BOUNDED OR ALMOST PERIODIC CLASSICAL SOLUTIONS FOR SOME NONLINEAR PARABOLIC EQUATIONS

By

Mitsuhiro NAKAO and Tokumori NANBU

(Received Oct. 31, 1975)

§0. Introduction

In this paper we shall consider the following nonlinear partial differential equation of parabolic type

$$(0.1) \quad \frac{\partial}{\partial t} u + \sum_{|\gamma|, |\beta| \leq m} (-1)^{|\gamma|} D_x^\gamma (A_{\gamma\beta}(x) D_x^\beta u) + B(x, u, \dots, D_x^{l_0} u) \\ = f(x, t)$$

in $\Omega \times [0, \infty)$ or $\Omega \times (-\infty, \infty)$ together with the boundary condition

$$(0.2) \quad D_x^\gamma u|_{\partial\Omega} = 0 \quad \text{for } |\gamma| \leq m-1 \quad (m \geq 1),$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$ and $D_x^\gamma u$ or $D_x^{|\gamma|} u$ stands for x -derivatives of order $|\gamma| = |(\gamma_1, \dots, \gamma_n)| = \sum_{i=1}^n \gamma_i$, $\gamma_i \geq 0$.

Throughout the paper, the functions considered are all real valued and we make the following assumptions:

$$H_1. \quad 2m \geq \left[\frac{n}{2} \right] + 1, \quad A_{\gamma\beta} \in C^{8m+|\gamma|-2}(\bar{\Omega})$$

and

$$C_0^2 \sum_{|\gamma|, |\beta| \leq m} A_{\gamma\beta}(x) D^\gamma u D^\beta u \geq \|u\|_{W_0^{m,2}}^2 \quad \text{for } u \in W_0^{m,2}$$

with some $C_0 > 0$.

$$H_2. \quad B(x, V_0, V_1, \dots, V_{l_0}) \text{ is } \max\left(3, \left[\frac{n}{2} \right] + 1\right) \text{ times continuously differentiable}$$

in its arguments, and satisfies the conditions

$$|B(x, V_0, \dots, V_{l_0})| \leq k_1 \left(\sum_{i=0}^{l_0} |V_i|^2 \right)^{\frac{1+\alpha}{2}},$$

$$\left| \frac{\partial B}{\partial V_i} \right| \leq k_1 \left(\sum_{i=0}^{l_0} |V_i|^2 \right)^{\frac{\alpha}{2}},$$

$$\left| \frac{\partial^2 B}{\partial V_i \partial V_j} \right| \leq k_1 \left(\sum_{i=0}^{l_0} |V_i|^2 \right)^{\frac{\alpha}{2}}$$

and

$$\left| \frac{\partial^3 B}{\partial V_i \partial V_j \partial V_k} \right| \leq k_1 \left(\sum_{i=0}^{l_0} |V_i|^2 \right)^{\frac{\tilde{\alpha}}{2}},$$

where k_1 is a positive constant, $\tilde{\alpha} = \max(0, \alpha - 1)$, $\tilde{\tilde{\alpha}} = \max(0, \alpha - 2)$, $l_0 = \max \left\{ 0, \min \left(\left[m - \frac{n}{3} \right], 2m - \left[\frac{n}{2} \right] - 1 \right) \right\}$, and α is a positive number such that

$$0 < \alpha < \frac{2(m-l_0)}{n-2(m-l_0)} \quad \text{if } l_0 > m - \frac{n}{2},$$

$0 < \alpha < \infty$ if $0 \leq l_0 \leq m - \frac{n}{2}$. Strictly speaking V_i is a vector $V_i = (\dots, V_i^{(v)}, \dots)$ and $|V_i|^2 = \sum_v |V_i^{(v)}|^2$, but here and henceforth, for brevity, we use the notation $\frac{\partial B}{\partial V_i}$, etc., for any partial derivatives with respect to the components of V_i .

$$H_3, \quad f \in \bigcap_{k=0}^3 C^{3-k}(R; W^{6m-k, 2} \cap W_0^{5m-k, 2})$$

and

$$M \equiv \sup_{t \in R} \max (\|f(t)\|_{L^2}, \|Af(t)\|_{L^2}, \|D_t f(t)\|_{L^2}, \|A^2 f(t)\|_{L^2}, \|AD_t f(t)\|_{L^2}, \\ \|D_t^2 f(t)\|_{L^2}, \|A^3 f(t)\|_{L^2}) < +\infty.$$

Recently, using L^2 -method, one of the present authors [4] has proved the existence of a bounded or almost periodic strong solution for the equation (0.1)–(0.2) with $m=1$, $l_0=0$. Subsequently, in [5] we have given an existence theorem of such solutions in the classical sense for similar equations of second order. For

the existence of global or periodic classical solutions we can refer Friedmann [1], Oleinik & Kruzhkov [6], Prodi [7], Smulev [9] and others.

However, if $m > 1$, the classical methods used in [1] [5] [6] [9] fail to our problem here, because the so-called maximum principle holds no longer to the equation (0.1)–(0.2), and we must employ another approach for the existence of classical solutions.

In this paper, combining the methods in [4] and Sather [8], we establish an existence theorem of a bounded or almost periodic classical solution for the problem (0.1)–(0.2) with $m \geq 1$. It should be noted that we make no monotonicity assumption on $B(x, u, \dots, D_x^{l_0} u)$ in $D_x^\gamma u$ ($|\gamma| \leq l_0$).

§1. Preliminaries

We employ usual notations for function spaces (see, e. g., Lions [2], Mizohata [3]). The following well known lemma is essential for our arguments.

LEMMA 1.1. (Sobolev)

Let $u \in W_0^{m,2}$. Then we have

(1) if $2m > n$, then $W_0^{m,2} \subset C^{m-\lfloor \frac{n}{2} \rfloor - 1}$

and

$$(1.1) \quad \sum_{0 \leq i \leq \lfloor m - \frac{n}{2} \rfloor} |D_x^i u(x)| \leq S \|u\|_{W_0^{m,2}} \quad \text{for } u \in W_0^{m,2}$$

and

(2) if $l \geq 0$, $m - \frac{n}{2} \leq l \leq m$, then $W_0^{m,2} \subset W_0^{l, \frac{2n}{n-2(m-l)}}$ and

$$(1.2) \quad \sum_{i=0}^l \|D_x^i u\|_{L^q} \leq S \|u\|_{W_0^{m,2}}, \quad \left(q \leq \frac{2n}{n-2(m-l)} \right)$$

where S is a certain constant.

Let $\langle \cdot, \cdot \rangle$ denote the duality relationship between $W^{-m,2}$ and $W_0^{m,2}$. We define the linear operator $A: W_0^{m,2} \rightarrow W^{-m,2}$ by

$$\langle Au, v \rangle = \int_{\Omega} \sum_{|\gamma|, |\beta| \leq m} (-1)^{|\gamma|} A_{\gamma\beta}(x) D^\gamma v D^\beta u dx$$

for $u, v \in W_0^{m,2}$, and put

$$\|u\|_{H_0^m}^2 = \langle Au, u \rangle \quad \text{for } u \in W_0^{m,2}.$$

Note that by the assumption H_1 and Lemma 1.1, the norm $\|\cdot\|_{H_0^m}$ is equivalent to $\|\cdot\|_{W_0^{m,2}}$ and (1.1), (1.2) can be written as

$$(1.1)' \quad \sum_{0 \leq i \leq [m-\frac{n}{2}]} |D_x^i u(x)| \leq C_0 S \|u\|_{H_0^m}$$

and

$$(1.2)' \quad \sum_{i=0}^l \|D_x^i u\|_{L^q} \leq C_0 S \|u\|_{H_0^m}.$$

Also we note that if $u \in W_0^{m,2}$, then

$$B(\cdot, u, \dots, D_x^{l_0} u) \in L^2$$

by H_1 .

§2. Approximate Solutions

We employ the Galerkin's method for our purpose. Let $\{\phi_j\}_{j=1}^\infty$ be the basis of $W_0^{m,2}$ consisting of the eigenfunctions of A , and consider the system of ordinary differential equations

$$(2.1) \quad \begin{aligned} (u'_k(t), \phi_j) + \langle Au_k(t), \phi_j \rangle + (B(\cdot, u_k, \dots, D_x^{l_0} u_k), \phi_j) \\ = (f(t), \phi_j), \quad j = 1, 2, \dots, k \end{aligned}$$

with

$$(2.2) \quad u_k(0) = 0,$$

where $u_k = \sum_{i=1}^k \lambda_i^k(t) \phi_i$.

Here we note that we may assume $\phi_j \in W^{8m,2} \cap W_0^{m,2}$ since $\partial\Omega$ is sufficiently smooth. (2.1) is equivalent to

$$(2.1)' \quad u'_k(t) + Au_k(t) + P_k B(\cdot, u_k, \dots, D_x^{l_0} u_k) = P_k f,$$

where P_k is the projection of the space L^2 into the subspace $[\phi_1, \dots, \phi_k]$ in L^2 . As is easily seen, $u_k(t)$ exists on some interval, say, $[0, T_k]$. From now on we shall estimate $u_k(t)$ in various norms. First, we show

LEMMA 2.1.

Suppose that there exist two numbers $a, b > 0$ such that

$$M_0 < a(1 - k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha) / C_0^2 S^2,$$

$$M_0 < ab(1 - k_1 C_0^{\alpha+1} S^{\alpha+1} b^{\alpha+1}) \quad \text{if } \alpha \geq 1$$

and

$$1 - k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha > 0,$$

where $M_0 = \sup_t \|f(t)\|_{L^2}$

$$L_0 = \begin{cases} b & \text{if } \alpha \geq 1 \\ \sqrt{2a(M_0 + b_1)} & \text{if } 1 > \alpha > 0 \end{cases}$$

with

$$b_1 = \frac{1}{2} (1 - \alpha) (a(\alpha + 1))^{\frac{\alpha+1}{1-\alpha}} (k_1 C_0^{\alpha+1} S^{\alpha+1})^{\frac{2}{1-\alpha}}.$$

Then $u_k(t)$ exists on $R^+ \equiv [0, \infty)$ and we have

$$(2.3) \quad \|u_k(t)\|_{L^2} < a \quad \text{and} \quad \|u_k(t)\|_{H_0^m} \leq L_0.$$

Proof.

It suffices to show (2.3) for $0 \leq t \leq T_k$. The proof is essentially the same as that of Lemma 2.1 in [4], and is given mainly for completeness.

By (2.1) we have first

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_k(t)\|_{L^2}^2 + \|u_k(t)\|_{H_0^m}^2 + (B(\cdot, u_k, \dots, D_x^{l_0} u_k), u_k(t)) \\ & = (f(t), u_k(t)) \end{aligned}$$

and, by Lemma 1.1,

$$\begin{aligned} (2.4) \quad \frac{1}{2} \frac{d}{dt} \|u_k(t)\|_{L^2}^2 & \leq M_0 \|u_k(t)\|_{L^2} - \|u_k(t)\|_{H_0^m}^2 \\ & \quad + k_1 C_0^{\alpha+2} S^{\alpha+2} \|u_k(t)\|_{H_0^m}^{\alpha+2}. \end{aligned}$$

Next, we have

$$(2.4) \quad (u'_k(t), Au_k(t)) + \|Au_k(t)\|_{L^2}^2 + (B(\cdot, u_k, \dots, D_x^{l_0} u_k), Au_k(t)) \\ = (f(t), Au_k(t))$$

and

$$(2.5) \quad \frac{1}{2} \frac{d}{dt} \|u_k(t)\|_{H_0^m}^2 \leq (\|f(t)\|_{L^2} + \|B(\cdot, u_k, \dots, D_x^{l_0} u_k)\|_{L^2} \\ - \|Au_k(t)\|_{L^2}) \|Au_k(t)\|_{L^2} \\ \leq (M_0 + K_1 C_0^{\alpha+1} S^{\alpha+1} \|u_k(t)\|_{H_0^m}^{\alpha+1} \\ - \|Au_k(t)\|_{L^2}) \|Au_k(t)\|_{L^2}.$$

We shall prove (2.3) by contradiction. Suppose that (2.3) was false. Then there would be numbers t_1 and (or) $t_2 \leq T_k$ such that

$$(2.6) \quad \|u_k(t)\|_{L^2} < a \quad \text{for } 0 \leq t < t_1 \quad \text{and} \quad \|u_k(t_1)\|_{L^2} = a$$

and (or)

$$(2.7) \quad \|u_k(t)\|_{H_0^m} \leq L_0 \quad \text{for } 0 \leq t \leq t_2 \quad \text{and} \quad \|u_k(t)\|_{H_0^m} > L_0$$

for $t_2 < t < t_2 + \varepsilon$, where ε is any positive number. If there are no number such as t_1 or t_2 , we take $t_1 = T_k$ or $t_2 = T_k$. (i) The case $t_2 \geq t_1$. In this case, for $0 \leq t \leq t_1$, we have by (2.4)

$$\frac{1}{2} \frac{d}{dt} \|u_k(t)\|_{L^2}^2 \leq M_0 \|u_k(t)\|_{L^2} - (1 - k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha) \|u_k(t)\|_{H_0^m}^2,$$

(noting $1 - k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha > 0$)

$$\leq \{M_0 - (1 - k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha) S^{-2} C_0^{-2}\} \|u_k(t)\|_{L^2} \times \\ \times \|u_k(t)\|_{L^2},$$

which implies

$$\|u_k(t)\|_{L^2} \leq M_0 S^2 C_0^2 / (1 - k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha) \\ < a \quad \text{for } 0 \leq t \leq t_1.$$

This is a contradiction to (2.6).

(ii) The case $t_1 > t_2$. In this case, for $0 \leq t \leq t_1$, we have

$$\begin{aligned}
 (2.8) \quad & \frac{1}{2} \frac{d}{dt} \|u_k(t)\|_{H_0^m}^2 \leq (M_0 + k_1 C_0^{\alpha+1} S^{\alpha+1} \|u_k(t)\|_{H_0^m}^{\alpha+1} \\
 & - \frac{1}{a} \|Au_k(t)\|_{L^2} \|u_k(t)\|_{L^2} \|Au_k(t)\|_{L^2} \\
 & \leq (M_0 + C_0^{\alpha+1} S^{\alpha+1} \|u_k(t)\|_{H_0^m}^{\alpha+1} - \frac{1}{a} \|u_k(t)\|_{H_0^m}^2) \times \\
 & \quad \times \|Au_k(t)\|_{L^2}.
 \end{aligned}$$

If $0 \leq \alpha < 1$, this and Young's inequality give

$$\frac{1}{2} \frac{d}{dt} \|u_k(t)\|_{H_0^m}^2 \leq (M_0 + b_1 - \frac{1}{2a} \|u_k(t)\|_{H_0^m}^2) \|Au_k(t)\|_{L^2},$$

which implies

$$\|u_k(t)\|_{H_0^m} \leq \sqrt{2a(M_0 + b_1)} = L_0 \quad \text{for } 0 \leq t < t_1.$$

This is a contradiction to (2.7).

If $1 \leq \alpha$, we have by (2.8)

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|u_k(t)\|_{H_0^m}^2 & \leq \{M_0 + (k_1 S^{\alpha+1} C_0^{\alpha+1} b^{\alpha-1} - a^{-1}) \|u_k(t)\|_{H_0^m}^2\} \times \\
 & \quad \times \|Au_k(t)\|_{L^2}
 \end{aligned}$$

for $0 \leq t \leq t_2$. From this we can conclude

$$\|u_k(t)\|_{H_0^m}^2 \leq M_0 / (a^{-1} - k_1 C_0^{\alpha+1} S^{\alpha+1} b^{\alpha-1}) < b^2$$

for $0 \leq t \leq t_2$, which contradicts also (2.7), since $\|u_k(t_2)\|_{H_0^m} = L_0 = b$. q. e. d.

REMARK 1.1.

Note that we can choose a and L_0 as small as is wanted if M_0 is sufficiently small.

LEMMA 2.2.

$$(2.9) \quad \|u'_k(t)\|_{L^2} \leq a_1 \quad \text{and} \quad \|u'_k(t)\|_{H_0^m} \leq L_1$$

for $t \in \mathbb{R}^+$, where

$$a_1 = \max\left(\frac{C_0^2 S^2 M}{1 - k_1(I_0 + 1)C_0^{\alpha+2}S^{\alpha+2}L_0^\alpha}, M\right)$$

and

$$L_1 = \max(M, \sqrt{2a_1M + a_1^2(k_1C_0^{\alpha+1}S^{\alpha+1}L_0^\alpha)^2}).$$

Proof.

Differentiating (2.1) with respect to t , we have

$$(2.10) \quad (u_k''(t), \phi_j) + \langle Au_k'(t), \phi_j \rangle + \left(\sum_{i=0}^{l_0} \frac{\partial B}{\partial V_i} D_x^i u_k', \phi_j\right) \\ = (f'(t), \phi_j).$$

By (2.10) we have first

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_k'(t)\|_{L^2}^2 &\leq \|f'(t)\|_{L^2} \|u_k'(t)\|_{L^2} - \|u_k'(t)\|_{H_0^m}^2 \\ &\quad + k_1 \int_{\Omega} \left(\sum_{i=0}^{l_0} |D_x^i u_k(t)|^2\right)^{\frac{\alpha}{2}} \times \\ &\quad \times \left(\sum_{i=0}^{l_0} |D_x^i u_k'(t)|^2\right)^{\frac{1}{2}} |u_k'(t)| dx \\ &\leq M \|u_k'(t)\|_{L^2} - \|u_k'(t)\|_{H_0^m}^2 \\ &\quad + k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha \|u_k'\|_{H_0^m} \|u_k'\|_{L^2} \\ &\leq (C_0 S M + k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha \|u_k'(t)\|_{H_0^m} \\ &\quad - \|u_k'(t)\|_{H_0^m}) \|u_k'(t)\|_{H_0^m} \end{aligned}$$

(noting $1 - C_0^{\alpha+2} k_1 S^{\alpha+2} L_0^\alpha > 0$)

$$(2.11) \quad \leq \{C_0 S M - (1 - k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha)/C_0 S \cdot \\ \cdot \|u_k'\|_{L^2}\} \|u_k'(t)\|_{H_0^m}.$$

On the other hand, we have by (2.1)'

$$(2.12) \quad \|u_k'(0)\|_{L^2} = \|D_t f(x, 0)\|_{L^2} \leq M.$$

From (2.11) and (2.12) we see

$$\|u'_k(t)\|_{L^2} \leq a_1 \quad \text{for } t \in R^+.$$

Next, we have, by (2.10) again,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u'_k(t)\|_{H_0^m}^2 + \|Au'_k(t)\|_{L^2}^2 \\ & \leq \|D_t f(t)\|_{L^2} \|Au'_k(t)\|_{L^2} + k_1 \int_{\Omega} \left(\sum_{i=0}^{l_0} |D_x^i u_k|^2 \right)^{\frac{\alpha}{2}} \times \\ & \quad \times \left(\sum_{i=0}^{l_0} |D_x^i u'_k(t)|^2 \right)^{\frac{1}{2}} |Au'_k(t)| dx \end{aligned}$$

and

$$\begin{aligned} (2.13) \quad & \frac{1}{2} \frac{d}{dt} \|u'_k(t)\|_{H_0^m}^2 \leq (M + k_1 C_0^{\alpha+1} S^{\alpha+1} L_0^g \|u'_k\|_{H_0^m} \\ & \quad - \|Au'_k\|) \|Au'_k(t)\|_{L^2} \\ & \leq (M + k_1 C_0^{\alpha+1} S^{\alpha+1} L_0^g \|u'_k\|_{H_0^m} \\ & \quad - \frac{1}{a_1} \|u'_k\|_{H_0^m}^2) \|Au'_k(t)\|_{L^2} \\ & \leq \left\{ M + \frac{a_1}{2} (k_1 C_0^{\alpha+1} S^{\alpha+1} L_0^g)^2 \right. \\ & \quad \left. - \frac{1}{2a_1} \|u'_k(t)\|_{H_0^m}^2 \right\} \|Au'_k(t)\|_{L^2}. \end{aligned}$$

Furthermore we have by (2.1)'

$$\begin{aligned} (2.14) \quad & \|u'_k(0)\|_{H_0^m}^2 \leq M \|Au'_k(0)\|_{L^2} = M \|AD_k f(0)\|_{L^2} \\ & \leq M \|Af(0)\|_{L^2} \leq M^2. \end{aligned}$$

By (2.13) and (2.14) we see

$$\|u'_k(t)\|_{H_0^m} \leq L_1. \quad \text{q.e.d.}$$

We proceed to further estimates of approximate solutions $\{u_k(t)\}$.

LEMMA 2.3.

We have

$$(2.15) \quad \|D_t^2 u_k(t)\|_{L^2} \leq a_2 \quad \text{and} \quad \|D_t^2 u_k(t)\|_{H_0^m} \leq L_2$$

for $t \in R^+$, where

$$a_2 = \max \left(2M, \frac{C_0^2 S^2 M + k_1 C_0^{4+\alpha} S^{4+\alpha} L_1^2 L_0^\alpha}{1 - k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha} \right).$$

and

$$L_2 = \max (2M, \sqrt{2a_2 M + a_2^2 \{k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha + k_1 C_0^{2+\alpha} S^{\alpha+2} L_1 L_0^\alpha\}^2}).$$

Proof.

Differentiating (2.10) with respect to t , we have

$$(2.16) \quad (D_t^3 u_k, \phi_j) + \langle A D_t^2 u_k(t), \phi_j \rangle + \left(\sum_{i=0}^{l_0} \frac{\partial B}{\partial V_i} D_x^i u_k + \right. \\ \left. + \sum_{i,j=0}^{l_0} \frac{\partial^2 B}{\partial V_i \partial V_j} D_x^i D_t u_k(t) D_x^j D_t u_k(t), \phi_j \right) = (D_t^2 f(t), \phi_j)$$

for $j = 1, \dots, k$.

First we have from above

$$(2.17) \quad \frac{1}{2} \frac{d}{dt} \|D_t^2 u_k(t)\|_{L^2}^2 \leq \|D_t^2 f\|_{L^2} \|D_t^2 u_k\|_{L^2} - \|D_t^2 u_k\|_{H_0^m}^2 + \\ + k_1 \int_{\Omega} \left(\sum_{i=0}^{l_0} |D_x^i u_k|^2 \right)^{\frac{\alpha}{2}} \left(\sum_{i=0}^{l_0} |D_x^i D_t^2 u_k|^2 \right)^{\frac{1}{2}} \times \\ \times |D_t^2 u_k| dx \\ + k_1 \int_{\Omega} \left(\sum_{i=0}^{l_0} |D_x^i u_k|^2 \right)^{\frac{\alpha}{2}} \left(\sum_{i=0}^{l_0} |D_x^i D_t u_k|^2 \right)^{\frac{1}{2}} \times \\ \times \left(\sum_{i=0}^{l_0} |D_x^i D_t u_k|^2 \right)^{\frac{1}{2}} |D_t^2 u_k| dx.$$

Since

[the third term of the right hand side of (2.17)]

$$\leq k_1 \|D_t^2 u_k(t)\|_{L^2} C_0^2 S^2 \|D_t u_k(t)\|_{H_0^m}^2 \int_{\Omega} \sum_{i=0}^{l_0} |D_x^i u_k|^{\frac{\alpha}{2} \frac{n}{4m-n-4l_0}} dx \\ \leq k_1 C_0^{2+\alpha} S^{2+\alpha} L_1^2 L_0^\alpha \|D_t^2 u_k(t)\|_{L^2},$$

we have from (2.17)

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|D_t^2 u_k(t)\|_{L^2}^2 &\leq (C_0 S M + k_1 C_0^{2+\alpha} S^{2+\alpha} L_0^\alpha \|D_t^2 u_k\|_{H_0^m} \\
 &\quad + k_1 C_0^{3+\alpha} S^{3+\alpha} L_1^2 L_0^\alpha - \|D_t^2 u_k\|_{H_0^m}) \times \\
 &\quad \times \|D_t^2 u_k(t)\|_{H_0^m} \\
 &\leq \{C_0 S M - (1 - k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha) C_0^{-1} S^{-1} \|D_t^2 u_k\|_{L^2} \\
 &\quad + k_1 C_0^{3+\alpha} S^{3+\alpha} L_1^2 L_0^\alpha\} \|D_t^2 u_m\|_{H_0^m}.
 \end{aligned}
 \tag{2.18}$$

Also, at $t=0$, we have

$$\begin{aligned}
 \|D_t^2 u_k(0)\|_{L^2} &= \left\| P_k \left\{ D_t f - A(D_t u_k)_0 - \Sigma \left(\frac{\partial B}{\partial V_i} D_x^i (D_t u_k) \right)_0 \right\} \right\|_{L^2} \\
 &\leq \|D_t f\|_{L^2} + \|A f\|_{L^2} \leq 2M.
 \end{aligned}$$

By (2.18) and (2.19) we see

$$\|D_t^2 u_k(t)\|_{L^2} \leq a_2,$$

which proves the former part of (2.15).

Next, we shall show the latter part of (2.15). By (2.16) again we have

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|D_t^2 u_k(t)\|_{H_0^m}^2 &+ \|A D_t^2 u_k(t)\|_{L^2}^2 \\
 &\leq (\|D_t^2 f\|_{L^2} + k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha \|D_t^2 u_k(t)\|_{H_0^m} \\
 &\quad + k_1 C_0^{1+\alpha} S^{1+\alpha} L_1 L_0^\alpha \|D_t^2 u_k\|_{L^2}) \|A D_t^2 u_k\|_{L^2},
 \end{aligned}$$

and hence

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|D_t^2 u_k(t)\|_{H_0^m}^2 &\leq (M + k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha \|D_t^2 u_k\|_{H_0^m} \\
 &\quad + k_1 C_0^{2+\alpha} S^{2+\alpha} L_1 L_0^\alpha \|D_t^2 u_k\|_{H_0^m} \\
 &\quad - a_2^{-1} \|D_t^2 u_k(t)\|_{H_0^m}^2) \|A D_t^2 u_k\|_{L^2} \\
 &\leq \left\{ M + \frac{a_2}{2} (k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha + k_1 C_0^{2+\alpha} S^{2+\alpha} L_1 L_0^\alpha)^2 \right. \\
 &\quad \left. - (2a_2)^{-1} \|D_t^2 u_k\|_{H_0^m}^2 \right\} \|A D_t^2 u_k\|_{L^2},
 \end{aligned}$$

which together with the estimate at $t=0$:

$$\begin{aligned} \|D_t^2 u_k(0)\|_{H_0^m}^2 &\leq 2M \|A(D_t^2 u_k)_0\|_{L^2} \\ &= 2M \left\| AP_k \left\{ D_t f - A(D_t u_k)_0 - \left(\sum \frac{\partial B}{\partial V_i} D_t D_x^i u_k \right)_0 \right\} \right\|_{L^2} \end{aligned}$$

(by H_1)

$$\leq 2M(\|A(D_t f)_0\|_{L^2} + \|A^2 f(0)\|_{L^2}) \leq 4M^2$$

implies the latter assertion of (2.15).

q. e. d.

Finally in this section we shall show

LEMMA 2.4.

$$(2.20) \quad \|D_t^3 u_k(t)\|_{L^2} \leq a_3 \quad \text{for } t \in \mathbb{R}^+,$$

where

$$\begin{aligned} a_3 = \max \{ &\sqrt{3M + k_1 C_0^2 S^2 M^2}, (C_0^2 S^2 M + k_1 C_0^{3+\alpha} S^{3+\alpha} L_1 L_2 L_0^\alpha + \\ &+ k_1 C_0^{5+\alpha} S^{5+\alpha} L^{3+\alpha}) / (1 - k_1 C_0^{2+\alpha} S^{2+\alpha} L_0^\alpha) \}. \end{aligned}$$

Proof.

Differentiation of (2.16) yields

$$\begin{aligned} (2.21) \quad (D_t^4 u_k(t), \phi_j) &+ \langle AD_t^3 u_k, \phi_j \rangle + (D_t^3 B(\cdot, u_k, \dots, D_x^{l_0} u_k), \phi_j) \\ &= (D_t^3 f(t), \phi_j), \quad j = 1, \dots, k. \end{aligned}$$

Then we have, as is usual,

$$\begin{aligned} (2.22) \quad \frac{1}{2} \frac{d}{dt} \|D_t^3 u_k(t)\|_{L^2}^2 &\leq \|D_t^3 f\|_{L^2} \|D_t^3 u_k\|_{L^2} - \|D_t^3 u_k\|_{H_0^m}^2 \\ &+ \int_{\Omega} |D_t^3 B(\cdot, u_k, \dots, D_x^{l_0} u_k)| |D_t^3 u_k| dx. \end{aligned}$$

Using the assumption $l_0 \leq \left[m - \frac{n}{3} \right]$, we see easily
|the last term of (2.22)|

$$\leq \int_{\Omega} \left(\sum_{i=0}^{l_0} \left| \frac{\partial B}{\partial V_i} \right| |D_x^i D_t^3 u_k| \right)$$

$$\begin{aligned}
& + \sum_{i,j=0}^{l_0} \left| \frac{\partial^2 B}{\partial V_i \partial V_j} \right| |D_x^i D_t^2 u_k| |D_x^i D_t u_k| \\
& + \sum_{i,j,l=0}^{l_0} \left| \frac{\partial^3 B}{\partial V_i \partial V_j \partial V_l} \right| |D_x^i D_t u_k| |D_x^i D_t u_k| |D_x^i D_t u_k| \Big) \times \\
& \times |D_t^3 u_k| dx \\
& \leq k_1 C_0^{\alpha+1} S^{\alpha+1} L_0^\alpha \|D_t^3 u_k\|_{L^2} \|D_t^3 u_k\|_{H_0^m} \\
& + k_1 c_0^{1+\tilde{\alpha}} S^{1+\tilde{\alpha}} L_0^{\tilde{\alpha}} L_1 L_2 \|D_t^3 u_k\|_{L^2} \\
& + k_1 C_0^{3+\tilde{\alpha}} S^{3+\tilde{\alpha}} L_0^{3+\tilde{\alpha}} \|D_t^3 u_k\|_{L^2}.
\end{aligned}$$

From (2.22) and (2.23) we have

$$\begin{aligned}
(2.24) \quad & \frac{1}{2} \frac{d}{dt} \|D_t^3 u_k(t)\|_{L^2}^2 \leq \{C_0 S M - (1 - k_1 C_0^{2+\alpha} S^{2+\alpha} L_0^\alpha) \times \\
& \times C_0^{-1} S_0^{-1} \|D_t^3 u_k\|_{L^2} + k_1 C_0^{2+\tilde{\alpha}} S^{2+\tilde{\alpha}} L_0^{\tilde{\alpha}} L_1 L_2 \\
& + k_1 C_0^{4+\tilde{\alpha}} S^{4+\tilde{\alpha}} L_0^{3+\tilde{\alpha}}\} \|D_t^3 u_k\|_{H_0^m}.
\end{aligned}$$

Also we have

$$\begin{aligned}
(2.25) \quad & \|D_t^3 u_k(0)\|_{L^2}^2 = \|P_k(D_t^2 f(0) - A(D_t^2 u_k)_0 - (D_t^2 B)_0)\|_{L^2}^2 \\
& \leq \|D_t^2 f(0)\|_{L^2}^2 + \|AP_k(-AD_t u_k - D_t B + D_t f)_0\|_{L^2}^2 \\
& + \left\| \left(\Sigma \frac{\partial^2 B}{\partial V_i \partial V_j} D_x^i D_t u_k D_x^i D_t u_k \right)_0 \right\|_{L^2}^2 \\
& + \left\| \left(\Sigma \frac{\partial B}{\partial V_i} D_x^i D_t^2 u_k \right)_0 \right\|_{L^2}^2 \\
& \leq 2M + \|AP_k A(-Au_k - P_k B + f)_0\|_{L^2}^2 \\
& + k_1 C_0^2 S^2 M^2 \\
& \leq 3M + k_1 C_0^2 S^2 M^2.
\end{aligned}$$

By (2.24) and (2.25) we obtain (2.20).

q. e. d.

§3. Existence of Bounded Solution

In this section we shall prove the existence of a bounded classical solution of the problem (0.1)–(0.2). Let $\{u_k\}$ be the approximate solutions constructed in the previous section. An immediate consequence of Lemmas 2.1–2.4 is that there exists a subsequence of $\{u_k\}$, which will be denoted by the same notation, and a function $u \in L^2_{loc}(\Omega \times R^+)$ such that

$$(3.1) \quad D_t^l u_k \longrightarrow D_t^l u \quad \text{in } L^2_{loc}(\Omega \times R^+) \quad \text{and a.e. in } \Omega \times R^+$$

$$D_x^\gamma u_k \longrightarrow D_x^\gamma u \quad \text{in } L^2_{loc}(\Omega \times R^+) \quad \text{and a.e. in } \Omega \times R^+$$

for $0 \leq l \leq 2$, $|\gamma| \leq m-1$.

Moreover, from the equations

$$D_t^l u_k(x, t) = (D_t^l u_k)_0(x) + \int_0^t D_t^{l+1} u_k(x, s) ds$$

$$D_x^\gamma u_k(x, t) = (D_x^\gamma u_k)_0(x) + \int_0^t D_x^\gamma D_t u_k(x, s) ds$$

we obtain, for $u_{kl} = u_k - u_l$,

$$(3.2) \quad \|D_t^j u_{k,l}\|_{L^2} \leq \|(D_t^j u_{k,l})_0\|_{L^2} + \sqrt{t} \sqrt{\int_\Omega \int_0^t |D_s^{j+1} u_{k,l}(x, s)|^2 ds dx}$$

$$\|D_x^\gamma u_{k,l}\|_{L^2} \leq \|(D_x^\gamma u_{k,l})_0\|_{L^2} + \sqrt{t} \sqrt{\int_\Omega \int_0^t |D_x^\gamma D_t u_{k,l}|^2 ds dx}$$

with $0 \leq j \leq 2$, $|\gamma| \leq m$, and hence by (3.1)

$$(3.3) \quad \begin{aligned} D_t^j u_k(\cdot, t) &\longrightarrow D_t^j u(\cdot, t) \text{ in } L^2(\Omega) \text{ uniformly} \\ &\text{on each compact set in } R^+ \quad (j=0, 1), \text{ and} \\ D_x^\gamma u_k(t) &\longrightarrow D_x^\gamma u(\cdot, t) \text{ in } L^2(\Omega) \text{ uniformly} \\ &\text{on each compact set in } R^+ \quad (|\gamma| \leq m-1). \end{aligned}$$

Further it is easy to see that

$$(3.4) \quad \begin{aligned} \|D_t^2 u(\cdot, t) - D_t^2 u(\cdot, s)\|_{L^2} &\leq \text{const. } |t-s|, \\ \|D_x^\gamma D_t^j(\cdot, t) - D_x^\gamma D_t^j u(\cdot, s)\|_{L^2} &\leq \text{const. } |t-s| \end{aligned}$$

with $|\gamma| \leq m, j=0, 1$.

Next, by the equation

$$Au_k = -u'_k - P_k B(\cdot, u_k, \dots, D_x^{l_0} u_k) + f$$

and by the estimates in the previous section we have

$$\begin{aligned} \|u_k(\cdot, t)\|_{2m} &\leq \text{const. } \|Au_k\|_{L^2} \\ &\leq \text{const. } (\|u'_k\|_{L^2} + \|B(\cdot, u_k, \dots, D_x^{l_0} u_k)\|_{L^2} + \|f\|_{L^2}) \\ &\leq \text{const. } < +\infty, \end{aligned}$$

and hence, with the aid of Sobolev's imbedding theorem (notice that $2m > \left[\frac{n}{2}\right] + 1$),

$$(3.5) \quad |D_x^\gamma u_k(x, t)| \leq \text{const. } < +\infty \quad \text{for } |\gamma| \leq l_0, \quad t \in R^+.$$

Similarly, we have

$$(3.6) \quad |D_t D_x^\gamma u_k(x, t)| \leq \text{const. } < +\infty \quad \text{for } |\gamma| \leq l_0, \quad t \in R^+.$$

The relations (3.3) together with (3.5) and (3.6) show in a straight forward manner

$$(3.7) \quad D_t^j B(\cdot, u_k, \dots, D_x^{l_0} u_k) \longrightarrow D_t^j B(\cdot, u, \dots, D_x^{l_0} u)$$

in $L^2(\Omega)$ uniformly on each compact set in R^+ for $j=0, 1$. Since we may assume by a standard argument

$$(3.8) \quad D_x^\gamma (D_t^j u_k(\cdot, t)) \longrightarrow D_x^\gamma (D_t^j u(\cdot, t))$$

weakly in $L_{loc}^\infty(R^+; L^2(\Omega))$ for $|\gamma| \leq m, j=0, 1$, we see easily from above results

$$(3.9) \quad D_t^{j+1} u + A D_t^j u + D_t^j B(\cdot, u, \dots, D_x^{l_0} u) = D_t^j f$$

in L^2 a. e. on R^+ ($j=0, 1$), and

$$(3.10) \quad u(x, 0) = 0, \quad u(t) \in W_0^{m,2}.$$

It remains to investigate the regularity of the weak solution u in (3.9)–(3.10). We note that the estimates (2.3), (2.9), (2.15) and (2.20) remain valid for u_m replaced by u . First, by (3.9) with $j=0$, the $W_0^{m,2}$ -valued function u is a weak solution of the elliptic equation

$$(3.11) \quad Au(\cdot, t) = f(\cdot, t) - D_t u(\cdot, u) - B(\cdot, u, \dots, D_x^{l_0} u)$$

for a.e. $t \in R^+$, and hence

$$(3.12) \quad \|u(\cdot, t)\|_{W^{2m,2}} \leq \text{const.} \cdot \|\text{the right hand side of (3.11)}\|_{L^2} \\ \leq \text{const.} < +\infty.$$

From (3.12), (3.3) and (3.7) we see easily

$$(3.13) \quad u \in C^0(R^+; W^{2m} \cap W_0^{m,2}),$$

and hence

$$(3.14) \quad D_x^\gamma u(x, t) \in C^0(\bar{\Omega} \times R^+) \quad \text{if} \quad 0 \leq |\gamma| \leq 2m - \left[\frac{n}{2} \right] - 1.$$

Equation (3.9) with $j=1$ shows that for a.e. $t \in R^+$ $W_0^{m,2}$ -valued function $D_t u(\cdot, t)$ is a weak solution of the equation

$$(3.15) \quad A(D_t u) = D_t f - \sum_{i=0}^{l_0} \left(\frac{\partial B}{\partial V_i} \cdot D_x^i D_t u \right) - D_t^2 u \in L^2,$$

and by (3.4) we have

$$(3.16) \quad D_t u \in C^0(R^+; W^{2m,2}),$$

which implies by Sobolev's lemma

$$(3.17) \quad D_t u \in C^0(\bar{\Omega} \times R^+).$$

Furthermore, by (3.13), we have $D_x^{l_0} u \in C^0(R^+; W^{[n/2]+1,2}) \subset C^0(\bar{\Omega} \times R^+)$, and hence (cf. Th. 7.1 in [3])

$$B(\cdot, u, \dots, D_x^{l_0} u) \in C^0(R^+; W^{[n/2]+1,2}).$$

Therefore with the aid of the regularity theory of elliptic equations we see, by (3.11),

$$u \in C^0(R^+; W^{2m+[n/2]+1,2}),$$

and

$$(3.18) \quad D_x^\gamma u(x, t) \in C^0(\bar{\Omega} \times R^+) \quad (0 \leq |\gamma| \leq 2m).$$

From (3.17) and (3.18) we conclude that the weak solution $u(x, t)$ is in fact a classical solution of the problem (0.1)–(0.2) on $\Omega \times R^+$ with the initial condition $u(x, 0) = 0$. The uniqueness of such solution is obvious. We summarize above result in the following

THEOREM 3.1.

Suppose $H_1 - H_3$. Then, under the assumption of Lemma 2.1, the initial-boundary value problem (0.1)–(0.2) with $u(x, 0) = 0$ admits a unique classical solution $u(x, t)$. Moreover the estimates (2.3), (2.9), (2.15) and (2.20) are valid for $u_m = u$.

Next, we shall prove the existence of bounded solution on $\Omega \times R$. To do so, we let $u_r(t)$ be the classical solution of the problem

$$(3.19) \quad \begin{aligned} & \frac{\partial u}{\partial t} + \sum_{|\gamma|, |\beta| \leq m} (-1)^{|\gamma|} D^\gamma (A_{\gamma\beta}(x) D^\beta u) + B(x, u, \dots, D_x^{l_0} u) \\ & = f(x, t) \quad \text{with} \\ & u(x, -r) = 0 \quad \text{and} \quad D^\gamma u(x, -r)|_{\partial\Omega} = 0, \quad 0 \leq |\gamma| \leq m-1, \end{aligned}$$

where $r = 1, 2, 3, \dots$

By Theorem 3.1, $u_r(x, t)$ exists uniquely for each r , and the estimates (2.3), (2.9), (2.15) and (2.20) hold for $u_m = u_r$. Similar arguments just as above give

$$D_t u_r \in C^0([-r, \infty), W^{2m, 2}), \quad u_r \in C^0([-r, \infty), W^{2m+[n/2]+1, 2}),$$

and

$$(3.20) \quad \|D_t u(\cdot, t)\|_{W^{2m, 2}} + \|u_r(\cdot, t)\|_{W^{2m+[n/2]+1, 2}} \leq \text{const.} < +\infty$$

for $r, -r \leq t < +\infty$.

Moreover we can see easily that $D_t u_r(\cdot, t)$ and $u_r(\cdot, t)$ are equicontinuous on each half interval $[-T, \infty)$ and for $r > T$ with respect to the topologies in $W^{2m, 2}$ and $W^{2m+[n/2]+1, 2}$, respectively. Thus we can conclude that an appropriate subsequence of $u_r(t)$, which will be denoted by the same notation for simplicity, and a function $u(\cdot, t) \in W^{2m+[n/2]+1, 2}$, $t \in R$, such that for $\forall T > 0$

$$(3.20) \quad u_r(\cdot, t) \longrightarrow u(\cdot, t) \quad \text{weakly in } L^\infty([-T, T], W^{2m+[n/2]+1, 2})$$

$$(3.21) \quad u'_\gamma(\cdot, t) \longrightarrow u'(\cdot, t) \quad \text{weakly in } L^\infty([-T, T], W^{2m, 2})$$

$$(3.22) \quad D^\gamma u_r(\cdot, t) \longrightarrow D^\gamma u(\cdot, t) \quad \text{strongly in } L^2 \text{ for } |\gamma| \leq 2m \\ \text{and a.e. on } \Omega \times [-T, T]$$

as $r(>T) \rightarrow \infty$, and moreover, by equicontinuity,

$$(3.23) \quad D_t u(t) \in C^0(R; W^{2m, 2}), \quad u(t) \in C^0(R; W^{2m+[n/2]+1, 2}).$$

It is easy to see that the limit function $u(x, t)$ is a classical bounded solution of the problem (0.1)–(0.2) in the sense that

$$(3.24) \quad |D_t u(x, t)| + |D^\gamma u(x, t)| \leq \text{const.} < +\infty \quad \text{for } |\gamma| \leq 2m, t \in R$$

Now we have proved the following main result.

THEOREM 3.2.

Under the same assumptions of Theorem 3.1, the problem (0.1)–(0.2) admits a bounded classical solution u . In particular (2.3), (2.9), (2.15) and (2.20) are valid for u .

§4. Local Uniqueness, Periodicity and Almost Periodicity

In this section we shall study a local uniqueness, periodicity, and almost periodicity of the bounded solution whose existence have been proved in Theorem 3.2.

First we shall prove

THEOREM 4.1.

Suppose that the inequality

$$(4.1) \quad 2^\alpha k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha < 1$$

holds. Then the problem (0.1)–(0.2) admits, at most, one (strong) solution in the ball $S_{L_0} = \{u \in L^\infty(R; W_0^{m, 2}) \mid \|u(t)\|_{H_0^m} \leq L_0\}$.

Proof.

The proof is essentially the same as that of Theorem II in [4] and given for completeness. Let $u_1(t), u_2(t)$ be two solutions in S_{L_2} and put $w(t) = u_1(t) - u_2(t)$.

Then we have

$$\begin{aligned}
 (4.2) \quad & \frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2}^2 \leq -\|w(t)\|_{H_0^m}^2 + \int_{\Omega} |B(\cdot, u_1, \dots, D_x^{l_0} u_1) \\
 & \quad - B(\cdot, u_2, \dots, D_x^{l_0} u_2)| |w(t)| dx \\
 & \leq -\|w(t)\|_{H_0^m}^2 + k_1 \int_{\Omega} \left\{ \sum_{i=0}^{l_0} (|D^i u_1| + |D^i u_2|)^2 \right\}^{\frac{\alpha}{2}} \\
 & \quad \left(\sum_{i=0}^{l_0} |D^i u_1 - D^i u_2| \right) |w| dx \\
 & \leq -\|w(t)\|_{H_0^m}^2 + 2^\alpha k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha \|w(t)\|_{H_0^m}^2 \\
 & \leq -(1 - 2^\alpha k_1 C_0^{\alpha+2} S^{\alpha+2} L_0^\alpha) / C_0^2 S_0^2 \|w(t)\|_{L^2}^2 \\
 & \leq 0.
 \end{aligned}$$

From (4.1) and (4.2) we can show easily that if $w(i) \neq 0$ for some i , then $\lim_{t \rightarrow -\infty} \|w(t)\|_{L^2} = \infty$ which is a contradiction. q. e. d.

COROLLARY 4.1.

In addition to the assumptions of Theorem 3.2, we assume $f(\cdot, t)$ is ω -periodic. Then there exists a positive constant $\overline{M}_0 > 0$ such that if $M_0 < \overline{M}_0$, the problem (0.1)–(0.2) admits a unique ω -periodic classical solution in S_{L_0} .

Proof.

By Remark 2.1 made after Lemma 2.1 and the estimate (2.3) for $u_m = u$ we can choose a positive number $\overline{M}_0$ such that the bounded solution in Theorem 3.2 belongs to S_{L_0} with L_0 satisfying (4.1). Since $f(t)$ is ω -periodic, $u(t + \omega)$ is also a bounded solution of our problem having the same bounds as $u(t)$. Hence, by Theorem 4.1, we have $u(t) = u(t + \omega)$ for $t \in \mathbb{R}$. q. e. d.

Finally, we shall show the almost periodicity of the bounded solution in Theorem 3.2 when $f(t)$ is almost periodic.

THEOREM 4.2.

In addition to the assumptions of Theorem 3.2, we assume $f(\cdot, t)$ is almost periodic in the norm $C^0(\overline{\Omega})$, and let $M_0 < \overline{M}_0$. Then $u(\cdot, t)$ and $D_t u(\cdot, t)$ are almost periodic in the norms $C^{2m}(\overline{\Omega})$ and $C^0(\overline{\Omega})$, respectively.

Proof.

Let $\{t_n\}$ be any real sequence. To show the almost periodicity of $u(\cdot, t)$ in $C^{2m}(\bar{\Omega})$, it suffices by Bochner's criterion to prove that there exists a subsequence $\{t_{\bar{n}}\}$ of $\{t_n\}$ such that $u(\cdot, t+t_{\bar{n}})$ is uniformly convergent on R in $C^{2m}(\bar{\Omega})$. If it were not the case, there exist two subsequences $\{t_{n_1}\}$ and $\{t_{n_2}\}$ of $\{t_n\}$, and a sequence $\{s_n\}$ such that

$$(4.3) \quad |u(s_n + t_{n_1}) - u(s_n + t_{n_2})|_{C^{2m}(\bar{\Omega})} \geq \delta > 0$$

for some $\delta > 0$.

Here, by the almost periodicity of $f(\cdot, t)$ we may assume, without loss of generality,

$$(4.4) \quad \lim_{n \rightarrow \infty} f(\cdot, t + s_n + t_{n_1}) = \lim_{n \rightarrow \infty} f(\cdot, t + s_n + t_{n_2}) = F(\cdot, t) \text{ in } C^0(\bar{\Omega})$$

uniformly on R for some $F(\cdot, t) \in C^0(R; C^0(\bar{\Omega}))$.

On the other hand, $\{u(\cdot, t + s_n + t_{n_1})\}$ and $\{u(\cdot, t + s_n + t_{n_2})\}$ satisfy, of course, the estimates (2.3), (2.9), (2.15) and (2.20). Therefore we can extract subsequences of them, which we denote by the same notations for simplicity, satisfying the same convergency properties as in (3.20)–(3.22) as $n \rightarrow \infty$. Let $U_1(t)$ and $U_2(t)$ be two limit functions, respectively. Then, with the aid of Ascoli-Arzelà's Lemma, we see in particular

$$(4.5) \quad \begin{aligned} u(\cdot, t + s_n + t_{n_1}) &\longrightarrow U_1(\cdot, t) \text{ in } C_{\text{loc}}^0(R^+; C^{2m}(\bar{\Omega})) \\ \text{and} \end{aligned}$$

$$u(\cdot, t + s_n + t_{n_2}) \longrightarrow U_2(\cdot, t) \text{ in } C_{\text{loc}}^0(R^+; C^{2m}(\bar{\Omega})).$$

Here we have used the fact that the imbedding $W^{2m+[n/2]+1,2} \rightarrow C^{2m}(\bar{\Omega})$ is compact.

From (4.3), (4.4) and (4.5), $U_1(\cdot, t)$ and $U_2(\cdot, t)$ are both solutions of the equation

$$\frac{\partial}{\partial t} U + \sum_{\substack{|\gamma| \leq m \\ |\beta| \leq m}} (-1)^{|\gamma|} D^\gamma (A_{\gamma\beta}(x) D^\beta U) + B(x, U, \dots, D_x^{l_0} U) = F(x, t)$$

on $\Omega \times R$ with $D^\gamma U|_{\partial\Omega} = 0$ for $|\gamma| \leq m-1$, and with

$$(4.6) \quad |U_1(\cdot, 0) - U_2(\cdot, 0)|_{C^{2m}(\bar{\Omega})} \geq \delta > 0.$$

But, by the local uniqueness of bounded solution, we know $U_1(\cdot, t) = U_2(\cdot, t)$, which contradicts (4.6). Thus the almost periodicity of $u(\cdot, t)$ have been proved. That of $u'(\cdot, t)$ follows easily from the equation (0.1) itself, and the proof of Theorem is now completed. q. e. d.

References

- [1] A. FRIEDMAN, *On quasi-linear parabolic equations of the second order*, J. Math. Mech. **7** (1958), p. 793–808.
- [2] J. L. LIONS, *Quelques Methodes de Resolution des Problemes aux Limites Nonlineaire*, Dunod, Paris, 1969.
- [3] S. MIZOHATA, *Theory of Partial Differential Equations*, Iwanami, 1965 (in Japanese).
- [4] M. NAKAO, *On boundedness, periodicity and almost periodicity of solutions of some nonlinear parabolic equations*, J. Differential Eqs. **19** (1975), p. 371–385.
- [5] M. NAKAO & T. NANBU, *On the existence of global, bounded, periodic, and almost periodic solutions of nonlinear parabolic equations*, Math. Rep. College Gen. Edu. Kyushu Univ. **X**, (to appear).
- [6] O. A. OLEINIK & S. N. KRUSHKOV, *Quasi-linear second-order parabolic equations with many independent variables*, Russian Math. Surveys **16**, No. 5 (1961), p. 105–146.
- [7] G. PRODI, *Problemi al contorno non lineari per equazione di tipo parabolio non lineari in due variabili-soluzioni periodiche*, Rend. Sem. Mat. Univ. Padova. **23** (1954), p. 25–85.
- [8] J. SATHER, *The existence of a global classical solution of the initial-boundary value problem for $\square u + u^3 = f$* , Arch. Rat. Mech. Anal. **22** (1966), p. 292–307.
- [9] I. I. SMULEV, *Periodic solutions of the first boundary problem for parabolic equations*, Amer. Math. Soc. Trans. (2) **79** (1969), p. 215–229.

Department of Mathematics
 Faculty of Science
 Kyushu University
 and
 Department of Mathematics
 College of General Education
 Kyushu University