TIME-VARYING SEQUENTIAL MACHINES IN A CATEGORY (II)

By

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1. Introduction

In [2], we gave a definition of a time-varying sequential machine in a category $\mathbf{K}$ having countable coproducts and showed that for every time-varying (periodic) sequential machine M , there always exists a time-invariant sequential machine M' which is equivalent to M , in the sense that M and M' have the same behavior, as well as minimal. In this paper we consider the problem of finding a non-redundant time-varying (periodic) sequential machine M'' which is equivalent to a given time-invariant minimal sequential machine M' . We shall show that for certain minimal time-invariant sequential machines, there always exist equivalent non-redundant time-varying (periodic) machines.

We shall use the terminology, definitions and results of [2] freely.

2. Time-varying analogs of time-invariant sequential machines

DEFINITION 2.1. A time-varying (periodic) sequential machine $M' = (Q', \delta', \tau', \beta')$, where $Q' = (Q'_k | k \in N^+)$, is called a time-varying (periodic) analog of a time-invariant sequential machine $M = (Q, \delta, \tau, \beta)$ if M' and M have the same behavior.

Obviously, the time-invariant sequential machine $M = (Q, \delta, \tau, \beta)$ considered as a time-varying (periodic) sequential machine is a time-varying (periodic) analog of itself. This analog is a trivial one. We are interested in constructing an analog different from the above mentioned one.

DEFINITION 2.2. A time-varying (periodic) analog $M' = (Q', \delta', \tau', \beta')$ of a time-invariant sequential machine $M = (Q, \delta, \tau, \beta)$ is called a non-redundant

time-varying (periodic) analog of M if, for all $k \in N^+$, Q'_k is a mono-subobject of Q , that is, there exists a monomorphism $i_k: Q'_k \rightarrow Q$ which is not an isomorphism.

In the sequel we suppose that $\mathbf{K}$ has an image factorization system ($\mathfrak{E}$ =epimorphisms, $\mathfrak{M}$ =equalizers) and X preserves $\mathfrak{E}$.

Let $M=(Q, \delta, \tau, \beta)$ be a time-invariant sequential machine with reachability map $r: IX^* \rightarrow Q$. Since $IX^* = \coprod_{n \geq 0} IX^n$, $r \cdot in_n: IX^n \rightarrow Q$, for all $n \geq 0$, is well defined. Let (r_n, i_n) , for all $n \geq 0$, be an $\mathfrak{E}-\mathfrak{M}$ factorization of $r \cdot in_n$.

DEFINITION 2.3. A time-invariant sequential machine $M=(Q, \delta, \tau, \beta)$ is said to satisfy condition (i) if, for all $n \geq 0$, i_n is not an isomorphism where (r_n, i_n) is an $\mathfrak{E}-\mathfrak{M}$ factorization of $r \cdot in_n$.

THEOREM 2.1. For every time-invariant minimal sequential machine $M=(Q, \delta, \tau, \beta)$ satisfying condition (i), there always exists a non-redundant time-varying analog.

PROOF. Let, for all $n \geq 0$, (r_n, i_n) be an $\mathfrak{E}-\mathfrak{M}$ factorization of $r \cdot in_n$, $r: IX^* \rightarrow Q$ being the reachability map of M . Thus there exists a family $(Q'_{n+1} | n \geq 0)$ of $\mathbf{K}$ -objects such that the diagram

$$\begin{array}{ccc} IX^n & \xrightarrow{r \cdot in_n} & Q \\ & \searrow r_n & \nearrow i_n \\ & Q'_{n+1} & \end{array}$$

commutes for all $n \geq 0$.

Since $i_n: Q'_{n+1} \rightarrow Q$, for all $n \geq 0$, is an equalizer, there exists a pair (α_n, γ_n) of $\mathbf{K}$ -morphisms and a $\mathbf{K}$ -object Q''_n such that the diagram

$$Q'_{n+1} \xrightarrow{i_n} Q \begin{array}{c} \xrightarrow{\alpha_n} \\ \xleftarrow{\gamma_n} \end{array} Q''_n$$

commutes for all $n \geq 0$.

We note that

$$\begin{aligned}
\alpha_n \cdot \delta \cdot i_{n-1} X \cdot r_{n-1} X &= \alpha_n \cdot \delta \cdot (i_{n-1} \cdot r_{n-1}) X \\
&= \alpha_n \cdot \delta \cdot (r \cdot in_{n-1}) X \\
&= \alpha_n \cdot r \cdot in_n = \alpha_n \cdot i_n \cdot r_n \\
&= \gamma_n \cdot i_n \cdot r_n = \gamma_n \cdot (r \cdot in_n) \\
&= \gamma_n \cdot \delta \cdot (r \cdot in_{n-1}) X \\
&= \gamma_n \cdot \delta \cdot (i_{n-1} \cdot r_{n-1}) X \\
&= \gamma_n \cdot \delta \cdot i_{n-1} X \cdot r_{n-1} X
\end{aligned}$$

for all $n \in N^+$, which, r_{n-1} being an epimorphism and X preserves epimorphisms, implies that

$$\alpha_n \cdot \delta \cdot i_{n-1} X = \gamma_n \cdot \delta \cdot i_{n-1} X$$

for all $n \in N^+$.

Since each i_n is an equalizer, there exists a unique $\delta'_n: Q'_n X \rightarrow Q'_{n+1}$ such that the diagram

$$\begin{array}{ccccc}
Q'_{n+1} & \xrightarrow{i_n} & Q & \xrightarrow[\gamma_n]{\alpha_n} & Q'' \\
& \nwarrow \delta'_n & \nearrow \delta & & \\
& & QX & & \\
& \nwarrow & \nearrow i_{n-1} X & & \\
& & Q'_n X & &
\end{array}$$

commutes for all $n \in N^+$. We set

$$\begin{aligned}
Q' &= (Q'_n | n \in N^+) \\
\delta' &= (\delta'_n: Q'_n X \longrightarrow Q'_{n+1} | n \in N^+) \\
\tau' &= r_0, \quad \text{where } i_0 \cdot r_0 = r \cdot in_0 = \tau \\
\beta' &= (\beta'_n = \beta \cdot i_{n-1}: Q'_n \longrightarrow Y | n \in N^+),
\end{aligned}$$

and consider the machine

$$M' = (Q', \delta', \tau', \beta').$$

We note that the family $(i_{n-1}: Q'_n \rightarrow Q | n \in N^+)$ of $\mathbf{K}$ -morphisms makes the diagrams

commutative for all $n \in N^+$, that is, this family of $\mathbf{K}$ -morphisms is a simulation $\psi = (i_{n-1} | n \in N^+): M' \rightarrow M$. Hence M and M' have the same behavior and, therefore, M' is a time-varying analog of M . Also $i_{n-1}: Q'_n \rightarrow Q$, $n \in N^+$, being an equalizer, is a monomorphism.

Since M satisfies condition (i), therefore, no member of the family $(i_{n-1}: Q'_n \rightarrow Q | n \in N^+)$ of $\mathbf{K}$ -morphisms is an isomorphism. Hence M' is a non-redundant time-varying analog of M .

REMARK 2.2. We remark that Theorem 2.1 is valid even if the given time-invariant sequential machine M is not minimal. Also Theorem 2.1 can be proved even if $\mathbf{K}$ has an image factorization system $(\mathfrak{E}, \mathfrak{M})$.

3. Periodic analogs of time-invariant sequential machines

DEFINITION 3.1. Let $M = (Q, \delta, \tau, \beta)$ be a time-invariant sequential machine with reachability map $r: IX^* \rightarrow Q$. Let Q' , δ' , τ' and β' be constructed in a manner similar to the proof of Theorem (2.1). The machine M is said to satisfy condition (ii) if

- (a) M satisfies condition (i),
- (b) for some positive integer T and non-negative integer m , there exists an isomorphism $\psi: Q'_{m+T+1} \rightarrow Q'_{m+1}$ rendering the diagram

commutative.

THEOREM 3.1. *For every time-invariant minimal sequential machine $M=(Q, \delta, \tau, \beta)$ satisfying condition (ii), there always exists a non-redundant periodic analog.*

PROOF. We set

$$\bar{Q} = (\bar{Q}_n | n \in N^+), \text{ where } \bar{Q}_{n+T} = \bar{Q}_n \text{ for all } n > m \text{ and}$$

$$\bar{Q}_n = Q'_n \text{ for all } n(1 \leq n \leq m+T),$$

$$\bar{\delta} = (\bar{\delta}_n | n \in N^+), \text{ where } \bar{\delta}_{n+T} = \bar{\delta}_n \text{ for all } n > m,$$

$$\bar{\delta}_n = \delta'_n \text{ for all } n(1 \leq n \leq m+T-1)$$

$$\text{and } \bar{\delta}_{m+T} = \psi \cdot \delta'_{m+T},$$

$$\bar{\tau} = r_0,$$

$$\bar{\beta} = (\bar{\beta}_n | n \in N^+), \text{ where } \bar{\beta}_{n+T} = \bar{\beta}_n \text{ for all } n > m \text{ and}$$

$$\bar{\beta}_n = \beta'_n \text{ for all } n(1 \leq n \leq m+T).$$

We consider the machine $\bar{M}=(\bar{Q}, \bar{\delta}, \bar{\tau}, \bar{\beta})$, which is periodic by definition. We also set

$$\psi = (\psi_n | n \geq 0), \text{ where } \psi_{n+T} = \psi_n \text{ for all } n \geq m$$

$$\text{and } \psi_n = i_n \text{ for all } n(0 \leq n \leq m+T-1).$$

We note that the diagrams

commute for all $n \in \mathbb{N}^+$. Thus $\psi: \bar{M} \rightarrow M$ is a simulation and, therefore, the machines M and $\bar{M}$ have the same behavior. Since M satisfies condition (ii), $\bar{M}$ is a non-redundant periodic analog of M .

REMARK 3.2. We remark that Theorem 3.1 is valid even if the given time invariant machine M is not minimal. Also Theorem 3.1 can be proved even if $\mathbf{K}$ has an image factorization system $(\mathfrak{E}, \mathfrak{M})$.

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References

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