ANALYTIC SOLUTIONS OF SOME SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS IN COMPLEX DOMAINS WITH VALUES IN BANACH SPACES

By

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(Received October 29, 1976)

Introduction

Let $\Omega_1, \Omega_2, \dots, \Omega_n$ be simply connected domains containing the origin in the complex plane and $A_{jk}(x)$ and $A_j(x)$ be holomorphic functions in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ for $j, k=1, 2, \dots, n$. J. Kajiwara [1] dealt with holomorphic solutions, integral representations of the holomorphic solutions and holomorphic periodic solutions of the Goursat's (or Riemann's) problem of the system

$$\frac{\partial u_j}{\partial x_j} = \sum_{k=1}^n A_{jk}(x) u_k + A_j(x) \quad (j=1, 2, \dots, n)$$

of partial differential equations in the polydomain $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$, and then sought holomorphic solutions in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ of the singular system

$$\begin{cases} \frac{\partial u_j}{\partial x_j} = \sum_{k=1}^n A_{jk}(x) u_k + A_j(x) & (j=1, 2, \dots, n-1) \\ x_n \frac{\partial u_n}{\partial x_n} = \sum_{k=1}^n A_{nk}(x) u_k + A_n(x). \end{cases}$$

Moreover, when $A_{jk}(x)$ and $A_j(x)$ are matrix-valued holomorphic functions in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ for $j, k=1, 2, \dots, n$, he discussed similarly the problems of the above two systems of partial differential equations in the paper [2].

Let E be a Banach space and $L(E)$ be the collection of continuous endomorphisms of E . Let $A_{jk}(x)$ and $A_j(x)$ be respectively analytic mappings of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ into $L(E)$ and E for $j, k=1, 2, \dots, n$. Then we shall extend the above results in this case in the present paper.

§1. Analytic mappings into Banach space

Now we state the main results of analytic mappings into Banach space. We shall refer to Ph. Noverraz [3].

Let E and F be Hausdorff locally convex complex vector spaces and F be quasi-complete. Let U be a domain in E .

DEFINITION 1.1. A mapping f of U into F is called analytic in U if f is Gâteaux-analytic and continuous.

LEMMA 1.2. If a mapping f of U into F is analytic in U , for any point $x \in U$ there exists an n -homogeneous polynomial $f^n(x; h)$ with respect to $h \in E$ for $n=0, 1, 2, \dots$ such that the series $\sum_{n=0}^{\infty} f^n(x; h)$ is uniformly summable with respect to h in a sufficiently small neighborhood of the origin in E and it is equal to $f(x+h)$.

Then we have

$$f^n(x; h) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(x + e^{ni\theta}h)}{e^{ni\theta}} d\theta$$

whenever $h+x$ belongs to the absolutely convex hull in U of x for $n=0, 1, 2, \dots$.

From now on we shall assume that $E = \mathbb{C}^n$ and F is a Banach space.

Let $r_1, r_2, \dots, r_n$ be positive numbers. We denote the set $\{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n; |z_k| < r_k, k=1, 2, \dots, n\}$ by $D(r_1, r_2, \dots, r_n)$.

LEMMA 1.3. If a mapping $f: D(r_1, r_2, \dots, r_n) \rightarrow F$ is analytic, then there exists an element $a_{v_1 v_2 \dots v_n}$ of F for $v_1, v_2, \dots, v_n = 0, 1, 2, \dots$ such that the function f has a power series expansion

$$f(z) = \sum_{v_1 v_2 \dots v_n = 0}^{\infty} a_{v_1 v_2 \dots v_n} z_1^{v_1} z_2^{v_2} \dots z_n^{v_n}$$

which converges absolutely and uniformly in any compact subset of $D(r_1, r_2, \dots, r_n)$.

PROOF. Let $x=0$ and $h=z$ in Lemma 1.2. We have

$$f^m(0; z) = f^m(0; z_1 \overset{n\text{-times}}{(1, 0, \dots, 0)} + \dots + z_n \overset{n\text{-times}}{(0, 0, \dots, 0, 1)}).$$

Since $f^m(0; z)$ is m -linear with respect to z , there exists an element $A_{v_1 v_2 \dots v_n}$ of

F for $v_1 + v_2 + \dots + v_n = m$ such that

$$f^m(0, z) = \sum_{v_1 + v_2 + \dots + v_n = m} A_{v_1 v_2 \dots v_n} z_1^{v_1} z_2^{v_2} \dots z_n^{v_n}.$$

Therefore we have the proposition since $f(z) = \sum_{m=0}^{\infty} f^m(0, z)$.

LEMMA 1.4. *Let $\Omega = \{z \in \mathbf{C}; |z| < r\}$ for a positive number r . If a mapping $f: \Omega \rightarrow F$ is analytic, then for points $x, y \in \Omega$ the integral value of $f(z)$ along a rectifiable curve in Ω jointing x with y is independent of the special choice of the curve.*

PROOF. Let C_1 and C_2 be such curves. Since f is analytic, by the Cauchy's integral theorem, we have

$$\int_{C_1} f(z) dz - \int_{C_2} f(z) dz = 0.$$

Let f be an analytic mapping of $D(r_1, r_2, \dots, r_n)$ into F . According to Lemma 1.3, we have

$$\frac{\partial f}{\partial x_j}(x_1, x_2, \dots, x_n) = \sum_{v_1 v_2 \dots v_n = 0}^{\infty} v_j a_{v_1 v_2 \dots v_n} x_1^{v_1} \dots x_{j-1}^{v_{j-1}} x_{j+1}^{v_{j+1}} \dots x_n^{v_n}$$

for any point $(x_1, x_2, \dots, x_n) \in D(r_1, r_2, \dots, r_n)$ and $j = 1, 2, \dots, n$. Of course the power series of the right side converges absolutely and uniformly in any compact set of $D(r_1, r_2, \dots, r_n)$.

LEMMA 1.5. *Let $\Omega = \{z \in \mathbf{C}; |z| < r\}$ for any positive integer r . If a mapping $f: \Omega - \{0\} \rightarrow F$ is analytic, then f is represented by the Laurent series expansion.*

PROOF. Let r_1, r_2 and r_3 be positive numbers such that $0 < r_2 < r_1 < r$. We put $C_1 = \{z \in \mathbf{C}; |z| = r_1\}$, $C_2 = \{z \in \mathbf{C}; |z| = r_2\}$ and $\Gamma = \{z \in \mathbf{C}; |z - \zeta| = r_3\}$ for any point $\zeta \in \Omega$, $r_2 < |\zeta| < r_1$. We can choose sufficiently small $r_3 > 0$ such that Γ is contained in the interior of C_1 and the interior of Γ does not meet the interior of C_2 .

By Cauchy's integral theorem, we have

$$\int_{\Gamma} \frac{f(z)}{z - \zeta} dz = \int_{C_1} \frac{f(z)}{z - \zeta} dz - \int_{C_2} \frac{f(z)}{z - \zeta} dz$$

we have

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-\zeta} dz = \sum_{n=0}^{\infty} a_n \zeta^n$$

for any point ζ in the interior of C_1 , where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z^{n+1}} dz \quad (n=0, 1, 2, \dots).$$

We have

$$-\frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-\zeta} dz = \sum_{n=1}^{\infty} a_{-n} \zeta^{-n}$$

for any point ζ in the exterior of C_2 , where

$$a_{-n} = \frac{1}{2\pi i} \int_{C_2} f(z) z^{n-1} dz \quad (n=0, 1, 2, \dots).$$

Since

$$\frac{1}{2\pi i} \int_r \frac{f(z)}{z-\zeta} dz = f(\zeta),$$

f is represented by the Laurent series

$$f(\zeta) = \sum_{n=-\infty}^{\infty} a_n \zeta^n$$

which converges absolutely and uniformly in any compact subset of $r_2 < |\zeta| < r_1$.

LEMMA 1.6. *Let E be a Banach space and $L(E)$ be the collection of continuous endomorphisms of E . Let Ω be an open set in $\mathbb{C}^n$. Let $T: \Omega \rightarrow L(E)$ be an analytic mapping. If there exists the inverse mapping $(T(x))^{-1}$ of $T(x)$ for all $x \in \Omega$, then the mapping $\Omega \ni x \rightarrow (T(x))^{-1} \in L(E)$ is analytic.*

PROOF. Let I be the identity mapping of E onto E . Since T is continuous, for every point $x_0 \in \Omega$ there exists a positive number δ such that

$$\|(T(x_0))^{-1}\| \|T(x) - T(x_0)\| < 1$$

for every point x with $|x - x_0| \leq \delta$, where the symbol $\|\cdot\|$ denotes the norm in E . We have

$$\begin{aligned} \|(T(x_0))^{-1}\| \|T(x) - T(x_0)\| &\geq \|(T(x_0))^{-1}(T(x) - T(x_0))\| \\ &= \|(T(x_0))^{-1}T(x) - I\|. \end{aligned}$$

Hence we have

$$\|(T(x_0))^{-1}T(x) - I\| < 1$$

for every point x with $|x - x_0| \leq \delta$. We denote $(T(x_0))^{-1}T(x)$ by $S(x)$. Then, since

$$S(x) = I + (T(x_0))^{-1}T(x) - I,$$

$$\|(T(x_0))^{-1}T(x) - I\| < 1$$

for x with $|x - x_0| \leq \delta$, it follows that

$$(S(x))^{-1} = I - ((T(x_0))^{-1}T(x) - I) + ((T(x_0))^{-1}T(x) - I)^2 - \dots$$

for $|x - x_0| \leq \delta$. Since the right side converges uniformly, $(S(x))^{-1}$ is analytic with respect to x in a neighborhood of x_0 . Since

$$(T(x))^{-1} = (S(x))^{-1}(T(x_0))^{-1},$$

the mapping $\Omega \ni x \rightarrow (T(x))^{-1} \in L(E)$ is analytic in a neighborhood of x_0 .

§2. Some systems of partial differential equations in complex domains with values in Banach spaces

We shall extend the results described in the paper [1], [2].

In this section E denotes exclusively a Banach space with the norm $\|\cdot\|$. Let $L(E)$ be the collection of continuous endomorphisms of E , equipped with the norm $\|\cdot\|$ induced from E . Therefore $L(E)$ is a Banach space.

LEMMA 2.1. *Let S be a domain in $\mathbb{C}^m$. We put $\Omega_j = \{z \in \mathbb{C}; |z| < 1\}$ for $j=1, 2, \dots, n$ and $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$. Let A_{jk} and A_j be, respectively, analytic mappings of $\Omega \times S$ into $L(E)$ and E for $j, k=1, 2, \dots, n$. Let g_j be an analytic mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n \times S$ into E for $j=1, 2, \dots, n$. Then the Goursat's problem*

$$(1) \quad \begin{cases} \frac{\partial u_j(x; s)}{\partial x_j} = \sum_{k=1}^n A_{jk}(x, s) u_k(x; s) + A_j(x, s) \\ u_j(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n; s) \\ = g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, s) \quad (j=1, 2, \dots, n) \end{cases}$$

with parameter $s \in S$ has an E^n -valued unique analytic solution $u(x; s) = (u_1(x; s), u_2(x; s), \dots, u_n(x; s))$ in $\Omega \times S$.

PROOF. Let D be a domain in $\mathbb{C}^n$. Let f and A be, respectively, analytic mappings of D into E and $L(E)$. We denote $(x_1, x_2, \dots, x_n) \in D$ by x and put $G(x) = A(x)f(x)$. For any fixed point $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$, by Lemma 1.3 there exist a positive number r , an element $a_{v_1 v_2 \dots v_n}$ of E and an element $A_{v_1 v_2 \dots v_n}$ of $L(E)$ for $v_1, v_2, \dots, v_n = 0, 1, 2, \dots$ such that f and A are represented by power series

$$f(x) = \sum_{v_1 v_2 \dots v_n = 0}^{\infty} a_{v_1 v_2 \dots v_n} (x_1 - x_1^0)^{v_1} (x_2 - x_2^0)^{v_2} \dots (x_n - x_n^0)^{v_n}$$

$$A(x) = \sum_{v_1 v_2 \dots v_n = 0}^{\infty} A_{v_1 v_2 \dots v_n} (x_1 - x_1^0)^{v_1} (x_2 - x_2^0)^{v_2} \dots (x_n - x_n^0)^{v_n}$$

in $|x_j - x_j^0| \leq r, j = 1, 2, \dots, n$, which converge uniformly in the polydisc $\{(x_1, x_2, \dots, x_n) \in \mathbb{C}^n; |x_j - x_j^0| \leq r, j = 1, 2, \dots, n\}$. Hence we have

$$\begin{aligned} G(x) &= A(x) \left(\sum_{v_1 v_2 \dots v_n = 0}^{\infty} a_{v_1 v_2 \dots v_n} (x_1 - x_1^0)^{v_1} (x_2 - x_2^0)^{v_2} \dots (x_n - x_n^0)^{v_n} \right) \\ &= \sum_{v_1 v_2 \dots v_n = 0}^{\infty} (A(x) (a_{v_1 v_2 \dots v_n})) (x_1 - x_1^0)^{v_1} (x_2 - x_2^0)^{v_2} \dots (x_n - x_n^0)^{v_n} \\ &= \sum_{v_1 v_2 \dots v_n = 0}^{\infty} \sum_{\mu_1 \mu_2 \dots \mu_n = 0}^{\infty} A_{\mu_1 \mu_2 \dots \mu_n} (a_{v_1 v_2 \dots v_n}) (x_1 - x_1^0)^{v_1 + \mu_1} (x_2 - x_2^0)^{v_2 + \mu_2} \dots \\ &\quad (x_n - x_n^0)^{v_n + \mu_n}, \end{aligned}$$

which converges absolutely and uniformly in $|x_j - x_j^0| \leq r, j = 1, 2, \dots, n$. Therefore, G is an analytic mapping of D into E , since G is continuous. By the fact and Lemma 1.4, we can define the system of integral equations

$$(2) \quad \begin{cases} u_j^{(0)}(x; s) = g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, s) \\ u_j^{(v+1)}(x; s) = \int_0^{x_j} \left(\sum_{k=1}^n A_{jk}(x_1, x_2, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n, s) u_k^{(v)}(x_1, \right. \\ \quad \left. x_2, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n, s) \right. \\ \quad \left. + A_j(x_1, x_2, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n, s) \right) d\lambda_j \\ \quad \left. + g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, s) \right) \\ \quad (j=1, 2, \dots, n, \quad v=1, 2, 3, \dots), \end{cases}$$

and can select the line segment jointing 0 with x_j as the integral contour of (2). Moreover, making use of Lemma 1.3, $u_j^{(v)}$ ($j=1, 2, \dots, n$, $v=0, 1, 2, \dots$) is an analytic mapping of $\Omega \times S$ into E . We have

$$(3) \quad u_j^{(v+1)}(x; s) - u_j^{(v)}(x; s) = \int_0^{x_j} \sum_{k=1}^n A_{jk}(x_1, x_2, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n, s) (u_k^{(v)}(x_1, x_2, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n, s) - u_k^{(v-1)}(x_1, x_2, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n, s)) d\lambda_j.$$

For any compact subset S' of S and numbers $\eta_1, \eta_2, \dots, \eta_n$ such that $0 \leq \eta_1, \eta_2, \dots, \eta_n < 1$, we put

$$W^{(v)}(\eta_1, \eta_2, \dots, \eta_n) = \max_{\substack{|x_j| \leq \eta_j \\ s \in S'}} \max_{1 \leq j \leq n} \|u_j^{(v+1)}(x; s) - u_j^{(v)}(x; s)\|$$

for $v=0, 1, 2, \dots$. Since the integral contour is the line segment, putting $\lambda_j = re^{i\theta}$ ($\theta = \arg x_j$), we have

$$\begin{aligned} & \|u_j^{(v+1)}(x; s) - u_j^{(v)}(x; s)\| \\ & \leq \int_0^{|x_j|} \sum_{k=1}^n \|A_{jk}(x_1, \dots, x_{j-1}, re^{i\theta}, x_{j+1}, \dots, x_n, s)\| \|u_k^{(v)}(x_1, \dots, x_{j-1}, re^{i\theta}, x_{j+1}, \dots, x_n, s) - u_k^{(v-1)}(x_1, \dots, x_{j-1}, re^{i\theta}, x_{j+1}, \dots, x_n, s)\| dr \end{aligned}$$

by Lemma 1.3. We obtain

$$\begin{aligned} W^{(v)}(\eta_1, \eta_2, \dots, \eta_n) & \leq M \sum_{j=1}^n \int_0^{\eta_j} W^{(v-1)}(\eta_1, \eta_2, \dots, \eta_{j-1}, t_j, \eta_{j+1}, \dots, \eta_n) dt_j \\ W^{(0)}(\eta_1, \eta_2, \dots, \eta_n) & \leq W^{(0)} \quad (v=1, 2, \dots), \end{aligned}$$

where M and $W^{(0)}$ are constant positive numbers. Hence we have

$$W^{(v)}(\eta_1, \eta_2, \dots, \eta_n) \leq W^{(0)} \frac{(nM(\eta_1 + \eta_2 + \dots + \eta_n))^v}{v!} \quad (v=1, 2, \dots).$$

It follows that

$$\|u_j^{(v+1)}(x; s) - u_j^{(v)}(x; s)\| \leq \frac{1}{v!} W^{(0)} (n^2 M)^v$$

for $|x| \leq r < 1$, $s \in S'$. Thus the sequence $\{u_j^{(v)}\}_{v=0}^\infty$ ($j=1, 2, \dots, n$) of analytic

mappings converges uniformly to an analytic mapping u_j ($j=1, 2, \dots, n$) of $\Omega \times S$ into E . $u(x; s) = (u_1(x; s), u_2(x; s), \dots, u_n(x; s))$ is an analytic solution of (1).

Let $v(x; s) = (v_1(x; s), v_2(x; s), \dots, v_n(x; s))$ be another analytic solution of (1). Similarly there exists a positive number V such that

$$\|u_j(x; s) - v_j(x; s)\| \leq \frac{V}{v!} (n^2 M)^v$$

for $|x| < r$, $s \in S'$, $j=1, 2, \dots, n$. This means that $u=v$. The uniqueness of the solution of (1) is proved.

THEOREM 2.2. *Let S and Ω_j be a domain in $\mathbb{C}^m$ and a simply connected domain in the complex plain for $j=1, 2, \dots, n$, respectively. Let a_j be an analytic mapping of S into Ω_j , $g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, s)$ be an analytic mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n \times S$ into E , A_j be an analytic mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n \times S$ into E and A_{jk} be an analytic mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n \times S$ into $L(E)$ for $j, k=1, 2, \dots, n$. Then the Goursat's problem*

$$(4) \quad \begin{cases} \frac{\partial u_j(x; s)}{\partial x_j} = \sum_{k=1}^n A_{jk}(x, s) u_k(x; s) + A_j(x, s) \\ u_j(x_1, x_2, \dots, x_{j-1}, a_j(s), x_{j+1}, \dots, x_n; s) = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n, s) \end{cases}$$

with parametre $s \in S$ has an E^n -valued unique analytic solution $u(x; s) = (u_1(x; s), u_2(x; s), \dots, u_n(x; s))$ in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n \times S$.

PROOF. We shall prove the theorem in case that $\Omega_j \neq \mathbb{C}$ for $j=1, 2, \dots, n$. By the Riemann's theorem, there exists a conformal mapping ϕ_j of Ω_j onto $\{x \in \mathbb{C}; |x| < 1\}$ for $j=1, 2, \dots, n$. Since ϕ_j is differentiable, our Goursat's problem is equivalent to the problem

$$\begin{cases} \frac{\partial U_j(\xi; s)}{\partial \xi_j} = \sum_{k=1}^n \phi_j'(\xi_j) A_{jk}(\phi_1(\xi_1), \phi_2(\xi_2), \dots, \phi_n(\xi_n), s) U_k(\xi; s) \\ \quad + A_j(\phi_1(\xi_1), \phi_2(\xi_2), \dots, \phi_n(\xi_n)) \\ U_j(\xi_1, \xi_2, \dots, \xi_{j-1}, \phi_j^{-1}(a_j(s)), \xi_{j+1}, \dots, \xi_n, s) \\ \quad = g_j(\phi_1(\xi_1), \dots, \phi_{j-1}(\xi_{j-1}), \phi_{j+1}(\xi_{j+1}), \dots, \phi_n(\xi_n), s) \end{cases}$$

with parametre $s \in S$ in $\prod_{j=1}^n \{x \in \mathbb{C}; |x| < 1\}$, where $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \prod_{j=1}^n \{x \in \mathbb{C}; |x| < 1\}$. Thus, it is sufficient to prove Theorem 2.2 in case that $\Omega_j = \{x_j \in \mathbb{C};$

$|x_j| < 1\}$ for $j=1, 2, \dots, n$. We shall solve the problem by the method of successive approximations

$$\begin{aligned} u_j^{(v+1)}(x, s) &= \int_{a_j(s)}^{x_j} \left(\sum_{k=1}^n A_{jk} u_k^{(v)} + A_j \right) (x_1, x_2, \dots, x_{j-1}, \lambda_j, x_{j+1}, \dots, x_n, s) d\lambda_j \\ &\quad + g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, s) \\ u_j^{(0)}(x, s) &= g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, s). \end{aligned}$$

Changing variables

$$y_j = \frac{x_j - a_j(s)}{1 - a_j(s)x_j} \quad (j=1, 2, \dots, n),$$

we can prove the uniform convergence of $\{u_j^{(v)}(x, s); v=1, 2, \dots\}$ by the method used in the proof of Lemma 2.1. The limits $u_j(x, s)$ of $\{u_j^{(v)}(x, s); v=1, 2, \dots\}$ give the unique holomorphic solution of the problem.

THEOREM 2.3. *Let Ω_j be a simply connected domain containing the origin in the complex plane for $j=1, 2, \dots, n$. Let A_{jk} be an analytic mapping of $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ into $L(E)$ and A_j be an analytic mapping of Ω into E for $j, k=1, 2, \dots, n$. Then, there exist an analytic mapping P_j of Ω into E , an analytic mapping R_j of Ω into $L(E)$ and an analytic mapping $K_{k\alpha_1\alpha_2\dots\alpha_l}^l$ of $\Omega \times \Omega_{\alpha_1} \times \Omega_{\alpha_2} \times \dots \times \Omega_{\alpha_l}$ into $L(E)$ for $j, k=1, 2, \dots, n, l=1, 2, \dots, n-1$ and for integers $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l \leq n$ such that for every analytic mapping g_j of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n$ into E for $j=1, 2, \dots, n$, the equation*

$$\begin{aligned} (5) \quad u_j(x) &= \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} K_{k\alpha_1\alpha_2\dots\alpha_l}^l(x; s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) \\ &\quad g_k(x_1, x_2, \dots, x_{\alpha_1-1}, s_{\alpha_1}, x_{\alpha_1+1}, \dots, x_{\alpha_l-1}, s_{\alpha_l}, x_{\alpha_l+1}, \dots, x_n) \\ &\quad ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} + P_j(x) + R_j(x) g_j(x) \quad (j=1, 2, \dots, n) \end{aligned}$$

is a unique analytic solution of the Goursat's problem

$$(6) \quad \begin{cases} \frac{\partial u_j(x)}{\partial x_j} = \sum_{k=1}^n A_{jk}(x) u_k(x) + A_j(x) \\ u_j(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ (j=1, 2, \dots, n) \end{cases}$$

in Ω .

PROOF. There exists the analytic solution P_j of the problem

$$(7) \quad \begin{cases} \frac{\partial P_j(x)}{\partial x_j} = \sum_{k=1}^n A_{jk}(x) P_k(x) + A_j(x) \\ P_j(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) = 0 \in E \\ (j=1, 2, \dots, n) \end{cases}$$

by Theorem 2.2. Let R_j be the analytic solution of the problem

$$(8) \quad \begin{cases} \frac{\partial R_j(x)}{\partial x_j} = A_{jj}(x) R_j(x) \\ R_j(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) = I \quad (j=1, 2, \dots, n) \end{cases}$$

in Ω , where I is the identity mapping of E . We shall define $K_{m\alpha_1\alpha_2\cdots\alpha_l}^k$ by induction with respect to l . We put

$$(9) \quad K_{m\alpha}^j(x; s_\alpha) = 0$$

in $\Omega \times \Omega_\alpha$ for $j \notin \{m, \alpha\}$. Let $(K_{m\alpha}^\alpha(x; s_\alpha), K_{m\alpha}^m(x; s_\alpha))$ be the analytic solution in $\Omega \times \Omega_\alpha$ of the Goursat's problem

$$(10) \quad \begin{cases} \frac{\partial K_{m\alpha}^\alpha}{\partial x_\alpha} = A_{\alpha\alpha} K_{m\alpha}^\alpha + A_{\alpha m} K_{m\alpha}^m \\ \frac{\partial K_{m\alpha}^m}{\partial x_m} = A_{m\alpha} K_{m\alpha}^\alpha + A_{mm} K_{m\alpha}^m \\ K_{m\alpha}^\alpha(x_1, \dots, x_{\alpha-1}, s_\alpha, x_{\alpha+1}, \dots, x_n; s_\alpha) \\ \quad = (A_{\alpha m} R_m)(x_1, \dots, x_{\alpha-1}, s_\alpha, x_{\alpha+1}, \dots, x_n) \\ K_{m\alpha}^m(x_1, \dots, x_{m-1}, 0, x_{m+1}, \dots, x_n, s_\alpha) = 0 \end{cases}$$

in $\Omega_\alpha \times \Omega_m$ with parametre $\{x_1, x_2, \dots, x_{\alpha-1}, x_{\alpha+1}, \dots, x_{m-1}, x_{m+1}, \dots, x_n, s_\alpha\}$. We assume that $K_{m\alpha_1\alpha_2\cdots\alpha_p}^k(x; s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_p})$ is defined when $p < l$. We put

$$(11) \quad K_{m\alpha_1\alpha_2\cdots\alpha_l}^j(x; s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) = 0$$

in $\Omega \times \Omega_{\alpha_1} \times \cdots \times \Omega_{\alpha_l}$ for $j \notin \{m, \alpha_1, \alpha_2, \dots, \alpha_l\}$. The Goursat's problem

$$\begin{aligned}
 (12) \quad & \left\{ \begin{aligned}
 & \frac{\partial K_{m\alpha_1\alpha_2\ldots\alpha_l}^{\alpha_v}}{\partial x_{\alpha_v}} = A_{\alpha_v m} K_{m\alpha_1\alpha_2\ldots\alpha_l}^m + \sum_{\mu=1}^l A_{\alpha_v \alpha_\mu} K_{m\alpha_1\alpha_2\ldots\alpha_l}^{\alpha_\mu} \\
 & \frac{\partial K_{m\alpha_1\alpha_2\ldots\alpha_l}^m}{\partial x_m} = A_{mm} K_{m\alpha_1\alpha_2\ldots\alpha_l}^m + \sum_{\mu=1}^l A_{m\alpha_\mu} K_{m\alpha_1\alpha_2\ldots\alpha_l}^{\alpha_\mu} \\
 & K_{m\alpha_1\alpha_2\ldots\alpha_l}^{\alpha_v}(x_1, \ldots, x_{\alpha_v-1}, s_{\alpha_v}, x_{\alpha_v+1}, \ldots, x_n, s_{\alpha_1}, \ldots, s_{\alpha_l}) \\
 & = A_{\alpha_v m}(x_1, \ldots, x_{\alpha_v-1}, s_{\alpha_v}, x_{\alpha_v+1}, \ldots, x_n) K_{m\alpha_1\alpha_2\ldots\alpha_{v-1}\alpha_{v+1}\ldots\alpha_l}^m(x_1, \\
 & \quad \ldots, x_{\alpha_v-1}, s_{\alpha_v}, x_{\alpha_v+1}, \ldots, x_n, s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_{v-1}}, s_{\alpha_{v+1}}, \ldots, s_{\alpha_l}) \\
 & \quad + \sum_{\mu=1}^l A_{\alpha_v \alpha_\mu}(x_1, \ldots, x_{\alpha_v-1}, s_{\alpha_v}, x_{\alpha_v+1}, \ldots, x_n) K_{m\alpha_1\alpha_2\ldots\alpha_{v-1}\alpha_{v+1}\ldots\alpha_l}^{\alpha_\mu} \\
 & \quad (x_1, x_2, \ldots, x_{\alpha_v-1}, s_{\alpha_v}, x_{\alpha_v+1}, \ldots, x_n, s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_{v-1}}, s_{\alpha_{v+1}}, \ldots, s_{\alpha_l}) \\
 & K_{m\alpha_1\alpha_2\ldots\alpha_l}^m(x_1, x_2, \ldots, x_{m-1}, 0, x_{m+1}, \ldots, x_n, s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}) = 0 \\
 & (v=1, 2, \ldots, l)
 \end{aligned} \right.
 \end{aligned}$$

with $(n-1)$ parametres $\{x_1, x_2, \ldots, x_n, s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\} - \{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_l}, x_m\}$ in $\Omega_{\alpha_1} \times \Omega_{\alpha_2} \times \cdots \times \Omega_{\alpha_1} \times \Omega_m$ has an $L(E)$ -valued unique analytic solution $(K_{m\alpha_1\alpha_2\ldots\alpha_l}^{\alpha_1}, K_{m\alpha_1\alpha_2\ldots\alpha_l}^{\alpha_2}, \ldots, K_{m\alpha_1\alpha_2\ldots\alpha_l}^{\alpha_l}, K_{m\alpha_1\alpha_2\ldots\alpha_l}^m)$. For the analytic mappings P_j, R_j and $K_{k\alpha_1\alpha_2\ldots\alpha_l}^j$ for $j, k=1, 2, \ldots, n, l=1, 2, \ldots, n-1$ and for integers $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l \leq n$ determined in this way, we shall show that (5) is an analytic solution of (6). By the definitions (9) and (11), it follows from (5) that

$$\begin{aligned}
 u_j(x) = & \sum_{\substack{\alpha_1 < \alpha_2 < \cdots < \alpha_l \\ j \notin \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \cdots \int_0^{x_{\alpha_l}} K_{j\alpha_1\alpha_2\ldots\alpha_l}^j(x, s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}) \\
 & g_j(x_1, \ldots, s_{\alpha_1}, \ldots, s_{\alpha_2}, \ldots, s_{\alpha_l}, \ldots, x_n) ds_{\alpha_1} ds_{\alpha_2} \cdots ds_{\alpha_l} \\
 & + \sum_{\substack{\alpha_1 < \alpha_2 < \cdots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \cdots \int_0^{x_{\alpha_{v-1}}} \int_0^{x_j} \int_0^{x_{\alpha_{v+1}}} \cdots \int_0^{x_{\alpha_l}} \\
 & K_{k\alpha_1\alpha_2\ldots\alpha_{v-1}j\alpha_{v+1}\ldots\alpha_l}^j(x, s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_{v-1}}, s_j, s_{\alpha_{v+1}}, \ldots, s_{\alpha_l}) \\
 & g_k(x_1, \ldots, s_{\alpha_1}, \ldots, s_{\alpha_{v-1}}, s_j, s_{\alpha_{v+1}}, \ldots, s_{\alpha_l}, \ldots, x_n) \\
 & ds_{\alpha_1} ds_{\alpha_2} \cdots ds_{\alpha_{v-1}} ds_j ds_{\alpha_{v+1}} \cdots ds_{\alpha_l}
 \end{aligned}$$

$$+ P_j(x) + R_j(x)g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

We have

$$\begin{aligned} \frac{\partial u_j(x)}{\partial x_j} &= \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} \frac{\partial K_{j\alpha_1\alpha_2\dots\alpha_l}^j}{\partial x_j}(x, s_{\alpha_1}, \dots, s_{\alpha_l}) \\ &\quad g_j(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_2}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} \\ &\quad + \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} \frac{\partial K_{k\alpha_1\alpha_2\dots\alpha_l}^j}{\partial x_j}(x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) \\ &\quad g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_2}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} \\ &\quad + \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_{v-1}}} \int_0^{x_{\alpha_v+1}} \dots \int_0^{x_{\alpha_l}} \\ &\quad K_{k\alpha_1\alpha_2\dots\alpha_{v-1}j\alpha_v+1\dots\alpha_l}^j(x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_{v-1}}, x_j, s_{\alpha_v+1}, \dots, s_{\alpha_l}) \\ &\quad g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_{v-1}}, \dots, x_j, \dots, s_{\alpha_v+1}, \dots, s_{\alpha_l}, \dots, x_n) \\ &\quad ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_{v-1}} ds_{\alpha_v+1} \dots ds_{\alpha_l} \\ &\quad + \frac{\partial P_j(x)}{\partial x_j} + \frac{\partial R_j(x)}{\partial x_j} g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ &\quad (j=1, 2, \dots, n). \end{aligned}$$

From (12), we have

$$\begin{aligned} &\sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} \frac{\partial K_{k\alpha_1\alpha_2\dots\alpha_l}^j}{\partial x_j}(x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) \\ &\quad g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_2}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} \\ &= \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} (A_{jk} K_{k\alpha_1\alpha_2\dots\alpha_l}^k + \sum_{\mu=1}^l A_{j\alpha_\mu} K_{k\alpha_1\alpha_2\dots\alpha_l}^{\alpha_\mu}) \\ &\quad (x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_2}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l}. \end{aligned}$$

Since we have

$$\sum_{k \neq j} K_{kj}^j(x, x_j) g_k(x) = \sum_{k \neq j} A_{jk}(x) R_k(x) g_k(x)$$

from (10), hence, by (12) we have

$$\begin{aligned} & \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_{v-1}}} \int_0^{x_{\alpha_v+1}} \dots \int_0^{x_{\alpha_l}} K_{k\alpha_1\alpha_2\dots\alpha_{v-1}j\alpha_v+1\dots\alpha_l}^j \\ & (x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_{v-1}}, x_j, s_{\alpha_v+1}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_{v-1}}, \dots, x_j, \dots, \\ & s_{\alpha_v+1}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_{v-1}} ds_{\alpha_v+1} \dots ds_{\alpha_l} \\ & = \sum_{k \neq j} A_{jk}(x) R_k(x) g_k(x) + \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 2 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_{v-1}}} \int_0^{x_{\alpha_v+1}} \dots \int_0^{x_{\alpha_l}} \\ & (A_{jk} K_{k\alpha_1\alpha_2\dots\alpha_{v-1}\alpha_v+1\dots\alpha_l}^k + \sum_{\mu=1}^l A_{j\alpha_\mu} K_{k\alpha_1\alpha_2\dots\alpha_{v-1}\alpha_v+1\dots\alpha_l}^{\alpha_\mu})(x, s_{\alpha_1}, s_{\alpha_2}, \dots, \\ & s_{\alpha_{v-1}}, s_{\alpha_v+1}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, x_j, \dots, s_{\alpha_l}, \dots, x_n) \\ & ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_{v-1}} ds_{\alpha_v+1} \dots ds_{\alpha_l} \\ & = \sum_{k \neq j} A_{jk}(x) R_k(x) g_k(x) + \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \neq j \\ 1 \leq l \leq n-2}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} (A_{jk} K_{k\alpha_1\alpha_2\dots\alpha_l}^k \\ & + \sum_{\mu=1}^l A_{j\alpha_\mu} K_{k\alpha_1\alpha_2\dots\alpha_l}^{\alpha_\mu})(x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_n) \\ & ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l}. \end{aligned}$$

From (7) and (8), we have

$$\begin{aligned} & \frac{\partial P_j(x)}{\partial x_j} + \frac{\partial R_j(x)}{\partial x_j} g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ & = \sum_{k=1}^n A_{jk}(x) P_k(x) + A_j(x) + A_{jj}(x) R_j(x) g_j(x). \end{aligned}$$

From (12), we have

$$\begin{aligned}
& \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} \frac{\partial K_{j\alpha_1\alpha_2\dots\alpha_l}^j}{\partial x_j}(x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_j \\
& (x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_2}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} \\
& = \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} (A_{jj} K_{j\alpha_1\alpha_2\dots\alpha_l}^j + \sum_{\mu=1}^l A_{j\alpha_\mu} K_{j\alpha_1\alpha_2\dots\alpha_l}^{\alpha_\mu}) \\
& (x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_j(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\frac{\partial u_j}{\partial x_j}(x) &= \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} (A_{jj} K_{j\alpha_1\alpha_2\dots\alpha_l}^j \\
& + \sum_{\mu=1}^l A_{j\alpha_\mu} K_{j\alpha_1\alpha_2\dots\alpha_l}^{\alpha_\mu})(x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_j(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_n) \\
& ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} + \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} (A_{jk} K_{k\alpha_1\alpha_2\dots\alpha_l}^k \\
& + \sum_{\mu=1}^l A_{j\alpha_\mu} K_{k\alpha_1\alpha_2\dots\alpha_l}^{\alpha_\mu})(x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_n) \\
& ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} + \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \neq j \\ 1 \leq l \leq n-2}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} (A_{jk} K_{k\alpha_1\alpha_2\dots\alpha_l}^k \\
& + \sum_{\mu=1}^l A_{j\alpha_\mu} K_{k\alpha_1\alpha_2\dots\alpha_l}^{\alpha_\mu})(x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_n) \\
& ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} + \sum_{k \neq j} A_{jk}(x) R_k(x) g_k(x) + \sum_{k=1}^n A_{jk}(x) P_k(x) + A_j(x) \\
& + A_{jj}(x) R_j(x) g_j(x).
\end{aligned}$$

Since

$$\begin{aligned}
& \{(k, \alpha_1, \alpha_2, \dots, \alpha_l); \alpha_1 < \alpha_2 < \dots < \alpha_l, j, k \in \{\alpha_1, \alpha_2, \dots, \alpha_l\}, k \neq j, 1 \leq l \leq n-2\} \\
& = \{(k, \alpha_1, \alpha_2, \dots, \alpha_l); \alpha_1 < \alpha_2 < \dots < \alpha_l, j, k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\}, k \neq j, 1 \leq l \leq n-1\}
\end{aligned}$$

and

$$\begin{aligned} & \{(j, \alpha_1, \alpha_2, \dots, \alpha_l); \alpha_1 < \alpha_2 < \dots < \alpha_l, j \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\}, 1 \leq l \leq n-1\} \\ & \cup \{(k, \alpha_1, \alpha_2, \dots, \alpha_l); \alpha_1 < \alpha_2 < \dots < \alpha_l, j, k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\}, k \neq j, 1 \leq l \leq n-1\} \\ & = \{(k, \alpha_1, \alpha_2, \dots, \alpha_l); \alpha_1 < \alpha_2 < \dots < \alpha_l, j, k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\}, 1 \leq l \leq n-1\}, \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial u_j(x)}{\partial x_j} &= \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j, k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} (A_{jk} K_{k\alpha_1\alpha_2\dots\alpha_l}^k \\ &+ \sum_{\mu=1}^l A_{j\alpha_\mu} K_{k\alpha_1\alpha_2\dots\alpha_l}^{\alpha_\mu})(x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_n) \\ &ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} + \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} (A_{jk} K_{k\alpha_1\alpha_2\dots\alpha_l}^k \\ &+ \sum_{\mu=1}^l A_{j\alpha_\mu} K_{k\alpha_1\alpha_2\dots\alpha_l}^{\alpha_\mu})(x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_n) \\ &ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} + \sum_{k=1}^n A_{jk}(x) R_k(x) g_k(x) + \sum_{k=1}^n A_{jk}(x) P_k(x) + A_j(x). \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j, k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} A_{jk} K_{k\alpha_1\alpha_2\dots\alpha_l}^k g_k + \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} A_{jk} K_{k\alpha_1\alpha_2\dots\alpha_l}^k g_k \\ &= \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} A_{jk} K_{k\alpha_1\alpha_2\dots\alpha_l}^k g_k \end{aligned}$$

and

$$\sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j, k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \left(\sum_{\mu=1}^l A_{j\alpha_\mu} K_{k\alpha_1\alpha_2\dots\alpha_l}^{\alpha_\mu} \right) + \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ j \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \left(\sum_{\mu=1}^l A_{j\alpha_\mu} K_{k\alpha_1\alpha_2\dots\alpha_l}^{\alpha_\mu} \right)$$

$$= \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ k \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1 \\ m \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\}}} A_{jk} K_{m\alpha_1\alpha_2\dots\alpha_l}^k.$$

Thus we have

$$\begin{aligned} \frac{\partial u_j(x)}{\partial x_j} &= \sum_{k=1}^n A_{kj}(x) \left(\sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} K_{k\alpha_1\alpha_2\dots\alpha_l}^k \right. \\ &\quad (x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} \\ &\quad + \sum_{\substack{\alpha_1 < \alpha_2 < \dots < \alpha_l \\ k \in \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ m \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ 1 \leq l \leq n-1}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} K_{m\alpha_1\alpha_2\dots\alpha_l}^k (x, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) \\ &\quad g_m(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} + P_k(x) + R_k(x) g_k(x) \Big) + A_j(x) \\ &= \sum_{k=1}^n A_{jk}(x) u_k(x) + A_j(x) \quad (j=1, 2, \dots, n). \end{aligned}$$

Substituting $x_j=0$ in (5), we have

$$K_{k\alpha_1\alpha_2\dots\alpha_l}^j(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) = 0$$

for $j \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\}$ by (12). Hence we obtain

$$u_j(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$$

for $j=1, 2, \dots, n$.

Next we shall discuss analytic periodic solutions of the Goursat's problem and analytic solutions of the singular systems.

LEMMA 2.4. *Let Ω_j be a simply connected domain containing the origin in the complex plane for $j=1, 2, \dots, n$. Assume that Ω_n is invariant under the translation $x_n \rightarrow x_n + a$. Let A_{jk} and A_j be, respectively, analytic mappings of $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ into $L(E)$ and E for $j, k=1, 2, \dots, n$ and g_j be an analytic mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n$ into E for $j=1, 2, \dots, n$. Assume that A_{jk} ($j, k=1, 2, \dots, n$), A_j ($j=1, 2, \dots, n$) and g_j ($j=1, 2, \dots, n-1$) have a period a with respect to the variable x_n . Then, the Goursat's problem*

$$(13) \quad \begin{cases} \frac{\partial u_j(x)}{\partial x_j} = \sum_{k=1}^n A_{jk}(x) u_k(x) + A_j(x) \\ u_j(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) = g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \end{cases} \quad (j=1, 2, \dots, n)$$

has an E^n -valued analytic solution $u(x) = (u_1(x), u_2(x), \dots, u_n(x))$ which has a period a with respect to the variable x_n if and only if the equation

$$(14) \quad u_n(x_1, x_2, \dots, x_{n-1}, a) = g_n(x_1, x_2, \dots, x_{n-1})$$

holds in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n-1}$.

PROOF. Suppose that the solution $u(x)$ in Ω of (14) satisfies in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n-1}$. We put

$$v_j(x_1, x_2, \dots, x_{n-1}, a) = u_j(x_1, x_2, \dots, x_{n-1}, x_n + a)$$

in Ω for $j=1, 2, \dots, n$. Then $v(x) = (v_1(x), v_2(x), \dots, v_n(x))$ is also an E^n -valued analytic solution of (13). Hence it follows from the uniqueness of the solution of (13) that $v(x) = u(x)$ in Ω .

LEMMA 2.5. Let Ω_j be a simply connected domain containing the origin in the complex plane for $j=1, 2, \dots, n$. Let $K_{\alpha_1 \alpha_2 \dots \alpha_l}(x_1, x_2, \dots, x_n; s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l})$ be an analytic mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n \times \Omega_{\alpha_1} \times \Omega_{\alpha_2} \times \dots \times \Omega_{\alpha_l}$ into $L(E)$ for integers $1 \leq l \leq n$, $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l \leq n$. Then, for any analytic mapping g of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ into E , the integral equation of Volterra's type of the second kind

$$(15) \quad f(x) + \sum_{\substack{1 \leq l \leq n \\ 1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l \leq n}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} K_{\alpha_1 \alpha_2 \dots \alpha_l}(x; s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l})$$

$$f(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_2}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} = g(x)$$

has an E -valued unique analytic solution $f(x)$ in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$.

PROOF. We shall prove Lemma 2.5 in case that $\Omega_j \neq \mathbb{C}$ for $j=1, 2, \dots, n$. By the Riemann's mapping theorem we may assume that $\Omega_j = \{x_j \in \mathbb{C}; |x_j| < \frac{1}{n}\}$ for $j=1, 2, \dots, n$. Let r be a positive number such that $r < \frac{1}{n}$. For any E -valued analytic mapping u in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$, we put

$$(Tu)(x) = \sum_{\substack{1 \leq l \leq n \\ 1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l \leq n}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} K_{\alpha_1 \alpha_2 \dots \alpha_l}(x; s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) \\ u(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_2}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l}$$

and

$$|u|(\xi_1, \xi_2, \dots, \xi_n) = \max_{|x_j| \leq \xi_j} \|u(x_1, x_2, \dots, x_n)\|$$

for $0 \leq \xi_j < \frac{1}{n}$, $j = 1, 2, \dots, n$. For some constant number $M > 0$, we have

$$|Tu|(\xi_1, \xi_2, \dots, \xi_n) \leq M \sum_{\substack{1 \leq l \leq n \\ 1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l \leq n}} \int_0^{\xi_{\alpha_1}} \int_0^{\xi_{\alpha_2}} \dots \int_0^{\xi_{\alpha_l}} |u|(\xi_1, \xi_2, \dots, \\ \lambda_{\alpha_1}, \dots, \lambda_{\alpha_2}, \dots, \lambda_{\alpha_l}, \dots, \xi_n) d\lambda_{\alpha_1} d\lambda_{\alpha_2} \dots d\lambda_{\alpha_l}$$

in $\{\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbf{R}^n; |\xi_j| < r, j = 1, 2, \dots, n\}$ since we can take the line segments starting from the origin as the integral contours. Let W be a number such that

$$\|g(x_1, x_2, \dots, x_n)\| \leq W$$

in $\{x = (x_1, x_2, \dots, x_n) \in \mathbf{C}^n; |x_j| < r, j = 1, 2, \dots, n\}$. Let $N = \sum_{1 \leq l \leq n} \binom{n}{l} M$. Then we have

$$|T^v g|(\xi_1, \xi_2, \dots, \xi_n) \leq W \frac{\{N(\xi_1 + \xi_2 + \dots + \xi_n)\}^v}{v!}$$

in $0 \leq \xi_1 < r, 0 \leq \xi_2 < r, \dots, 0 \leq \xi_n < r$. Therefore the series

$$f(x) = \sum_{v=0}^{\infty} (-1)^v (T^v g)(x)$$

converges uniformly in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$. $f(x)$ is of course an E -valued unique analytic solution of (15).

Let Ω be a simply connected domain containing the origin in the complex plane and A be an analytic mapping of Ω into $L(E)$. We denote by

$$\exp\left(\int_0^x A(\lambda) d\lambda\right)$$

the analytic solution u in Ω of the problem

$$(16) \quad \begin{cases} \frac{du}{dx} = A(x)u \\ u(0) = \text{the identity mapping of } E \end{cases}$$

for the sake of brevity of notations.

THEOREM 2.6. *Let Ω_j be a simply connected domain containing the origin in the complex plane for $j=1, 2, \dots, n$. Assume that $\Omega_n = \{x+a; x \in \Omega_n\}$ for a complex number a . Let A_j and A_{jk} be, respectively, analytic mappings of $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ into E and $L(E)$ for $j, k=1, 2, \dots, n$, and g_j be an analytic mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n$ into E for $j=1, 2, \dots, n-1$. Assume that A_j ($j=1, 2, \dots, n$), A_{jk} ($j, k=1, 2, \dots, n$) and g_j ($j=1, 2, \dots, n-1$) have a period a with respect to the variable x_n , and that*

$$(17) \quad \exp\left(\int_0^a A_{nn}(x_1, x_2, \dots, x_{n-1}, \lambda) d\lambda\right) - I \in GL(E)$$

in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n-1}$, where I is the identity mapping of E and $GL(E)$ is the collection of invertible elements of $L(E)$. Then the problem

$$(18) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = \sum_{k=1}^n A_{jk} u_k + A_j & (j=1, 2, \dots, n) \\ u_j(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) = g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ (j=1, 2, \dots, n-1) \end{cases}$$

has an E^n -valued unique analytic solution $u(x) = (u_1(x), u_2(x), \dots, u_n(x))$ in Ω which has a period a with respect to the variable x_n .

PROOF. By (8), (16) and (17), we have

$$R_n(x_1, x_2, \dots, x_{n-1}, a) - I \in GL(E)$$

in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n-1}$. Since the inverse linear mapping $(R_n(x_1, x_2, \dots, x_{n-1}, a) - I)^{-1}$ of $R_n(x_1, x_2, \dots, x_{n-1}, a) - I$ is analytic in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n-1}$ by Lemma 1.6, from Lemma 2.5 the integral equation

$$f(x_1, x_2, \dots, x_{n-1}) + \sum_{\substack{1 \leq l \leq n-1 \\ n \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\} \\ \alpha_1 < \alpha_2 < \dots < \alpha_l}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} (R_n(x_1, x_2, \dots, x_{n-1}, a) - I)^{-1}$$

$$\begin{aligned}
& K_{n\alpha_1\alpha_2\ldots\alpha_l}^n(x_1, x_2, \ldots, x_{n-1}, a; s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l})f(x_1, x_2, \ldots, s_{\alpha_1}, \ldots, s_{\alpha_2}, \ldots, \\
& s_{\alpha_1}, \ldots, x_{n-1})ds_{\alpha_1}ds_{\alpha_2}\cdots ds_{\alpha_l} \\
& = (I - R_n(x_1, x_2, \ldots, x_{n-1}, a))^{-1}P_n(x_1, x_2, \ldots, x_{n-1}, a) + \sum_{\substack{1 \leq l \leq n-1 \\ k \notin \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \\ n \in \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \\ \alpha_1 < \alpha_2 < \cdots < \alpha_l}} \\
& \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \cdots \int_0^{x_{\alpha_l}} (I - R_n(x_1, x_2, \ldots, x_{n-1}, a))^{-1}K_{k\alpha_1\alpha_2\ldots\alpha_l}^n(x_1, x_2, \ldots, \\
& x_{n-1}, a; s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l})g_k(x_1, x_2, \ldots, s_{\alpha_1}, \ldots, s_{\alpha_2}, \ldots, s_{\alpha_l}, \ldots, x_{n-1}, \\
& a)ds_{\alpha_1}ds_{\alpha_2}\cdots ds_{\alpha_l}
\end{aligned}$$

has an E -valued unique analytic solution $g_n(x_1, x_2, \ldots, x_{n-1})$ in $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_{n-1}$, where we use the notations in Theorem 2.3. Let $u(x) = (u_1(x), u_2(x), \ldots, u_n(x))$ be the E^n -valued analytic solution in Ω of the problem (6). By Theorem 2.3, we have

$$\begin{aligned}
u_n(x) &= P_n(x) + R_n(x)g_n(x) + \sum_{\substack{1 \leq l \leq n-1 \\ n \notin \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \\ \alpha_1 < \alpha_2 < \cdots < \alpha_l}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \cdots \int_0^{x_{\alpha_l}} K_{n\alpha_1\alpha_2\ldots\alpha_l}^n \\
& (x; s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l})g_n(x_1, \ldots, s_{\alpha_1}, \ldots, s_{\alpha_2}, \ldots, s_{\alpha_l}, \ldots, x_{n-1})ds_{\alpha_1}ds_{\alpha_2}\cdots ds_{\alpha_l} \\
& + \sum_{\substack{1 \leq l \leq n-1 \\ \alpha_1 < \alpha_2 < \cdots < \alpha_l \\ k \notin \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \\ n \in \{\alpha_1, \alpha_2, \ldots, \alpha_l\}}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \cdots \int_0^{x_{\alpha_l}} K_{k\alpha_1\alpha_2\ldots\alpha_l}^n(x; s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}) \\
& g_k(x_1, \ldots, s_{\alpha_1}, \ldots, s_{\alpha_2}, \ldots, s_{\alpha_l}, \ldots, x_n)ds_{\alpha_1}ds_{\alpha_2}\cdots ds_{\alpha_l},
\end{aligned}$$

and hence we have

$$u_n(x_1, x_2, \ldots, x_{n-1}, a) = g_n(x_1, x_2, \ldots, x_{n-1})$$

by the method of the choice of g_n . By Lemma 2.4 $u(x)$ is the unique analytic solution of the problem (18) which has a period a with respect to the variable x_n .

THEOREM 2.7. Let Ω_j be a simply connected domain containing the origin in the complex plane for $j=1, 2, \ldots, n$. Assume that $\Omega_n = \{x+a; x \in \Omega_n\}$ for a complex number a . Let A_{jk} and A_j be, respectively, analytic mappings of $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ into $L(E)$ and E for $j, k=1, 2, \ldots, n$, and g_j be an analytic

mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n$ into E for $j=1, 2, \dots, n$. Assume that A_{jk} ($j=1, 2, \dots, n$), A_j ($j=1, 2, \dots, n$) and g_j ($j=1, 2, \dots, n-1$) have a period a with respect to the variable x_n and that

$$A_{nk}(x)=0$$

in Ω for $k \neq p, n$,

$$\exp\left(\int_0^a A_{nn}(x_1, x_2, \dots, x_{n-1}, \lambda) d\lambda\right) = I$$

in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n-1}$,

$$\int_0^a (A_{pn} A_{np})(x_1, x_2, \dots, x_{n-1}, s) ds \in GL(E)$$

in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n-1}$, and

$$A_{nn}(x)T = TA_{nn}(x)$$

for all $x \in \Omega$ and $T \in L(E)$.

Then the problem (18) has an E^n -valued unique analytic solution $u(x) = (u_1(x), u_2(x), \dots, u_n(x))$ in Ω which has a period a with respect to the variable x_n if and only if the equation

$$(19) \quad T(g_1, g_2, \dots, g_{n-1})(x_1, x_2, \dots, x_{p-1}, 0, x_{p+1}, \dots, x_{n-1}, a) = 0$$

holds in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{p-1} \times \Omega_{p+1} \times \dots \times \Omega_{n-1}$, where

$$T(g_1, g_2, \dots, g_{n-1})(x) = P_n(x) + \sum_{\substack{1 \leq k \leq n-1 \\ 1 \leq l \leq n-1 \\ k \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\}}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} K_{k\alpha_1\alpha_2\dots\alpha_l}^n$$

$$(x; s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_k(x_1, \dots, s_{\alpha_1}, \dots, s_{\alpha_2}, \dots, s_{\alpha_l}, \dots, x_n) ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l}.$$

PROOF. Suppose that the problem (18) has an E^n -valued analytic solution $u(x) = (u_1(x), u_2(x), \dots, u_n(x))$ in Ω which has a period a with respect to the variable x_n . Since we have

$$K_{n\alpha_1\alpha_2\dots\alpha_l}^n(x; s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) = \int_0^{x_n} A_{nn}(x_1, x_2, \dots, x_{n-1}, \lambda) K_{n\alpha_1\alpha_2\dots\alpha_l}^n(x_1, x_2, \dots, x_{n-1}, \lambda; s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) d\lambda$$

by the integral representation of (12) when $p \notin \{\alpha_1, \alpha_2, \dots, \alpha_l\}$, we have

$$K_{n\alpha_1\alpha_2\ldots\alpha_l}^n(x; s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}) = 0$$

in $\Omega \times \Omega_{\alpha_1} \times \Omega_{\alpha_2} \times \cdots \times \Omega_{\alpha_l}$ when $p \notin \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, by (10)

$$(20) \quad K_{np}^p(x; x_p) = A_{pn}(x)R_n(x)$$

and moreover by (10)

$$(21) \quad K_{np}^n(x; s_p) = \int_0^{x_n} (A_{np}(x_1, x_2, \ldots, x_{n-1}, \lambda) K_{np}^p(x_1, x_2, \ldots, x_{n-1}, \lambda; s_p) \\ + A_{nn}(x_1, x_2, \ldots, x_{n-1}, \lambda) K_{np}^n(x_1, x_2, \ldots, x_{n-1}, \lambda; s_p)) d\lambda.$$

Since $A_{nn}(x)$ is commutative with every element of $L(E)$ with respect to the composition of mappings for every point $x \in \Omega$,

$$\int_0^{x_n} A_{nn}(x_1, x_2, \ldots, x_{n-1}, \lambda) d\lambda$$

and

$$\exp\left(\int_0^{x_n} A_{nn}(x_1, x_2, \ldots, x_{n-1}, \lambda) d\lambda\right)$$

are also commutative with every element of $L(E)$ for all $(x_1, x_2, \ldots, x_{n-1}) \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_{n-1}$. Hence by (21) we have

$$(22) \quad K_{np}^n(x; s_p) = \exp\left(\int_0^{x_n} A_{nn}(x_1, x_2, \ldots, x_{n-1}, \lambda) d\lambda\right) \int_0^{x_n} A_{np}(x_1, x_2, \ldots, \\ x_{n-1}, s) K_{np}^p(x_1, x_2, \ldots, x_{n-1}, s; s_p) \exp\left(-\int_0^s A_{nn}(x_1, x_2, \ldots, x_{n-1}, \lambda) d\lambda\right) ds.$$

By (8), (20) and (22), we have

$$(23) \quad K_{np}^n(x; x_p) = \exp\left(\int_0^{x_n} A_{nn}(x_1, x_2, \ldots, x_{n-1}, \lambda) d\lambda\right) \int_0^{x_n} (A_{np} A_{pn}) \\ (x_1, x_2, \ldots, x_{n-1}, s) ds.$$

The condition (14) becomes the following integral equation of Volterra's type of the first kind:

$$\int_0^{x_p} K_{np}^n(x_1, x_2, \ldots, x_{n-1}, a; s_p) g_n(x_1, x_2, \ldots, x_{p-1}, s_p, x_{p+1}, \ldots, x_{n-1}) ds_p$$

$$\begin{aligned}
 (24) \quad & + \sum_{\substack{1 \leq l \leq n-2 \\ n, p \notin \{\alpha_1, \alpha_2, \dots, \alpha_p\}}} \int_0^{x_p} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} K_{np\alpha_1\alpha_2\dots\alpha_l}^n(x_1, x_2, \dots, \\
 & x_{n-1}, a; s_p, s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_n(x_1, \dots, s_{\alpha_1}, \dots, s_p, \dots, s_{\alpha_l}, \dots, x_{n-1}) \\
 & ds_p ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} \\
 & = -T(g_1, g_2, \dots, g_{n-1})(x_1, x_2, \dots, x_{n-1}, a).
 \end{aligned}$$

Therefore we have

$$T(g_1, g_2, \dots, g_{n-1})(x_1, x_2, \dots, x_{p-1}, 0, x_{p+1}, \dots, x_{n-1}, a) = 0$$

in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{p-1} \times \Omega_{p+1} \times \dots \times \Omega_{n-1}$ when $x_p = 0$.

Conversely, suppose that the condition (19) holds in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{p-1} \times \Omega_{p+1} \times \dots \times \Omega_{n-1}$. Differentiating the both sides of (24) with respect to the variable x_p , we have

$$\begin{aligned}
 (25) \quad & K_{np}^n(x_1, x_2, \dots, x_{n-1}, a; x_p) g_n(x_1, x_2, \dots, x_p, \dots, x_{n-1}) \\
 & + \int_0^{x_p} \frac{\partial K_{np}^n(x_1, x_2, \dots, x_{n-1}, a; s_p)}{\partial x_p} \\
 & \quad g_n(x_1, x_2, \dots, x_{p-1}, s_p, x_{p+1}, \dots, x_{n-1}) ds_p \\
 & + \sum_{\substack{1 \leq l \leq n-2 \\ n, p \notin \{\alpha_1, \alpha_2, \dots, \alpha_p\}}} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} K_{np\alpha_1\alpha_2\dots\alpha_l}^n(x_1, x_2, \dots, x_{n-1}, a; \\
 & s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_l}) g_n(x_1, \dots, s_{\alpha_1}, \dots, s_p, \dots, s_{\alpha_l}, \dots, x_{n-1}) ds_p ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} \\
 & + \sum_{\substack{1 \leq l \leq n-1 \\ n, p \notin \{\alpha_1, \alpha_2, \dots, \alpha_p\}}} \int_0^{x_p} \int_0^{x_{\alpha_1}} \int_0^{x_{\alpha_2}} \dots \int_0^{x_{\alpha_l}} \frac{\partial K_{np\alpha_1\alpha_2\dots\alpha_l}^n}{\partial x_p}(x_1, x_2, \dots, \\
 & x_{n-1}, a; s_{\alpha_1}, s_{\alpha_2}, \dots, s_p, \dots, s_{\alpha_l}) g_n(x_1, \dots, s_{\alpha_1}, \dots, s_p, \dots, s_{\alpha_l}, \dots, x_{n-1}) \\
 & ds_p ds_{\alpha_1} ds_{\alpha_2} \dots ds_{\alpha_l} = -\frac{\partial}{\partial x_p} T(g_1, g_2, \dots, g_{n-1})(x_1, x_2, \dots, x_{n-1}, a).
 \end{aligned}$$

Since $K_{np}^n(x_1, x_2, \dots, x_{n-1}, a; x_p) \in GL(E)$ in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n-1}$ by the condition of the theorem and (23), (25) is an integral equation of Volterra's type of the second kind. Hence (25) has an E -valued analytic solution g_n in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n-1}$ by Lemma 2.5. The analytic mapping g_n satisfies (24) by (19).

THEOREM 2.8. Let m be a non-negative integer, Ω_j be a simply connected domain containing the origin in the complex plane for $j=1, 2, \dots, n+1$, ε be a positive number such that $\{y \in \mathbb{C}; |y| \leq \varepsilon\} \subset \Omega_{n+1}$, and a be a complex number with $0 < |a| \leq \varepsilon$. Let A_{jk} and A_j be analytic mappings of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n+1}$ into $L(E)$ and E , respectively, for $j, k=1, 2, \dots, n+1$, and g_j be an analytic mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n \times (\Omega_{n+1} - \{0\})$ into E for $j=1, 2, \dots, n$. Assume that

$$\exp\left(\int_0^{2\pi i} \frac{A_{n+1n+1}(x, ae^t)}{a^m e^{tm}} dt\right) - I \in GL(E)$$

for all $x \in \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$. Then the problem

$$(26) \quad \begin{cases} \frac{\partial u_j(x, x_{n+1})}{\partial x_j} = \sum_{k=1}^{n+1} A_{jk}(x, x_{n+1})u_k + A_{jk}(x, x_{n+1}) & (j=1, 2, \dots, n) \\ (x_{n+1})^{m+1} \frac{\partial u_{n+1}}{\partial x_{n+1}} = \sum_{k=1}^{n+1} A_{n+1k}(x, x_{n+1})u_k + A_{n+1}(x, x_{n+1}) \\ u_j(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{n+1}) \\ = g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}) & (j=1, 2, \dots, n) \end{cases}$$

where $x = (x_1, x_2, \dots, x_n) \in \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ and $x_{n+1} \in \Omega_{n+1} - \{0\}$, has a unique single E^{n+1} -valued analytic solution $u(x, x_{n+1}) = (u_1(x, x_{n+1}), u_2(x, x_{n+1}), \dots, u_{n+1}(x, x_{n+1}))$ in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n \times (\Omega_{n+1} - \{0\})$.

PROOF. For a fixed complex number a with $0 < |a| < \varepsilon$, we put

$$\begin{cases} x_{n+1} = ae^z \\ U_j(x, z) = u_j(x, ae^z) & (j=1, 2, \dots, n+1). \end{cases}$$

The problem (26) is equivalent to the problem

$$(27) \quad \begin{cases} \frac{\partial U_j}{\partial x_j}(x, z) = \sum_{k=1}^{n+1} A_{jk}(x, ae^z)U_k(x, z) + A_j(x, ae^z) & (j=1, 2, \dots, n) \\ \frac{\partial U_{n+1}}{\partial z}(x, z) = \sum_{k=1}^{n+1} \frac{A_{n+1k}(x, ae^z)}{a^m e^{mz}} U_k(x, z) + \frac{A_{n+1}(x, ae^z)}{a^m e^{mz}} \\ U_j(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n, z) \\ = g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, ae^z) & (j=1, 2, \dots, n) \end{cases}$$

in $\{(x, z) \in \mathbb{C}^{n+1}; x \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n, ae^z \in \Omega_{n+1} - \{0\}\}$. $\{z \in \mathbb{C}; ae^z \in \Omega_{n+1} - \{0\}\}$ is invariant under the translation $z \rightarrow z + 2\pi i$, contains the origin in the complex number and is a simply connected domain. Since

$$\exp\left(\int_0^{2\pi i} \frac{A_{n+1n+1}(x, ae^\lambda)}{a^m e^{m\lambda}} d\lambda\right) - I \in GL(E)$$

in $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ by the hypothesis, (27) has an E^{n+1} -valued analytic solution $U(x, z) = (U_1(x, z), U_2(x, z), \dots, U_{n+1}(x, z))$ in $\{(x, z) \in \mathbb{C}^{n+1}; x \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n, ae^z \in \Omega_{n+1} - \{0\}\}$, which has a period $2\pi i$, by Theorem 2.6. Hence (26) has a single-valued analytic solution $u(x, x_{n+1}) = (u_1(x, x_{n+1}), u_2(x, x_{n+1}), \dots, u_{n+1}(x, x_{n+1}))$ in $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n \times (\Omega_{n+1} - \{0\})$.

THEOREM 2.9. Let $\Omega_j = \{x_j \in \mathbb{C}; |x_j| < r\}$ for $j=1, 2, \dots, n+1$, and a is a complex number with $0 < |a| < r$. Let A_{jk} and A_j be analytic mappings of $\bar{\Omega}_1 \times \bar{\Omega}_2 \times \cdots \times \bar{\Omega}_{n+1}$ into $L(E)$ and E respectively for $j, k=1, 2, \dots, n+1$, and g_j be an analytic mapping of $\bar{\Omega}_1 \times \bar{\Omega}_2 \times \cdots \times \bar{\Omega}_{j-1} \times \bar{\Omega}_{j+1} \times \cdots \times \bar{\Omega}_{n+1}$ into E for $j=1, 2, \dots, n+1$. Then there exist a positive integer m and a positive number M satisfying the following property: The (many-valued) analytic solution $u(x, x_{n+1}) = (u_1(x, x_{n+1}), u_2(x, x_{n+1}), \dots, u_n(x, x_{n+1}))$ of the problem

$$(28) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = \sum_{k=1}^{n+1} A_{jk} u_k + A_j & (j=1, 2, \dots, n) \\ x_{n+1} \frac{\partial u_{n+1}}{\partial x_{n+1}} = \sum_{k=1}^{n+1} A_{n+1k} u_k + A_{n+1} \\ u_j(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{n+1}) \\ \quad = g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}) & (j=1, 2, \dots, n) \\ u_{n+1}(x, a) = g_{n+1}(x) & (\text{all } x \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n) \end{cases}$$

in $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n \times (\Omega_{n+1} - \{0\})$ satisfies

$$(29) \quad \|u_j(x, x_{n+1})\| \leq \frac{M \exp(m |\arg x_{n+1}|)}{|x_{n+1}|^m} \quad (j=1, 2, \dots, n+1)$$

in $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n \times (\Omega_{n+1} - \{0\})$.

PROOF. We put

$$\begin{cases} x_{n+1} = ae^z \\ U_j(x, z) = u_j(x, ae^z) \quad (x \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n) \end{cases} \quad (j=1, 2, \dots, n).$$

The problem (28) is equivalent to the problem

$$(30) \quad \left\{ \begin{array}{l} \frac{\partial U_j(x, z)}{\partial x_j} = \sum_{k=1}^{n+1} A_{jk}(x, ae^z) U_k(x, z) + A_j(x, ae^z) \quad (j=1, 2, \dots, n) \\ \frac{\partial U_{n+1}(x, z)}{\partial z} = \sum_{k=1}^{n+1} A_{n+1k}(x, ae^z) U_k(x, z) + A_{n+1}(x, ae^z) \\ U_j(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n, z) \\ \quad = g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, ae^z) \quad (j=1, 2, \dots, n) \\ U_{n+1}(x, 0) = g_{n+1}(x). \end{array} \right.$$

By Theorem 2.2, (30) has an E^{n+1} -valued analytic solution $U(x, z) = (U_1(x, z), U_2(x, z), \dots, U_{n+1}(x, z))$ in $\{(x, z) \in \mathbb{C}^{n+1}; x \in \Omega_1 \times \Omega_2 \times \dots \times \Omega_n, ae^z \in \Omega_{n+1} - \{0\}\}$. The solution $U(x, z)$ is obtained by the method of successive approximations:

$$\left\{ \begin{array}{l} U_j^{(v+1)}(x, z) = \int_0^{x_j} \left\{ \sum_{k=1}^{n+1} A_{jk}(x_1, x_2, \dots, x_{j-1}, s_j, x_{j+1}, \dots, x_n, ae^z) \right. \\ \quad U_k^{(v)}(x_1, x_2, \dots, x_{j-1}, s_j, x_{j+1}, \dots, x_n, ae^z) \\ \quad \left. + A_j(x_1, x_2, \dots, x_{j-1}, s_j, x_{j+1}, \dots, x_n, ae^z) \right\} ds_j \\ \quad + g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, ae^z) \quad (j=1, 2, \dots, n) \\ U_{n+1}^{(v+1)}(x, z) = \int_0^z \left\{ \sum_{k=1}^{n+1} A_{n+1k}(x, ae^\lambda) U_k^{(v)}(x, \lambda) + A_{n+1}(x, e^\lambda) \right\} d\lambda \\ \quad + g_{n+1}(x) \quad (v=1, 2, \dots) \\ U_j^{(0)}(x, z) = g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n, e^z) \quad (j=1, 2, \dots, n) \\ U_{n+1}^{(0)}(x, z) = g_{n+1}(x). \end{array} \right.$$

We have

$$\begin{aligned} & U_j^{(v+1)}(x, z) - U_j^{(v)}(x, z) \\ &= \int_0^{x_j} \left\{ \sum_{k=1}^{n+1} A_{jk}(x_1, x_2, \dots, x_{j-1}, s_j, x_{j+1}, \dots, x_n, ae^z) (U_j^{(v)}(x_1, x_2, \dots, x_{j-1}, \right. \\ & \quad \left. s_j, x_{j+1}, \dots, x_n, z) - U_j^{(v-1)}(x_1, x_2, \dots, x_{j-1}, s_j, x_{j+1}, \dots, x_n, z)) \right\} ds_j \end{aligned}$$

$$U_{n+1}^{(v+1)}(x, z) - U_{n+1}^{(v)}(x, z) = \int_0^z \left\{ \sum_{k=1}^{n+1} A_{n+1k}(x, ae^\lambda) (U_k^{(v)}(x, \lambda) - U_k^{(v-1)}(x, \lambda)) \right\} d\lambda$$

for $v=1, 2, \dots$. For positive numbers $\xi_1, \xi_2, \dots, \xi_n$ with $0 < \xi_j < r$ for $j=1, 2, \dots, n+1$, we put

$$W^{(v)}(\xi_1, \xi_2, \dots, \xi_{n+1}) = \max_{\substack{|x_j| \leq \xi_j \\ |z| \leq \xi_{n+1} \\ \operatorname{Re} z \leq 0}} \max_{1 \leq j \leq n+1} (\|U_j^{(v+1)}(x, z) - U_j^{(v)}(x, z)\|).$$

There exists a positive number M such that

$$W^{(v)}(\xi) \leq M \left\{ \sum_{k=1}^{n+1} \int_0^{\xi_j} W^{(v-1)}(\xi_1, \xi_2, \dots, \xi_{j-1}, \lambda_j, \xi_{j+1}, \dots, \xi_{n+1}) d\lambda_j \right\}$$

for $v=1, 2, \dots$. There exists a non-negative number $W^{(0)}$ such that $W^{(0)}(\xi) \leq W^{(0)}$ in $0 \leq \xi_j < r, j=1, 2, \dots, n+1$. Hence we have

$$W^{(v)}(\xi) \leq \frac{W^{(0)}((n+1)M(\xi_1 + \xi_2 + \dots + \xi_{n+1}))^v}{v!}$$

in $0 \leq \xi_j < r$ for $j=1, 2, \dots, n+1$. Since we have

$$U(x, z) \leq W^{(0)} \exp \{(n+1)M(|x_1| + |x_2| + \dots + |x_n| + |z|)\}$$

in $\{(x, z) \in \mathbb{C}^{n+1}; |x_j| < r, j=1, 2, \dots, n, \operatorname{Re} z \leq 0\}$ and $U(x, z) = u(x, ae^z)$, we obtain (29) from the equation $z = \log |x_{n+1}| - \log |a| + i(\arg x_{n+1} - \arg a)$.

THEOREM 2.10. *Let Ω_j be a simply connected domain containing the origin in the complex plane for $j=1, 2, \dots, n+1$. Let A be an analytic mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n+1}$ into $L(E)$. If*

$$\exp(2\pi i A(x, 0)) - I \in \operatorname{GL}(E)$$

for all $x \in \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$, then for every relatively compact subdomain Ω' , containing the origin in $\mathbb{C}^n$, in $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$, there exists a number $\varepsilon = \varepsilon(\Omega') > 0$ with $\{|x_{n+1}| \leq \varepsilon\} \subset \Omega_{n+1}$ such that

$$\exp \left(\int_0^{2\pi i} A(x, ae^t) dt \right) - I \in \operatorname{GL}(E)$$

for all $x \in \Omega'$ and $|a| < \varepsilon$.

PROOF. Let a_0 be a positive number such that $\{y \in \mathbf{C}; |y| \leq a_0\} \subset \Omega_{n+1}$. The E -valued solution of the problem

$$\begin{cases} \frac{dU(x, z)}{dz} = A(x, ae^z)U(x, z) \\ U(x, 0) = I \end{cases}$$

in $\{(x, z, a) \in \mathbf{C}^{n+2}; x \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n, \operatorname{Re} z < 1, |a| < a_0\}$ is denoted by

$$\exp\left(\int_0^z A(x, ae^t)dt\right),$$

which is analytic with respect to the variables x, z and a . Thus

$$B(x, a) = \exp\left(\int_0^{2\pi i} A(x, ae^t)dt\right) - I$$

is analytic with respect to the variables x and a . We have

$$B(x, 0) = \exp(2\pi i A(x, 0)) - I \in \operatorname{GL}(E)$$

for all $x \in \Omega$. Since $\bar{\Omega}' \times \{0\}$ is a compact subset of $\Omega \times \Omega_{n+1}$, $\{B(x, 0); x \in \bar{\Omega}'\}$ is compact in $\operatorname{GL}(E)$. Moreover since $\operatorname{GL}(E)$ is an open set in $L(E)$, $B^{-1}(\operatorname{GL}(E))$ is a neighborhood of the origin in $\Omega \times \Omega_{n+1}$, which contains $\bar{\Omega}' \times \{0\}$. Hence there exists a positive number ε such that $B(x, a) \in \operatorname{GL}(E)$ for all $x \in \bar{\Omega}'$ and $|a| < \varepsilon$.

THEOREM 2.11. Let Ω_j be a simply connected domain containing the origin in the complex plane for $j=1, 2, \dots, n+1$. Let A_{jk} and A_j be analytic mappings of $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_{n+1}$ into $L(E)$ and E , respectively, for $j, k=1, 2, \dots, n+1$, and g_j be an analytic mapping of $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_{j-1} \times \Omega_{j+1} \times \cdots \times \Omega_{n+1}$ into E for $j=1, 2, \dots, n$. Assume that

$$\exp(2\pi i A_{n+1, n+1}(x, 0)) - I \in \operatorname{GL}(E)$$

for all $x \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$. Then the problem

$$(31) \quad \begin{cases} \frac{\partial u_j}{\partial x_j} = \sum_{k=1}^{n+1} A_{jk} u_k + A_j & (j=1, 2, \dots, n) \\ x_{n+1} \frac{\partial u_{n+1}}{\partial x_{n+1}} = \sum_{k=1}^{n+1} A_{n+1, k} u_k + A_{n+1} \\ u_j(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{n+1}) \\ \quad = g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}) & (j=1, 2, \dots, n) \end{cases}$$

has a unique single E^{n+1} -valued analytic solution $u(x_1, x_2, \dots, x_{n+1}) = (u_1(x_1, x_2, \dots, x_{n+1}), u_2(x_1, x_2, \dots, x_{n+1}), \dots, u_{n+1}(x_1, x_2, \dots, x_{n+1}))$ in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n \times (\Omega_{n+1} - \{0\})$.

PROOF. Let Ω'_j be a relatively compact subdomain of Ω_j containing the origin for $j=1, 2, \dots, n$. We put $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ and $\Omega' = \Omega'_1 \times \Omega'_2 \times \dots \times \Omega'_n$. By Theorem 2.10, there exists a positive number $\varepsilon = \varepsilon(\Omega')$ with $\{|x_{n+1}| \leq \varepsilon\} \subset \Omega_{n+1}$ such that

$$\exp\left(\int_0^{2\pi i} A_{n+1, n+1}(x, ae^t) dt\right) - I \in GL(E)$$

for all $x \in \Omega'$ and $|a| < \varepsilon$. Now we take a fixed complex number a with $0 < |a| < \varepsilon$. By Theorem 2.8, (31) has a unique E^{n+1} -valued analytic solution $u(x, x_{n+1}; \Omega') = (u_1(x, x_{n+1}; \Omega'), u_2(x, x_{n+1}; \Omega'), \dots, u_{n+1}(x, x_{n+1}; \Omega'))$ in $\Omega' \times (\Omega_{n+1} - \{0\})$. Since Ω'_j is an arbitrary relatively compact subdomain of Ω_j for $j=1, 2, \dots, n$, by the uniqueness of the solution and the Identity theorem, (31) has a unique single E^{n+1} -valued analytic solution $u(x, x_{n+1}) = (u_1(x, x_{n+1}), u_2(x, x_{n+1}), \dots, u_{n+1}(x, x_{n+1}))$ in $\Omega \times (\Omega_{n+1} - \{0\})$.

THEOREM 2.12. Under the hypotheses of Theorem 2.11, assume that

$$A_{n+1, n+1}(x, 0) + mI \in GL(E)$$

for all $x \in \Omega$ and non-negative integers m . Then the analytic solution of (31) is analytic in $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{n+1}$.

PROOF. Let r be a positive number such that $\{x_j \in \mathbf{C}; |x_j| \leq r\} \subset \Omega_j$ for $j=1, 2, \dots, n+1$. We put $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$. Since $u_j(x, x_{n+1})$ is analytic in $D = \Omega \times \{x_{n+1} \in \mathbf{C}; 0 < |x_{n+1}| \leq r\}$, by Lemma 1.5, it has the Laurent-Hartogs series expansion

$$u_j(x, x_{n+1}) = \sum_{v=-\infty}^{\infty} u_{jv}(x) (x_{n+1})^v \quad (j=1, 2, \dots, n+1)$$

for all $(x, x_{n+1}) \in D$ where $u_{jv}(x)$ is an analytic mapping of Ω into E for $v=0, \pm 1, \pm 2, \dots$. A_{jk} , A_j and g_j yield to the Hartogs series expansions

$$A_{jk}(x, x_{n+1}) = \sum_{v=0}^{\infty} A_{jkv}(x) (x_{n+1})^v$$

$$A_j(x, x_{n+1}) = \sum_{v=0}^{\infty} A_{j,v}(x) (x_{n+1})^v$$

$$g_j(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}) \\ = \sum_{v=0}^{\infty} g_{jv}(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n)(x_{n+1})^v$$

for all $(x, x_{n+1}) \in D$ and $j, k=1, 2, \dots, n+1$, where A_{jkv} is an analytic mapping of Ω into $L(E)$, $A_{j,v}$ is an analytic mapping of Ω into E and g_{jv} is an analytic mapping of $\Omega_1 \times \Omega_2 \times \dots \times \Omega_{j-1} \times \Omega_{j+1} \times \dots \times \Omega_n$ into E for $j, k=1, 2, \dots, n+1$ and $v=0, 1, 2, \dots$. Here we have suppose that $g_{n+1} \equiv 0$. By Theorem 2.9, there is a positive integer m such that

$$u_{jv}(x) = 0$$

in $\{x \in \mathbb{C}^n; |x_j| < r, j=1, 2, \dots, n\}$ for $v < -m$. By the Identity theorem, $u_{jv}(x) = 0$ in Ω for $v < -\infty$. Substituting the Hartogs series expansions in (31), we have

$$(32) \quad \begin{cases} \frac{\partial u_{jv}(x)}{\partial x_j} = \sum_{k=1}^{n+1} \sum_{\mu=0}^{v+m} A_{jk\mu}(x) u_{k, v-\mu}(x) + A_{j,v}(x) & (j=1, 2, \dots, n) \\ vu_{n+1, v}(x) = \sum_{k=1}^{n+1} \sum_{\mu=0}^{v+m} A_{n+1, k\mu}(x) u_{n+1, v-\mu}(x) + A_{n+1, v}(x) \\ u_{jv}(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) \\ = g_{jv}(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n) & (j=1, 2, \dots, n), \end{cases}$$

where $A_{j,v} = g_{jv} = 0$ in Ω for $v < 0$.

Now we suppose that $u_{jv}(x) \not\equiv 0$ in Ω for some j and $v < 0$. Let $-m$ be the smallest integer among those v . We may assume that (32) holds for this $m > 0$. Substituting $v = -m$ in (32), we have

$$(33) \quad \begin{cases} \frac{\partial u_{j, -m}(x)}{\partial x_j} = \sum_{k=1}^{n+1} A_{jk0}(x) u_{k, -m}(x) & (j=1, 2, \dots, n) \\ -mu_{n+1, -m}(x) = \sum_{k=1}^{n+1} A_{n+1, k0}(x) u_{k, -m}(x) \\ u_{j, -m}(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) = 0 & (j=1, 2, \dots, n) \end{cases}$$

Since $A_{n+1, n+1, 0}(x) = A_{n+1, n+1}(x, 0)$ and $A_{n+1, n+1}(x, 0) + mI \in GL(E)$ in Ω , we have

$$(34) \quad u_{n+1, -m}(x) = - \sum_{k=1}^n (A_{n+1, n+1, 0}(x) + mI)^{-1} A_{n+1, k0}(x) u_{k, -m}(x)$$

in Ω . Substituting (34) in (33), we have

$$\begin{cases} \frac{\partial u_{j,-m}(x)}{\partial x_j} = \sum_{k=1}^n (A_{jk0}(x) - (A_{n+1n+10}(x) + mI)^{-1} A_{n+1k0}(x)) u_{k,-m}(x) \\ u_{j,-m}(x_1, x_2, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) = 0 \end{cases}$$

in Ω for $j=1, 2, \dots, n$. By the uniqueness of the solution of the Goursat's problem, we have

$$u_{j,-m}(x) = 0$$

in Ω for $j=1, 2, \dots, n$. Hence, $u_{n+1,-m}(x) = 0$ in Ω by (34), and thus it contradicts $u_{n+1,-m} \not\equiv 0$. Hence $m \leq 0$. Thus $u_j(x, x_n)$ is an analytic mapping of $\Omega \times \Omega_{n+1}$ into E for $j=1, 2, \dots, n+1$.

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