ON SOME BOUNDS FOR THE VALUE OF DIRICHLET'S L-FUNCTION $L(s, \chi)$ AT THE POINT $s=1$

By

Shigeru KANEMITSU

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§ 1. Introduction

In this paper we shall concern ourselves with a continuation for our former paper [5] and some applications. We shall obtain an upper bound for the value of $L(s, \chi)$, Dirichlet's L -function, at the point $s=1$ with a primitive odd character χ . Combining it with the former result, we shall have an upper bound for the product of the class number and the regulator of an abelian field over the rationals. If we use bounds for $L(1, \chi)$, then we are able to prove more easily the Brauer-Siegel theorem [6] in this special case of absolute abelian fields. Moreover, we shall have two kinds of average values of $L(1, \chi)$ over primitive odd characters χ .

§ 2. An upper bound for $L(1, \chi)$

In a similar way in the proof to Theorem 1 of [5], we have, in general, the following

THEOREM 1. *If χ is a primitive, non-principal Dirichlet character defined modulo f and $1 \leq m_0 < f$, $1 \leq n$, then we have*

$$(1) \quad \sum_{m=1}^{m_0} (m_0 - m)^{n-1} \chi(m) = (-1)^n \frac{1}{n} \{ (B_\chi - m_0)^n - B_\chi^n \} \\ + (-1)^n f^{n-1} (n-1)! \tau(\chi) \frac{1}{(2\pi i)^n} \sum_{l=-\infty}^{\infty} \frac{e^{-2\pi i(m_0/f)l} - 1}{l^n} \bar{\chi}(l)$$

If we assume further that χ and n are both odd, then as before, we have

$$(2) \quad \sum_{m=1}^{m_0} (m_0 - m)^{n-1} \chi(m) = -\frac{1}{n} \{ (B_\chi - m_0)^n - B_\chi^n \}$$

$$+ f^{n-1} (n-1)! \tau(\chi) \frac{4}{(2\pi i)^n} \sum_{l=1}^{\infty} \frac{\sin^2 \pi(m_0/f)l}{l^n} \bar{\chi}(l).$$

Next we take, in particular, $n=3$, and the formula (2) reduces to

$$\sum_{m=1}^{m_0} (m_0 - m)^2 \chi(m) = -B_{\chi}^1 m_0^2 - \frac{f^2 \tau(\chi)}{\pi^3 i} \sum_{l=1}^{\infty} \frac{\sin^2 \pi(m_0/f)l}{l^3} \bar{\chi}(l),$$

where we have used the fact that $B_{\chi}^2 = B_{\chi}^0 = 0$. Noting that $B_{\chi}^1 = -\frac{1}{\pi i} \tau(\chi) L(1, \bar{\chi})$, we have

$$(3) \quad \sum_{m=1}^{m_0} (m_0 - m)^2 \chi(m) = \frac{\tau(\chi)}{\pi i} m_0^2 L(1, \bar{\chi}) - \frac{f^2 \tau(\chi)}{\pi^3 i} \sum_{l=1}^{\infty} \frac{\sin^2 \pi(m_0/f)l}{l^3} \bar{\chi}(l).$$

It follows from this that

$$\begin{aligned} \frac{1}{\pi} f^{1/2} m_0^2 |L(1, \bar{\chi})| &\leq \sum_{m=1}^{m_0-1} m^2 + \frac{1}{\pi^3} f^{5/2} \sum_{l=1}^{\infty} \frac{\sin^2 \pi(m_0/f)l}{l^3} \\ &\leq \frac{1}{6} (m_0 - 1) m_0 (2m_0 - 1) + \frac{f^{5/2}}{\pi^3} \left(\pi^2 \frac{m_0^2}{f^2} + \sum_{l=2}^{\lfloor f/m_0 \rfloor + 1} \frac{\pi^2 (m_0/f)^2 l^2}{l^3} + \sum_{l=\lfloor f/m_0 \rfloor + 2}^{\infty} \frac{1}{l^3} \right) \\ &\leq \frac{1}{6} (m_0 - 1) m_0 (2m_0 - 1) + \frac{1}{\pi} f^{1/2} m_0^2 \left(1 + \sum_{l=2}^{\lfloor f/m_0 \rfloor + 1} \frac{1}{l} \right) + \frac{f^{5/2}}{\pi^3} \int_{\lfloor f/m_0 \rfloor + 1}^{\infty} \frac{dx}{x^3} \\ &\leq \frac{1}{6} (m_0 - 1) m_0 (2m_0 - 1) + \frac{1}{\pi} f^{1/2} m_0^2 \left\{ \log \left(\left\lceil \frac{f}{m_0} \right\rceil + 1 \right) + 1 \right\} \\ &\quad + \frac{1}{\pi^3} f^{5/2} \frac{1}{2} \left(\left\lceil \frac{f}{m_0} \right\rceil + 1 \right)^{-2}. \end{aligned}$$

Consequently,

$$|L(1, \bar{\chi})| \leq \frac{\pi}{3} \left(1 - \frac{1}{m_0} \right) \left(\frac{m_0}{f^{1/2}} - \frac{1}{2f^{1/2}} \right) + \log \left(\left\lceil \frac{f}{m_0} \right\rceil + 1 \right) + 1 + \frac{1}{2\pi^2}.$$

Taking $m_0 = \lfloor f^{1/2} \rfloor + 1$, we obtain a bound of Hua's type [4], [9]:

$$(4) \quad |L(1, \bar{\chi})| < \frac{1}{2} \log f + 3.$$

As to the class number $h(d)$ of an imaginary quadratic field, we obtain the following bound as already stated in [9]:

$$(5) \quad h(d) = \frac{w|d|^{1/2}}{2\pi} L(1, \chi) \leq \frac{w}{4\pi} |d|^{1/2} (\log |d| + 6).$$

Combining the inequality (4) with the preceding one, we eventually have

THEOREM 2. *If χ is a primitive, non-principal Dirichlet character to the modulus f , then it holds that*

$$(6) \quad |L(1, \chi)| < \frac{1}{2} \log f + 3.$$

For a prime modulus p we can say a little more about the value $L(1, \chi)$ as in [7]. Namely, let χ be a real, primitive odd character to the large modulus p and let n_χ denote the least n for which $\chi(n) \neq 0, 1$. Then for $1 \leq m_0 < p$, we know by (3) that

$$\left| \sum_{m=1}^{m_0} (m_0 - m)^2 \chi(m) \right| = \frac{p^{1/2}}{\pi^3} \left| \pi^2 m_0^2 L(1, \chi) - p^2 \sum_{l=1}^{\infty} \frac{\sin^2 \pi(m_0/p)l}{l^3} \chi(l) \right|.$$

We distinguish two cases:

Case 1. For some m_0 , $\frac{p}{2n_\chi} \leq m_0 < p$, the following inequality holds:

$$\pi^2 m_0^2 L(1, \chi) \leq p^2 \sum_{l=1}^{\infty} \frac{\sin^2 \pi(m_0/p)l}{l^3} \chi(l).$$

In this case we have, by using the bound $n_\chi < p^{\frac{1}{4\sqrt{e}+\varepsilon}}$ [1], [8],

$$(7) \quad L(1, \chi) \leq \log \left(\left\lfloor \frac{p}{m_0} \right\rfloor + 1 \right) + 2 \leq \log n_\chi + 3 \leq \left(\frac{1}{4\sqrt{e}} + \varepsilon \right) \log p + 3.$$

Case 2. For some m_0 , $\frac{p}{2n_\chi} \leq m_0 < p$, the following inequality is valid:

$$\pi^2 m_0^2 L(1, \chi) > p^2 \sum_{l=1}^{\infty} \frac{\sin^2 \pi(m_0/p)l}{l^3} \chi(l).$$

In this case we have ($1 \leq X < n_\chi$)

$$\sum_{l=1}^{\infty} \frac{\sin^2 \pi(m_0/p)l}{l^3} \chi(l) \geq \sum_{l=1}^X \frac{4m_0^2}{p^2 l} - \int_X^{\infty} x^{-3} dx \geq 4 \frac{m_0^2}{p^2} \log(X+1) - 2^{-1} X^{-2}.$$

We must choose X to satisfy the condition $\pi \frac{m_0}{p} X \leq \frac{\pi}{2}$, so that we take X to be $n_\chi - 1$. Thus we obtain

$$(8) \quad L(1, \chi) \geq \frac{4}{\pi^2} \log n_\chi - \frac{p^2}{2\pi^2 m_0^2 (n_\chi - 1)^2} \geq \frac{4}{\pi^2} \left(\log n_\chi - \frac{3}{4} \right).$$

Either (7) or (8) is valid (or perhaps both).

Note. We have assumed throughout that χ is primitive, but as for imprimitive χ we have, for instance, the following result: let χ be an imprimitive character to the modulus q and let it be induced by χ_1 modulo $f|q$. Then

$$(9) \quad |L(1, \chi)| = |L(1, \chi_1)| \prod_{p|q} \left| 1 - \frac{\chi_1(p)}{p} \right| \ll (\log f) \left(\log \frac{q}{f} \right),$$

since we have trivially $\prod_{p|q} \left| 1 - \frac{\chi_1(p)}{p} \right| \leq \prod_{p|q/f} \left(1 + \frac{1}{p} \right) \leq \prod_{p|q/f} \left(1 - \frac{1}{p} \right)^{-1} = \frac{q/f}{\phi(q/f)}$ and we know that $\phi(m) \gg \frac{m}{\log \log m}$.

§3. An application

Let k be an abelian field over the rationals Q of degree n and let h and R denote the class number and the regulator of k , respectively. Then, by the decomposition theorem of class field theory, we have the class number formula [3]

$$(10) \quad hR = \begin{cases} \frac{\sqrt{d_k}}{2^{n-1}} \prod_{\chi \neq \chi_1} L(1, \chi), & \text{if } k \text{ is real,} \\ \frac{w\sqrt{d_k}}{(2\pi)^{n/2}} \prod_{\chi \neq \chi_1} L(1, \chi), & \text{if } k \text{ is imaginary,} \end{cases}$$

where χ_1 denotes the principal character, d_k is the absolute value of the discriminant of k , and w is the number of roots of unity contained in k . Each χ in the product of (10) is a primitive Dirichlet character corresponding to the modulus $f(\chi)$. And by Artin's conductor-discriminant formula [2], we know that

$$(11) \quad \prod_{\chi} f(\chi) = d_k.$$

By (5) and (10) we have the inequality

$$(12) \quad hR \leq 4 \frac{n^2}{2^n} \sqrt{d_k} \prod_{\chi \neq \chi_1} \left\{ \frac{1}{2} \log f(\chi) + 3 \right\} \\ \ll \sqrt{d_k} \prod_{\chi \neq \chi_1} \left\{ \frac{1}{2} \log f(\chi) + 3 \right\},$$

where we used the elementary estimate $w \leq 2n^2$, and the implied constant in the

symbol $\ll$ is absolute, and as usual $\ll$ is Vinogradov's symbol, which is an equivalent to Landau's O -symbol.

For any $\alpha \geq 1$ we have from (12)

$$(13) \quad hR \ll \alpha^n d_k^{1/2 + 1/2\alpha},$$

which is a better estimate by the constant factor than the one in [6]. The inequality (13) clarifies the part of upper estimate in the Brauer-Siegel theorem for abelian fields. As for the part of lower bounds, we shall only need to assume the lower bound for the residue of the Dedekind zeta function, but other arguments hold good [6], since we have already had a bound $|L(1, \chi)| \leq \frac{1}{2} \log f + 3$, which clearly gives the bound of the following type:

$$|L(1, \chi)| \leq \alpha \log f^{1/2\alpha} \quad \text{for any } \alpha \geq 1.$$

§4. Average results

First we shall obtain an average value of $L(1, \chi)$ over primitive odd characters χ to a fixed modulus f . In the formula (1) we take $n=2$ to have a particular one:

$$(14) \quad \sum_{m=1}^{m_0} (m_0 - m) \chi(m) = \frac{\tau(\chi)}{\pi i} m_0 L(1, \bar{\chi}) + \frac{\tau(\chi)}{2\pi^2} fi \sum_{l=1}^X \frac{\sin 2\pi(m_0/f)l}{l^2} \bar{\chi}(l) + O\left(\frac{f^{3/2}}{X}\right),$$

where $X \geq 1$ and will be chosen later. From (14) it follows that

$$(15) \quad fm_0^2 \left| L(1, \bar{\chi}) - \frac{1}{2\pi} \frac{m_0}{f} \sum_{l=1}^X \frac{\sin 2\pi(m_0/f)l}{l^2} \chi(l) \right|^2 \leq \pi^2 \left| \sum_{m=1}^{m_0} (m_0 - m) \bar{\chi}(m) \right|^2 + \frac{f^3}{4\pi^2 X^2}.$$

We apply the sieve of the following type [8]:

$$(16) \quad \sum_{\chi \bmod f} \left| \sum_{m=M+1}^{M+N} a_m \chi(m) \right|^2 \leq \varphi(f) \left(1 + \left[\frac{N-1}{f} \right] \right) \sum_{\substack{m=M+1 \\ (m, f)=1}}^{M+N} |a_m|^2,$$

where a_m are any complex numbers, M and N (≥ 1) are arbitrary integers and φ denotes Euler's function. If $N \leq f$, this holds with equality. Applying (16)

to (15), we obtain

$$\begin{aligned}
& f m_0^2 \sum_{\substack{\chi \bmod f \\ \chi \text{ odd}}}^* \left| L(1, \chi) - \frac{1}{2\pi} \frac{f}{m_0} \sum_{l=1}^X \frac{\sin 2\pi(m_0/f)l}{l^2} \chi(l) \right|^2 \\
& \leq 2\pi^2 \sum_{\substack{\chi \bmod f \\ \chi \text{ odd}}}^* \left| \sum_{m=1}^{m_0} (m_0 - m) \bar{\chi}(m) \right|^2 + \frac{f^3}{2\pi^2 X^2} \sum_{\chi \bmod f}^* 1 \\
& \leq 2\pi^2 \sum_{\chi \bmod f} \left| \sum_{m=1}^{m_0} (m_0 - m) \chi(m) \right|^2 + \frac{f^3 \varphi(f)}{2\pi^2 X^2} \\
& \leq 2\pi^2 \varphi(f) \left(1 + \left\lceil \frac{m_0 - 1}{f} \right\rceil \right) \sum_{m=1}^{m_0-1} m^2 + \frac{f^3 \varphi(f)}{2\pi^2 X^2} \\
& \leq 2\pi^2 \varphi(f) \left(m_0^3 + \frac{f^3}{4\pi^4 X^2} \right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \sum_{\substack{\chi \bmod f \\ \chi \text{ odd}}}^* \left| L(1, \chi) - \frac{1}{2\pi} \frac{f}{m_0} \sum_{l=1}^X \frac{\sin 2\pi(m_0/f)l}{l^2} \chi(l) \right|^2 \\
& \leq \varphi(f) \left(2\pi^2 \frac{m_0}{f} + \frac{f^2}{2\pi^2 X^2 m_0^2} \right).
\end{aligned}$$

Taking $X = \left[\frac{f}{m_0} \right]^{3/2}$, we have the following

THEOREM 3. *If $\sum_{\substack{\chi \bmod f \\ \chi \text{ odd}}}^*$ denotes the summation over primitive, odd characters, then we have*

$$\begin{aligned}
(17) \quad & \sum_{\substack{\chi \bmod f \\ \chi \text{ odd}}}^* \left| L(1, \chi) - \frac{f}{2\pi m_0} \sum_{l=1}^{\lceil m_0/f \rceil^{3/2}} \frac{\sin 2\pi(m_0/f)l}{l^2} \chi(l) \right|^2 \\
& \leq \varphi(f) \left(2\pi^2 + \frac{1}{2\pi^2} \right) \frac{m_0}{f},
\end{aligned}$$

for any m_0 , $1 \leq m_0 < f$.

To obtain an average value when the modulus also varies we need a general form of (14). In fact, for $A \geq 1$, we can immediately show that

$$(18) \quad \sum_{m=1}^A (A-m) \chi(m) = A \frac{\tau(\chi)}{\pi i} L(1, \bar{\chi}) + \frac{\tau(\chi)}{2\pi^2} f i \sum_{l=1}^{\infty} \frac{\sin 2\pi(m_0/f)l}{l^2} \bar{\chi}(l),$$

where $m_0 = A - f \left[\frac{A}{f} \right]$. From this, as in the above, the following inequality is deduced:

$$\begin{aligned} fA^2 \left| L(1, \chi) - \frac{f}{2\pi A} \sum_{l=1}^X \frac{\sin 2\pi(m_0/f)l}{l^2} \chi(l) \right|^2 \\ \leq 2\pi^2 \left| \sum_{m=1}^A (A-m) \bar{\chi}(m) \right|^2 + \frac{f^3}{2\pi^2 X^2}, \end{aligned}$$

($X \geq 1$), so that, for given positive integer F , if we take $A^3 X^2 = F^2$, then it holds that

$$\begin{aligned} (19) \quad \left| L(1, \chi) - \frac{f}{2\pi A} \sum_{l=1}^X \frac{\sin 2\pi(m_0/f)l}{l^2} \chi(l) \right|^2 \\ \leq 2\pi^2 f^{-1} A^{-2} \left| \sum_{m=1}^A (A-m) \bar{\chi}(m) \right|^2 + \frac{f^2}{2\pi^2 A^2 X^2}. \end{aligned}$$

In this case we use Gallagher's sieve [8]:

$$(20) \quad \sum_{f \leq F} \frac{f}{\varphi(f)} \sum_{\chi \bmod f}^* \left| \sum_{m=M+1}^{M+N} a_m \chi(m) \right|^2 \leq (F^2 + \pi N) \sum_{m=M+1}^{M+N} |a_m|^2.$$

Applying (20) to (19), we obtain

$$\begin{aligned} \sum_{f \leq F} \frac{f^2}{\varphi(f)} \sum_{\substack{\chi \bmod f \\ \chi \text{ odd}}}^* \left| L(1, \chi) - \frac{f}{2\pi A} \sum_{l=1}^X \frac{\sin 2\pi(m_0/f)l}{l^2} \chi(l) \right|^2 \\ \leq 2\pi^2 A^{-2} \sum_{f \leq F} \frac{f}{\varphi(f)} \sum_{\chi \bmod f}^* \left| \sum_{m=1}^A (A-m) \chi(m) \right|^2 + \frac{F^4}{2\pi^2 A^2 X^2} \\ \leq 2\pi^2 A^{-2} (F^2 + A) \sum_{m=1}^{A-1} m^2 + \frac{F^4 A^{-2} X^{-2}}{2\pi^2} \\ \leq 2\pi^2 A (F^2 + A) + \frac{F^4 A^{-2} X^{-2}}{2\pi^2} \end{aligned}$$

Hence, choosing X to satisfy the condition $A^3 X^2 = F^2$ we have, for A , $1 \leq A \leq F^{2/3}$

THEOREM 4. *It holds that*

$$(21) \quad \sum_{f \leq F} \frac{f^2}{\varphi(f)} \sum_{\chi \bmod f}^* \left| L(1, \chi) - \frac{f}{2\pi A} \sum_{l=1}^{\lceil F^2/A^3 \rceil^{1/2}} \frac{\sin 2\pi(m_0/f)l}{l^2} \chi(l) \right|^2 \ll AF^2.$$

§5. Remark

For $0 \leq \alpha < 1$ and $0 \leq x$ we put

$$S_\alpha(x) = \sum_{m \leq x} (x-m)^\alpha \chi(m) \quad (S_\alpha(x) = 0 \text{ if } x < 1).$$

We have $\frac{d}{dx} S_\alpha(x) = \alpha S_{\alpha-1}(x)$ for $\alpha \geq 1$.

We shall make use of Theorem 1 with $n=1$ to obtain a formula relating $S_\alpha(x)$ with the Fourier series expansion. Indeed, we have

$$\begin{aligned} S(x) &= S_0(x) = \sum_{m=1}^{[x]} \chi(m) = \sum_{m=1}^{m_0} \chi(m) \\ &= -\tau(\chi) \frac{1}{2\pi i} \sum_{l=-\infty}^{\infty} \frac{e^{-2\pi i(m_0/f)l} - 1}{l} \bar{\chi}(l), \end{aligned}$$

where $m_0 = [x] - f \left[\frac{[x]}{f} \right]$ so that

$$(22) \quad S(x) = -\tau(\chi) \frac{1}{2\pi i} \sum_{l=-\infty}^{\infty} \frac{e^{-2\pi i([x]/f)l} - 1}{l} \bar{\chi}(l).$$

Therefore it holds that

$$\begin{aligned} (23) \quad S_\alpha(x) &= \alpha \int_0^x (x-t)^{\alpha-1} S(t) dt \\ &= \alpha \left\{ \sum_{n=0}^{[x]-1} S(n) \int_n^{n+1} (x-t)^{\alpha-1} dt + S([x]) \int_{[x]}^x (x-t)^{\alpha-1} dt \right\} \\ &= \sum_{n \leq x} S(n) \{ (x-n)^\alpha - (x-n-1)^\alpha \} + S(x) (\{x\} - 1)^\alpha \\ &= -\frac{\tau(\chi)}{2\pi i} \sum_{l=-\infty}^{\infty} \frac{\bar{\chi}(l)}{f} \left[\sum_{n \leq x} (e^{-2\pi i(n/f)l} - 1) \{ (x-n)^\alpha - (x-n-1)^\alpha \} \right. \\ &\quad \left. + (e^{-2\pi i([x]/f)l} - 1) (\{x\} - 1)^\alpha \right]. \end{aligned}$$

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