ON A SIMULATION AND AN OPTIMAL INPUT OF AUTOMATA IN A MONOIDAL CATEGORY

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What is a simulation of automata? A formalized concept of a simulation is introduced as a "weak simulation", based on the previous works on a category of automata as H. Ehrig and J. A. Goguen have developed in [1] and [2].

Some author use the term "simulation" as an automata morphism. But, in this report, "simulations" mean weak simulations which preserve transitions. These are slightly different from automata morphisms.

In § 3, we shall prove several categorical properties which are similar to ones stated in [2].

In § 4, limiting the monoidal category under consideration to the category $(\mathbf{Set}, \times)$, we study the relationship of encoders and simulations, and finally we prove an existence theorem for "minimal" input sets.

§ 1. A category of automata with fixed input and output objects

In this section, a little survey of Ehrig's theory is presented. When we want to discuss on simulations, input and output objects have to be variant. But this is based on what are stated in the following.

Let $(\mathbf{K}, \otimes, U)$ be a monoidal category. We define an automaton in $(\mathbf{K}, \otimes)$ as a diagram of the form

$$A: O \xleftarrow{l} S \otimes I \xrightarrow{d} S, \text{ denoted by } (S, d, l).$$

An automata morphism is a $\mathbf{k}$ -morphism $f: S \rightarrow S'$ which makes (1.1) commutative.

$$(1.1) \quad \begin{array}{ccccc} A : S & \xleftarrow{d} & S \otimes I & \xrightarrow{l} & O \\ \downarrow f & & \downarrow f \otimes I & & \parallel \\ A' : S' & \xleftarrow{d'} & S' \otimes I & \xrightarrow{l'} & O \end{array}$$

These form a category of automata in the monoidal category $(\mathbf{K}, \otimes)$ with fixed input and output objects I and O .

We assume the followings in order to study “deterministic” automata.

A1: $\mathbf{K}$ has countable coproducts.

Assumptions. A2: $(\mathbf{K}, \otimes)$ is closed (and symmetric).

A3: $\mathbf{K}$ has a canonical $\mathcal{E}$ - $\mathcal{M}$ -factorization system,

where $\mathcal{E}[\mathcal{M}]$ is the class of epimorphisms [monomorphisms] which includes all isomorphisms and is closed under compositions.

First, we try to make our general automata running; that is, a machine morphism $M(A)$ has to be defined for each automaton A .

For $n > 0$, $l_n : S \otimes I^n \rightarrow O$ are defined as

$$l_n : (S \otimes I^n \xrightarrow{d \otimes I^{n-1}} S \otimes I^{n-1} \rightarrow \dots \rightarrow S \otimes I^2 \xrightarrow{d \otimes I} S \otimes I \xrightarrow{l} O).$$

By the assumption A2, $S \otimes I^n \xrightarrow{S \otimes u_n} S \otimes I^+$ are coproduct injections, where u_n are injections of the coproduct $I^+ = \coprod_{n=0}^{\infty} I^n$ with $I^1 = I$ and $I^n = I \otimes I^{n-1}$. Hence there exists a unique morphism $l^+ : S \otimes I^+ \rightarrow O$ with $l^+(S \otimes u_n) = l_n$ for each $n > 0$. A machine morphism $M(A)$ is defined as a unique morphism with

$$(1.2) \quad \begin{array}{ccc} 0 & \xleftarrow{ev} & \langle I^+, 0 \rangle \otimes I^+ \\ \uparrow i^+ & \nearrow M(A) \otimes I^+ & \\ S \otimes I^+ & & \end{array}$$

$l^+ = ev(M(A) \otimes I^+)$ as in (1.2), where ev is the counit of the adjunction $\mathbf{K} \xrightleftharpoons[-\otimes I^+]{\langle I^+, - \rangle} \mathbf{K}$.

Following theorems are stated in [1].

THEOREM 1.1. (Characterization of machine morphisms)

A $\mathbf{K}$ -morphism $g : S \rightarrow \langle I^+, O \rangle$ is a machine morphism of an automaton $A = (S, d, l)$, if and only if, $l = ev_1 \circ (g \otimes I)$ and $g : (S, d) \rightarrow (\langle I^+, O \rangle, L)$ is a $\mathbf{K}$ -Medve-morphism, where L and i are defined as follows.

$$(1.3) \quad \begin{array}{ccc} I^{n+1} = I \otimes I^n & \xrightarrow{I \otimes u_n} & I \otimes I^+ \\ & \searrow u_{n+1} & \nearrow \exists: i \\ & & I^+ \end{array}$$

$$(1.4) \quad \begin{array}{ccc} 0 & \xleftarrow{ev} & \langle I^+, 0 \rangle \otimes I^+ \\ \uparrow ev & & \uparrow \\ \langle I^+, 0 \rangle \otimes I^+ & & \uparrow \\ \langle I^+, 0 \rangle \otimes i & & \uparrow \\ \langle I^+, 0 \rangle \otimes I \otimes I^+ & \nearrow \exists: L \otimes I^+ & \end{array}$$

((1.3) uses the coproduct property, and (1.4) is the adjunction)

Remark that we identify an element $(x_1(x_2 \cdots (x_{n+1}, x_n) \cdots))$ of I^n with a string $x_1 x_2 \cdots x_n$ over I in the case $\mathbf{K} = \mathbf{Set}$.

Next, we define a behavior of an automaton. A behavior of an automaton A is a left factor $m(A)$ of a canonical $\mathfrak{E}$ - $\mathfrak{M}$ -factorization of a machine morphism $M(A)$, as in figure (1.5). In the case of $\mathbf{Set}$, $E(A) = \{M(A)(s) : s \in S\}$, where $(M(A)(s) : I^+ \rightarrow O) = l^+(s, -)$

$$(1.5) \quad \begin{array}{ccc} S & \xrightarrow{M(A)} & \langle I^+, 0 \rangle \\ & \searrow e(A) & \nearrow m(A) \\ & & E(A) \end{array}$$

We sometimes identify the behavior $m(A)$ with its domain $E(A)$.

THEOREM 1.2 (Characterization of behaviors)

A $\mathbf{K}$ -morphism $m : B \rightarrow \langle I^+, O \rangle$ is a behavior of an automaton, if and only if, (1) m is a canonical representative of $\mathfrak{M}$ and (2) m is closed under left shift L : that is, m is a machine morphism.

THEOREM 1.3. For each behavior $B \xrightarrow{m} \langle I^+, O \rangle$, an automaton $\bar{A} = (B, L', ev_1(m \otimes I))$ is minimal; that is, if $E(A) \subseteq E(\bar{A})$, then there exists a unique automata morphism $f : A \rightarrow \bar{A}$, which is a reduction (i.e. $f \in \mathfrak{E}$) in the case $E(A) = E(\bar{A})$. In addition, $\bar{A}$ is observable: that is, the machine morphism $M(A) \in \mathfrak{M}$. (In fact $M(\bar{A}) = m \in \mathfrak{M}$)

§2. Categorical remarks

We use the term “closed” as follows:

A monoidal category is called closed if there exists a natural isomorphism $\phi: \text{Mor}_{\mathbf{K}}(-, < -, - >) \rightarrow \text{Mor}_{\mathbf{K}}(- \otimes -, -)$, where $\text{Mor}_{\mathbf{K}}(-, < -, - >)$ and $\text{Mor}_{\mathbf{K}}(- \otimes -, -)$ are set-valued functors: $\mathbf{K}^{OP} \times \mathbf{K}^{OP} \times \mathbf{K} \rightarrow \text{Set}$. Especially, $< -, - >: \mathbf{K}^{OP} \times \mathbf{K} \rightarrow \mathbf{K}$.

FACT 2.1. For a $\mathbf{K}$ -morphism $f: K' \rightarrow K$, $< f, O >: < K, O > \rightarrow < K', O' >$ is the adjoint morphism of $ev(< K, O > \otimes f)$, where ev is a counit of the adjunction $\mathbf{K} \xrightleftharpoons[\text{K, } ->]{- \otimes \mathbf{K}} \mathbf{K}$.

In addition to the assumptions A1, A2 and A3, A4 and A5 must be assumed in order to study simulations and weak simulations.

A4: $< f, g > \in \mathfrak{M}$, if $f \in \mathfrak{E}$ and $g \in \mathfrak{M}$.

A5: $f^+ \in \mathfrak{E}$, if $f \in \mathfrak{E}$,

where f^+ is defined as follows for each $\mathbf{K}$ -morphism f (See (2.1)).

$$(2.1) \quad \begin{array}{ccc} A^n & \xrightarrow{f^n} & B^n \\ u_n^A \downarrow & \cdots & \downarrow u_n^B \\ A^+ & \xrightarrow{\exists ! f^+} & B^+ \end{array}$$

Using the coproduct property, f^+ is defined as the unique morphism $A^+ \rightarrow B^+$ with $u_n^B f^n = f^+ u_n^A$ for each $n \geq 1$. Remark that correspondences $A \mapsto A^+$, $f \mapsto f^+$ can be extended to a functor $()^+: \mathbf{K} \rightarrow \mathbf{K}$. It is easy to verify that $(\text{Set}, \times)$, $(\text{Mod}_R, \otimes)$ and $(\text{Top}, \otimes)$ satisfy A1 ~ A5. Necessary informations are presented as follows.

	Set	Mod _R	Top
objects	sets	<i>R</i> -modules	topological spaces
morphisms	functions	<i>R</i> -linear functions	continuous functions
$\otimes$	cartesian product	tensor product	bi-product topology
$\mathfrak{E}$	surjections	surjective <i>R</i> -linear functions	surjective continuous functions
$\mathfrak{M}$	injections	injective linear functions	injective continuous functions with relative topology

§3. Simulations and weak simulations

In general, a “simulation” consists of encoder, decoder and state assignment as in figure (3.1).

We may associate a set of diagrams (3.2):

(3.2)

$$\begin{array}{ccc}
 I'^+ & \xrightarrow{M(A')(f(s))} & O' \\
 \uparrow \forall_{s \in S}: \phi_I^+ & & \downarrow \phi_O \\
 I^+ & \xrightarrow{M(A)(s)} & O
 \end{array}$$

Moreover, (3.2) is equivalent to (3.3) by using homofunctor $\text{Mor}_{\text{set}}(\phi^+, \phi_O): \text{Mor}_{\text{set}}(I'^+, O') \rightarrow \text{Mor}_{\text{set}}(I^+, O)$

We shall call (3.3) a weak simulation and study transition preserving weak simulations (called simulations).

(3.3)

$$\begin{array}{ccc}
 S' & \xrightarrow{M(A')} & \langle I'^+, O' \rangle \\
 \uparrow f & & \downarrow \langle \phi_I^+, \phi_O \rangle = \text{Mor}(\phi_I^+, \phi_O) \\
 S & \xrightarrow{M(A)} & \langle I^+, O \rangle
 \end{array}$$

NOTATIONS. For $\mathbf{K}$ -morphisms $\phi_I: I \rightarrow I'$ and $\phi_O: O' \rightarrow O$, a $\mathbf{K}$ -morphism $(\langle \phi_I^+, \phi_O \rangle: \langle I'^+, O' \rangle \rightarrow \langle I^+, O \rangle)$ is denoted by t_ϕ for short, where $\phi = (\phi_I, \phi_O)$ is a $\mathbf{K}^{OP} \times \mathbf{K}$ -morphism.

DEFINITION 3.1. We say that an automaton $A = (I, O, S, d, l)$ simulates an automaton $A' = (I', O', S', d', l')$ if there exists a $\mathbf{K}^{OP} \times \mathbf{K}$ -morphism $\phi = (\phi_I, \phi_O): (I, O) \rightarrow (I', O')$ and a $\mathbf{K}$ -morphism $f: S' \rightarrow S$ such that (3.4) is commutative.

$$(3.4) \quad \begin{array}{ccccc} S & \xleftarrow{d} & S \otimes I & \xrightarrow{l} & O \\ \uparrow f & & \uparrow f \otimes \phi_I & & \downarrow \phi_O \\ S' & \xleftarrow{d'} & S' \otimes I' & \xrightarrow{l'} & O' \end{array}$$

In this case, ϕ_I and ϕ_O are called an encoder and a decoder respectively, and we denote this by $\phi - A' f A$ or $(\phi, f): A' \rightarrow A$.

A composition of simulations $\phi - A' f A$ and $\phi' - A'' f' A'$ is defined as $\phi' \phi - A''(ff')A$, where $\phi' \phi$ is a $\mathbf{K}^{OP} \times \mathbf{K}$ -composition; that is, $\phi' \phi = (\phi'_I \phi_I, \phi'_O \phi_O)$. Remark that these automata and simulations form a category $\mathbf{K-S}$.

For each simulation $\phi - A' f A$, it is easy to prove

$$(3.5): \quad \phi_O l_n(f \otimes \phi_I^n) = l'_n \quad (n > 1), \quad \text{and}$$

$$(3.6): \quad \phi_O l^+(f \otimes \phi_I^+) = l'^+.$$

$$(3.7) \quad \begin{array}{ccccc} S \otimes I^+ & \xrightarrow{l^+} & 0 \\ \uparrow f \otimes \phi_I^+ & \searrow M(A) \otimes I^+ & \downarrow \langle I^+, 0 \rangle \otimes I^+ & \nearrow \epsilon_{I^+} & \downarrow \phi_O \\ & & \langle I^+, 0 \rangle \otimes I^+ & & 0 \\ & & \downarrow \langle I^+, \phi_O \rangle \otimes I^+ & & \\ & & \langle I^+, 0' \rangle \otimes I^+ & \searrow \epsilon_{O'} & \\ S' \otimes I^+ & \xrightarrow{l'^+} & 0' \\ \uparrow M(A') \otimes I^+ & \searrow & \downarrow \langle I^+, 0' \rangle \otimes I^+ & \nearrow \epsilon_{O'} & \\ & & \langle I^+, 0' \rangle \otimes I^+ & & \end{array}$$

(*)

Using (3.7), we have the commutativity of (*). Hence, we have

$$\begin{aligned} ev'_{O'}(M(A') \otimes I^+) &= ev_{O'}((\langle I^+, \phi_O \rangle M(A)f) \otimes \phi_I^+) \\ &= ev_{O'}[(\langle I^+, 0' \rangle \otimes \phi_I^+) \cdot (\langle I^+, \phi_O \rangle M(A)f) \otimes I^+] \\ &= ev'_{O'}[(\langle \phi_I^+, 0' \rangle \otimes I^+) \circ ((\langle I^+, \phi_O \rangle M(A)f) \otimes I^+)] \\ &= ev'_{O'}((\langle \phi_I^+, \phi_O \rangle M(A)f) \otimes I^+). \end{aligned}$$

Hence we have

$$(3.7): M(A') = t_\phi M(A) f.$$

DEFINITION 3.2. We say that an automaton A weakly simulates an automaton A' when there exists a $\mathbf{K}^{OP} \times \mathbf{K}$ -morphism $\phi = (\phi_I, \phi_O): (I, O) \rightarrow (I', O')$ and a $\mathbf{K}$ -morphism $f: S' \rightarrow S$ such that $M(A') = t_\phi M(A) f$ holds.

If $\phi \preceq A' f A$ holds, then

$$\begin{aligned} l'^+ &= ev'_{O'}(M(A') \otimes I'^+) = ev'_{O'}[(\langle \phi_I^+, O' \rangle \langle I^+, \phi_O \rangle M(A) f) \otimes I'^+] \\ &= ev_{O'}[\langle I^+, O' \rangle \otimes \phi_I^+][\langle I^+, \phi_O \rangle M(A) f \otimes I'^+] \\ &= ev_{O'}[\langle I^+, \phi_O \rangle M(A) f \otimes \phi_I^+] \\ &= \phi_O l^+(f \otimes \phi_I^+), \text{ using (3.7) and Fact 2.1.} \end{aligned}$$

Vice versa, if $l'^+ = \phi_O l^+(f \otimes \phi_I^+)$, then $\phi \preceq A' f A$. Thus we have

PROPOSITION 3.1. $\phi \preceq A' f A$, if and only if, $l'^+ = \phi_O l^+(f \otimes \phi_I^+)$.
Moreover,

$$\begin{aligned} l' &= l'^+(S' \otimes u'_1) = \phi_O l^+(f \otimes \phi_I^+)(S' \otimes u'_1) \\ &= \phi_O l^+(S \otimes u_1)(f \otimes \phi_I) \\ &= \phi_O l(f \otimes \phi_I) \end{aligned}$$

holds, and hence we have:

PROPOSITION 3.2. $\phi - A' f A$, if and only if, $\phi \preceq A' f A$ and (f, ϕ_I) is a $\mathbf{K}$ -Medvev-morphism.

In the following, we restrict $\mathbf{K}^{OP} \times \mathbf{K}$ -morphisms $\phi = (\phi_I, \phi_O)$ to $\phi_I \in \mathfrak{E}$ and $\phi_O \in \mathfrak{M}$. Now we study the categorical properties of weak simulations. If $\phi \preceq A' f A$, then we have a unique diagonal morphism ψ with $\psi e(A') = e(A) f$ and $t_\phi m(A) \psi = m(A')$ by the assumption A4 and the “Diagonal fill-in lemma” of the $\mathfrak{E} - \mathfrak{M}$ factorization. We denote $t_\phi m(A) \psi = m(A')$ by $E(A') \leq_\phi E(A)$.

$$(3.9) \quad \begin{array}{ccccc} S & & \xrightarrow{M(A)} & & \langle I^+, 0 \rangle \\ & \searrow e(A) & & \nearrow m(A) & \\ & & E(A) & & \\ & \uparrow \exists! \psi & & & \\ & & E(A') & & \\ & \nwarrow e(A') & & \searrow m(A') & \\ S' & & \xrightarrow{M(A')} & & \langle I'^+, O' \rangle \end{array}$$

FACT 3.3. *If $f \in \mathfrak{E}$ then $E(A) \cong E(A')$.*

Conversely, if A is observable and $E(A') \leq_{\phi} E(A)$ holds, then $e(A)$ is an isomorphism and hence there exists a unique $\mathbf{K}$ -morphism $f: S' \rightarrow S$ with $t_{\phi} M(A)f = M(A')$: that is, $\phi \leq A' f A$ holds for the morphism f .

PROPOSITION 3.4. *If $\phi \leq A' f A$, then $E(A') \leq_{\phi} E(A)$ holds. And $E(A') \leq_{\phi} E(A)$ implies the existence of a unique $f: S' \rightarrow S$ with $\phi \leq A' f A$, if A is observable.*

Let $\mathbf{K-WS}$ be a category of automata and weak simulations, and $\mathbf{B}$ be a (behavior) category whose objects are behaviors for some automata and morphisms are $B \leq_{\phi} B'$. Compositions of (behavior) morphisms are defined by $B'' \leq_{\phi'} B' = (B' \leq_{\phi} B)(B'' \leq_{\phi'} B')$.

$$(3.10) \quad \begin{array}{ccc} B & \xrightarrow{m} & \langle I^+, O \rangle \\ \exists! \psi \uparrow & & \downarrow t_{\phi} \\ B' & \xrightarrow{m'} & \langle I'^+, O' \rangle \end{array}$$

A (behavior) functor $\mathbf{E}^w: \mathbf{K-WS} \rightarrow \mathbf{B}$ is defined as $\mathbf{E}^w(A) = E(A)$ and $\mathbf{E}^w(\phi \leq A' f A) = E(A) \leq_{\phi} E(A')$. It is easy to verify that an observable automaton, which realizes a given behavior m , is cofree over m with respect to $\mathbf{E}^w$; that is, next proposition holds.

PROPOSITION 3.5. *The behavior functor $\mathbf{E}^w$ has a right inverse and a right adjoint functor.*

Now, we prove a property of observable automata and extend proposition 3.5.

LEMMA 3.6. *$\phi \leq A' f A$ if and only if $\phi \leq A' f A$, for each observable automaton A .*

PROOF. According to Proposition 3.2, we have only to prove that (ϕ_I, f) is a $\mathbf{K-Medve}_v$ -morphism if $\phi \leq A' f A$ holds.

$$\begin{aligned} l'^+ &= ev'(M(A') \otimes I'^+) = \phi_O ev(M(A) \otimes I^+)(f \otimes \phi_I^+) \\ &= \phi_O l^+(f \otimes \phi_I^+) \text{ (by Proposition 3.1).} \end{aligned}$$

$$\begin{array}{c}
 \xrightarrow{(M(A) \otimes I^+)(S \otimes i)} \\
 (3.11) \quad S \otimes I \otimes I^+ \xrightarrow{M(A) \otimes I \otimes I^+} \langle I^+, O \rangle \otimes I \otimes I^+ \xrightarrow{\langle I^+, O \rangle \otimes i} \langle I^+, O \rangle \otimes I^+ \\
 \downarrow d \otimes I^+ \quad \text{Theorem 1.1} \quad L \otimes I^+ \quad \text{definition of } L \quad \downarrow ev \\
 S \otimes I^+ \xrightarrow{M(A) \otimes I^+} \langle I^+, O \rangle \otimes I^+ \xrightarrow{ev} O
 \end{array}$$

Using (3.11), we have

$$\begin{aligned}
 ev'(M(A') \otimes I'^+)(d' \otimes I'^+) &= ev'(M(A') \otimes I'^+)(S' \otimes i') \\
 &= \phi_o ev(M(A) \otimes I^+)(f \otimes \phi_I^+)(S' \otimes i') \\
 &= \phi_o ev(M(A) \otimes I^+)(f \otimes (i \circ (\phi_I \otimes \phi_I^+))) \text{ by (3.12)} \\
 &= \phi_o ev(M(A) \otimes I^+)(S \otimes i)(f \otimes \phi_I \otimes \phi_I^+) \\
 &= \phi_o ev(M(A) \otimes I^+)(d \otimes I^+)(f \otimes \phi_I \otimes \phi_I^+), \text{ where}
 \end{aligned}$$

(3.12) is

$$\begin{array}{ccc}
 I' \otimes I'^+ & \xrightarrow{\phi_I \otimes \phi_I^+} & I \otimes I^+ \\
 \swarrow I' \otimes u_n & & \searrow I \otimes u_n \\
 I' \otimes I^n & \xrightarrow{\phi_I \otimes \phi_I^n} & I \otimes I^n \\
 \parallel \text{definition of } I' & & \parallel \text{definition of } I \\
 I'^{n+1} & \xrightarrow{\phi_I^{n+1}} & I^{n+1} \\
 \swarrow u_{n+1} & & \searrow u_{n+1} \\
 I'^+ & \xrightarrow{\phi_I^+} & I^+
 \end{array}$$

, by using the coproduct property.

Hence

$$\begin{aligned}
 ev'(M(A') \otimes I'^+)(d' \otimes I'^+) &= \phi_o ev(M(A) \otimes I^+)(f \otimes \phi_I^+)(d' \otimes I'^+) \\
 &= \phi_o ev(M(A) \otimes I^+)(d \otimes I^+)(f \otimes \phi_I \otimes \phi_I^+)
 \end{aligned}$$

For $\phi_o \in \mathfrak{M}$,

$$ev((M(A)fd') \otimes I^+)(S' \otimes \phi_I^+) = ev(((M(A)d(f \otimes \phi_I)) \otimes I^+)(S' \otimes \phi_I^+))$$

Because of $\phi_I^+ \in \mathfrak{E}$ by A4, we have

$$M(A)fd' = M(A)d(f \otimes \phi_I).$$

And finally $fd' = d(f \otimes \phi_I)$ holds, since A is observable.

Next theorems are direct conclusions of the lemma.

THEOREM 3.7. *A behavior functor $E: \mathbf{K-S} \rightarrow \mathbf{B}$ defined by (3.13) has a right inverse and a right adjoint functor.*

$$(3.13) \quad \begin{array}{ccc} \mathbf{K-WS} & \xrightarrow{E^W} & \mathbf{B} \\ \uparrow u & \searrow \scriptstyle{:=} & \\ \mathbf{K-S} & \xrightarrow{E} & \end{array}$$

THEOREM 3.8. *Consider a weak simulation $\phi \stackrel{w}{\sim} A' f A$, and let $g: A \rightarrow \bar{A}$ be a unique morphism to an observable automaton $\bar{A}$ equivalent to A . We have $\phi = A'(g \circ f)\bar{A}$. (Refer to (3.14))*

$$(3.14) \quad \begin{array}{ccc} A & \xrightarrow{\text{minimization}} & \bar{A} \\ \uparrow \text{weak simulation} & \searrow \scriptstyle{:=} & \\ A' & \xrightarrow{\text{simulation}} & \end{array}$$

PROPOSITION 3.9. *An automaton A is simulated by a type IO automaton, if and only if, there exists a type IO behavior $B \xrightarrow{m} \langle I^+, O \rangle$ and a $\mathbf{K}^{OP} \times \mathbf{K}$ -morphism ϕ such that $E(A) \leq_{\phi} B$.*

§4. Encoders, simulations and optimal inputs

In this section, we discuss the relations between encoders and simulations. It is justified that functions of decoders are not so important unless we discuss output behaviors. Unfortunately, results of this section are only limited to $(\mathbf{Set}, \times)$. In the following, we may omit the assumption in the previous section that decoder ϕ_O belong to the class $\mathfrak{M}$, but encoders ϕ_I are still assumed to belong to $\mathfrak{E}$; that is, encoders are surjective functions.

Consider a weak simulation $\phi \stackrel{w}{\sim} A' f A$, then there exists a function $\psi: E(A') \rightarrow E(A)$ with $t_{\phi} m(A)\psi = m(A')$ by taking a coretraction c of $e(A')$ as in a figure

(4.1):

$$(4.1) \quad \begin{array}{ccccc} S & & M(A) & & \langle I^+, 0 \rangle \\ & \searrow e(A) & & \nearrow m(A) & \\ & E(A) & & & \\ & \uparrow \psi & & & \\ & E(A') & & \nearrow m(A') & \\ & \nwarrow e(A') & & \searrow & \\ S' & & M(A') & & \langle I^+, 0' \rangle \end{array}$$

$\begin{array}{c} \uparrow f \\ \downarrow t_\phi \end{array}$
 $\begin{array}{c} \leftarrow c \\ \rightarrow c \end{array}$
 $\begin{array}{c} \leftarrow e \\ \rightarrow e \end{array}$
 $\begin{array}{c} \leftarrow e \\ \rightarrow e \end{array}$

Let $\bar{A}$ be an observable automaton $(E(A), L', ev_1(m(A) \times I))$. An automaton $\bar{A}'$ is also defined. Using a characterization of machine function, we have $M(\bar{A}) = m(A)$ and $M(\bar{A}') = m(A')$. Hence $t_\phi m(A)\psi = m(A')$ implies $\phi \stackrel{w}{=} \bar{A}'\psi\bar{A}$. Moreover, the following diagram (4.2) commutes by proposition 3.1. Remark that the state object of $\bar{A}$ is $E(A)$ and etc.

$$(4.2) \quad \begin{array}{ccc} E(A) \times I^+ & \xrightarrow{m(A) \times I^+} & \langle I^+, 0 \rangle \times I^+ \xrightarrow{ev} O \\ \psi \times \phi_I \uparrow & & \phi_O \downarrow \\ E(A') \times I'^+ & \xrightarrow{m(A') \times I'^+} & \langle I'^+, 0' \rangle \times I'^+ \xrightarrow{ev'} O' \end{array}$$

Let $K \xrightarrow[P_2]{P_1} I'$ be a kernel pair of an encoder ϕ_I : that is, $K = \{(a, b) \in I' \times I' : \phi_I(a) = \phi_I(b)\}$. The diagram (4.2) implies the diagram (4.3).

$$(4.3) \quad E(A') \times K^+ \xrightarrow[E(A') \times P_2^+]{E(A) \times P_1^+} E(A') \times I'^+ \xrightarrow{m(A') \times I'^+} \langle I'^+, 0' \rangle \times I'^+ \xrightarrow{ev'} O$$

Remark that (4.3) is equivalent to the condition (4.4).

$$(4.4) \quad \forall f \in E(A'): [\phi_I^+(w) = \phi_I^+(w') \text{ implies } f(w) = f(w')]$$

DEFINITION 4.1. For a given automaton $A' = (I', O', S', d', l')$, a surjective function $\phi: I' \rightarrow I$ is called a well morphism over A' if (4.3) or (4.4) holds.

FACT 4.1. If $\phi \stackrel{w}{=} A f A'$ for some f , then ϕ_I with $\phi = (\phi_I, \phi_O)$ is a well morphism over an automaton A .

FACT 4.2. A function ϕ is a well morphism over A , if and only if, (4.5) commutes.

$$(4.5) \quad S \times K^+ \xrightarrow[S \times P_2^+]{S \times P_1^+} S \times I^+ \xrightarrow{l^+} O, \text{ where } K \xrightarrow[P_2]{P_1} I \text{ is a kernel pair of } \phi.$$

(Proof of the Fact 4.2 depends on the equation

$$\begin{aligned} l^+(S \times P_1^+) &= ev(m(A) \times I^+)(e(A) \times I^+)(S \times P_1^+) \\ &= ev(m(A) \times I^+)(E(A) \times P_1^+)(e(A) \times K^+). \end{aligned}$$

Now, we have a question for what conditions a well morphism becomes an encoder of a simulation. Next lemma answers this question.

LEMMA 4.3. *For each well morphism $\phi: I \rightarrow \bar{I}$ over A , there exists an automaton $\bar{A}_c$ with an input set $\bar{I}$ (defined as in (4.6)) which weakly simulates A with an encoder ϕ . And hence, a simulation $(\phi, 1_O) - Ag \bar{\bar{A}}_c$ is obtained, where g is a minimization of $\bar{A}_c$.*

(4.6)

, where c is a coretraction of ϕ and $\bar{A}_c := (\bar{I}, O, S, \bar{d}, \bar{l})$.

PROOF. By the definition of $\bar{A}_c$, $(c, 1_O) - \bar{A}_c 1_S A$ holds, hence we have

(4.7): $l^+ = l^+(S \times c^+)$.

Consider the kernel pair diagram (4.8).

(4.8)

There exists a unique function g with $P_1 g = c\phi$ and $P_2 g = 1_I$, because of $\phi c\phi = \phi$. We have

$$\begin{aligned} l^+(S \times (c\phi)^+) &= l^+(S \times (P_1 g)^+) = l^+(S \times P_1^+)(S \times g^+) \\ &= l^+(S \times P_2^+)(S \times g^+) \\ &= l^+(S \times (P_2 g)^+) = l^+(S \times I^+) \\ &= l^+ \text{ by} \end{aligned}$$

using the Fact 4.2. Moreover,

$$\begin{aligned} l^+(S \times \phi^+) &= l^+(S \times c^+)(S \times \phi^+) \text{ (by (4.7))} \\ &= l^+(S \times (c\phi)^+) \\ &= l^+ : \end{aligned}$$

that is, $(\phi, 1_O) \models Al_s \bar{A}_c$ holds.

Finally, we have the simulation $(\phi, 1_O) - Ag \bar{A}_c$ by taking the minimization $g: \bar{A}_c \rightarrow \bar{\bar{A}}_c$ (Refer to Theorem 3.8).

Example of the Lemma 4.3.

Consider an infinite automaton A defined by (4.9) and a function $\phi: \{a, b\} \rightarrow \{x\}$; that is $\phi(a) = \phi(b) = x$.

It is obvious that ϕ is a well morphism over A . The construction of the simulation with an encoder ϕ is as follows:

Delete b -edges preserving outputs, if we choose c as $c(x) = a$.

The automaton $\bar{A}_c$ is as follows:

The following is the final form:

According to Fact 4.1 and Lemma 4.3, we have

THEOREM 4.4. *An automaton A' with an input set I' can be simulated by an automaton A with an input set I , if and only if, there exists a well morphism $\phi: I' \rightarrow I$ over A' .*

It is emphasized that a well morphism becomes an encoder of simulation. Thus we can assert that well morphisms and simulations are same concepts in a sense. Moreover simulations (essentially) depend on encoders.

Now we study an “optimal input”. For this purpose, let us examine a next example.

Example or Motivation of Optimal Input Sets.

Let I be a set $\{1, 2, 3\}$, and ϕ and ψ two well morphisms over an automaton A such that $\phi(1) = \phi(2)$ and $\psi(2) = \psi(3)$.

A relation $\sim_\phi$ over I^+ is defined as

$$w \sim_\phi w' \quad \text{iff} \quad \phi^+(w) = \phi^+(w'), \quad \text{for each well morphism.}$$

Then, the pushout τ of ϕ and ψ (that is, $\tau(1) = \tau(2) = \tau(3)$) is a well morphism also.

For example,

$$w = 321 \underset{\phi}{\sim} 322 \underset{\psi}{\sim} 222 \underset{\phi}{\sim} 122 \underset{\psi}{\sim} 123 = w'.$$

τ is a pushout, especially, cointersection of two epimorphisms in **Set**.

(4.12)

LEMMA 4.5. *A (large) cointersection of a (class) set of well morphisms over an automaton A is also a well morphism over A .*

PROOF. First, a large cointersection is a cointersection, since **Set** is cowell-powered. Thus we have only to prove in the case of set but not class.

Let $\{\phi_i\}_{i \in J}$ be a set of well morphisms over A , and $\bar{\phi}: I \rightarrow \bar{I}$ be a cointersection of these as in the following figure, where u_j is an injection of colimit.

(4.13)

Moreover, let $\{A_j = (I_j, O, S, d_j, l_j)\}_{j \in J}$ be a family of automata defined by lemma 4.2 for each well morphism ϕ_j . Since A_j weakly simulates A , we have $l_j^+(S \times \phi_j^+) = l^+$ for each $j \in J$. Hence we have $(l_j)_n(S \times \phi_j^n) = l_n$ for $n \geq 1$ and $j \in J$. Since $- \times -$ preserves cointersections, we have unique function $h_n: S \times I^n \rightarrow O$ with $(l_j)_n = h_n(S \times u_j^n)$ for each $n \geq 1$.

(4.14)

Let h^+ be a unique function with $h^+(S \times v_n) = h_n$ for each $n \geq 1$, using the coproduct injection $\bar{I}^n \xrightarrow{v_n} \bar{I}^+$, and $\bar{K} \xrightarrow{\bar{p}_1} \bar{I}$ be a kernel pair of $\bar{\phi}$.

$$(4.15) \quad \begin{array}{ccccccc} S \times \bar{K}^+ & \xrightarrow{S \times \beta_1^+} & S \times I^+ & \xrightarrow{S \times \phi^-} & S \times I^+ & \xrightarrow{h^-} & 0 \\ & \xrightarrow{S \times \beta_2^+} & & & & & \\ & & \searrow^{S \times \phi_j^+} & & \nearrow_{I_j^+} & & \\ & & S \times I_j^+ & & & & \\ & & = & & & & \\ & & \downarrow & & & & \\ & & I^+ & & & & \end{array}$$

Using a diagram (4.15), we have a commutative diagram (4.16).

$$(4.16) \quad S \times \bar{K}^+ \xrightarrow[S \times \bar{P}_2^+]{S \times \bar{P}_1^+} S \times I^+ \xrightarrow{I^+} O.$$

Thus $\bar{\phi}$ is a well morphism over A by Fact 4.2.

Now we can state the existence of an optimal input set. An automaton A is called i -reduced, if well morphisms over A are isomorphisms.

THEOREM 4.6. *For each automaton A , there exists an i -reduced automaton which simulates A .*

PROOF. The large cointersection of all well morphisms over A is also a well morphism over A by the lemma 4.5. Hence there exists an automaton $\bar{A}$ with an input set $\bar{I}$ such that $(\bar{\phi}, 1_o) - Ag\bar{A}$ by the lemma 4.3. Let $\phi: \bar{I} \rightarrow I'$ be a well morphism over $\bar{A}$, then there exists an automaton A' with $(\phi, 1_o) - \bar{A}g'A'$. The composition of simulations $(\phi\bar{\phi}, 1_o) - Ag'gA'$ implies that $\phi\bar{\phi}$ is a well morphism over A by Fact 4.1. Let u be an injection of cointersection for $\phi\bar{\phi}$ as in the figure (4.17).

$$(4.17) \quad I \xrightarrow{\bar{\phi}} \bar{I} \xrightarrow[\bar{\phi}]{\phi} I' \xrightarrow{u} \bar{I}$$

That is, $u\phi\bar{\phi}=\bar{\phi}$ holds. Since $\bar{\phi}$ is a surjection, $u\phi=1$ holds. Thus ϕ is an injection. Thus ϕ is an isomorphism.

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