_i - \tau) - (x_i - t) \frac{\partial}{\partial t} [k(x_n, t) u(t - \tau)] \right. \right. \\
 & \left. \left. + \frac{(x_i - t)^2}{2} \frac{\partial^2}{\partial t^2} [k(x_n, t) u(t - \tau)] \right] dt - \frac{h}{2} k(x_n, x_i) u(x_{i-m}) \right] \\
 & + \sum_{i=1}^{n-1} \left[\int_{x_i}^{x_{i+1/2}} \left[k(x_n, x_i) u(x_i - \tau) + (t - x_i) \frac{\partial}{\partial t} [k(x_n, t) u(t - \tau)] \right. \right. \\
 & \left. \left. + \frac{(t - x_i)^2}{2} \frac{\partial^2}{\partial t^2} [k(x_n, t) u(t - \tau)] \right] dt - \frac{h}{2} k(x_n, x_i) u(x_{i-m}) \right] \\
 & + O(h^2)
 \end{aligned}$$

where $x_{i-1/2} < c_i < x_i$ and $x_i < c'_i < x_{i+1/2}$.

Simplifying, we get

$$\begin{aligned}
 (4.4) \quad \delta_n = & \sum_{i=1}^{n-1} \left[\frac{\partial}{\partial t} [k(x_n, t) u(t - \tau)] \int_{x_i}^{x_{i+1/2}} (t - x_i) dt \right] \\
 & + \sum_{i=1}^{n-1} \int_{x_{i-1/2}}^{x_i} \left[\frac{\partial^2}{\partial t^2} [k(x_n, t) u(t - \tau)] \frac{(t - x_i)^2}{2} \right] dt \\
 & + \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1/2}} \left[\frac{\partial^2}{\partial t^2} [k(x_n, t) u(t - \tau)] \frac{(t - x_i)^2}{2} \right] dt.
 \end{aligned}$$

Due to our assumptions, $\frac{\partial^2}{\partial t^2} [k(x_n, t) u(t - \tau)]$ is bounded on $I''' = [0, \tau) \cup (\tau, a]$. Therefore, there exists a positive constant M such that

$$\left| \frac{\partial^2}{\partial t^2} [k(x_n, t) u(t - \tau)] \right| \leq M \quad \text{for } t \in I'''.$$

Since the first term in (4.4) has the value zero, therefore, taking modulus on both sides, we have

$$\delta_n = M \sum_{i=1}^{n-1} \int_{x_{i-1/2}}^{x_{i+1/2}} (t - x_i)^2 / 2 dt + O(h^2)$$

Finally, after integration we have,

$$\delta_n = O(h^2),$$

because

$$M(n-1)h^3/24 < Mh^2Nh/24 < aMh^2/24.$$

THEOREM 2. *The approximation scheme (4.2) is convergent with order two.*

Proof. Subtracting (4.2) from (4.1), we get

$$\begin{aligned} \varepsilon_n = & \frac{h}{2}k(x_n, x_0)\varepsilon_{-m} + h \sum_{i=1}^{n-1} k(x_n, x_i)\varepsilon_{i-m} + \frac{h}{2}k(x_n, x_n)\varepsilon_{n-m} \\ & + \delta_n \quad n = 1, 2, \dots, N, \end{aligned}$$

which yields

$$(4.5) \quad |\varepsilon_n| < Bh \sum_{i=1}^n |\varepsilon_{i-m}| + |\delta_n|.$$

Since $\delta_n = O(h^2)$, therefore, it follows by induction from (4.5) that $\varepsilon_n = O(h^2)$, which proves the theorem.

5. The Midpoint Method

Let us choose h such that $h = \tau / \left(m + \frac{1}{2}\right)$ where m is a fixed positive integer. Discretizing (3.1) and applying the midpoint quadrature formula, we get

$$\begin{aligned} (5.1) \quad u(x_n) = & f(x_n) + h \sum_{i=0}^{n-1} k(x_n, x_{i+\frac{1}{2}})u(x_{i+\frac{1}{2}} - \tau) \\ & + \delta_n \quad n = 1, 2, \dots, N. \end{aligned}$$

Replacing the value of τ and neglecting δ_n , we have the approximation scheme

$$(5.2) \quad U_n = f(x_n) + h \sum_{i=0}^{n-1} k(x_n, x_{i+\frac{1}{2}})U_{i-m} \quad n = 1, 2, \dots, N.$$

THEOREM 3. *The approximation scheme (5.2) is convergent with order two.*

Proof. The proof of this theorem can be carried out in a fashion similar to the previous case.

6. Numerical Example

In this section we give a numerical example to illustrate the above methods.

EXAMPLE.

Consider the integro-difference equation

$$u(x) = x + \int_0^x (x-t)u(t-2)dt, \quad x > 0$$

with initial condition

$$u(x) = x, \quad -2 \leq x \leq 0.$$

Its exact solution is

$$u(x) = x - x^2/2 + x^3/6, \quad 0 < x \leq 2,$$

$$u(x) = (x-2)^5/120 - (x-2)^4/12 + (x-2)^3/6 - (x-2) - 2/3, \quad 2 < x \leq 4.$$

A selection of numerical results is presented in table 1, table 2 and table 3.

Table 1. Example by the rectangular method

x	Exact Solution	h=.005		h=.01	
		Approximate Solution	Error	Approximate Solution	Error
0.5	0.2708333	0.2683313	0.25021×10^{-2}	0.2658250	0.50083×10^{-2}
1.0	0.1666667	0.1616625	0.50042×10^{-2}	0.1566500	0.100167×10^{-1}
1.5	-0.1875000	-0.1950063	0.75063×10^{-2}	-0.2025250	0.150250×10^{-1}
2.0	-0.6666667	-0.6766750	0.100083×10^{-1}	-0.6867000	0.200333×10^{-1}
2.5	-1.1507813	-1.1633954	0.126142×10^{-1}	-1.1760296	0.252484×10^{-1}
3.5	-1.9627604	-1.9830863	0.203249×10^{-1}	-2.0034388	0.406784×10^{-1}
4.0	-2.4000000	-2.4266833	0.266833×10^{-1}	-2.4533997	0.533997×10^{-1}

Table 2. Example by the trapezoidal method

x	Exact Solution	h=.01		h=.02	
		Approximate Solution	Error	Approximate Solution	Error
0.3	0.2145000	0.2144950	0.50×10^{-5}	0.2144800	0.200×10^{-4}
0.9	0.2115000	0.2114850	0.150×10^{-4}	0.2114400	0.600×10^{-4}
1.5	-0.1875000	-0.1875250	0.250×10^{-4}	-0.1876000	0.1000×10^{-3}
2.1	-0.7665083	-0.7665432	0.349×10^{-4}	-0.7666479	0.1397×10^{-3}
2.7	-1.3281078	-1.3281501	0.423×10^{-4}	-1.3282771	0.1694×10^{-3}
3.3	-1.8075673	-1.8076173	0.501×10^{-4}	-1.8077676	0.2003×10^{-3}
3.9	-2.3031668	-2.3032303	0.635×10^{-4}	-2.3034207	0.2540×10^{-3}

Table 3. Example by the midpoint method

x	Exact Solution	$h = .00998$	
		Approximate Solution	Error
0.2993	0.2141666	0.2141691	0.25×10^{-5}
0.9077	0.2084136	0.2084211	0.75×10^{-5}
1.5062	-0.1929648	-0.1929523	0.125×10^{-4}
2.1047	-0.7712232	-0.7712058	0.174×10^{-4}
2.7032	-1.3308922	-1.3308711	0.211×10^{-4}
3.3017	-1.8089085	-1.8088836	0.249×10^{-4}
3.9002	-2.3034008	-2.3033692	0.316×10^{-4}

7. Concluding Remarks

We have shown that approximations to integro-difference equations can be obtained by using simple quadrature formulas. Numerical results show that the midpoint and trapezoidal methods are equally and reasonably accurate, while the rectangular method is less satisfactory. From computational point of view, the trapezoidal method is much easier to apply than the midpoint method because the choice of meshsize is easier in this case.

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