MINIMAL REALIZATIONS OF IMPLICIT λ -MACHINES

By

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1. Introduction

In [2], M. A. Arbib and E. G. Manes gave a definition of an X -machine in a category and, using category theory, have shown the existence of minimal realization of the machine, that is, for every response morphism f there exists a minimal X -machine which realizes f uniquely up to isomorphism.

In this paper, we show that each response morphism f has a unique, up to isomorphism, implicit λ -machine which realizes it. Implicit λ -machine is general enough to include X -machine as a special case.

We assume that the reader has gone through [2] and knows (or can look up in [4]) the basic categorical notions used but not defined in this paper.

2. Preliminaries

Let (T, e, m) be a monad in an arbitrary category $\mathbf{K}$. For the monad (T, e, m) , $\mathbf{K}^T$ denotes the category of T -algebras and homomorphisms between them. We now state the following proposition without proof. Proof can be found in [4].

PROPOSITION 1. *For every $\mathbf{K}$ -object A , (AT, Am) is a T -algebra. Moreover, given any T -algebra (B, ζ) and any $\mathbf{K}$ -morphism $f: A \rightarrow B$ there exists a unique $\mathbf{K}^T$ -morphism $f^*: (AT, Am) \rightarrow (B, \zeta)$ such that the diagram*

commutes in $\mathbf{K}$.

DEFINITION 1 ([3]). Given a monad (T, e, m) and a functor $X: \mathbf{K} \rightarrow \mathbf{K}$, a distributive law λ of X over (T, e, m) is a correspondence which assigns to each $\mathbf{K}$ -object A a morphism $A\lambda: ATX \rightarrow AXT$ such that the diagrams

$$\begin{array}{ccc} ATX & \xrightarrow{A\lambda} & AXT \\ & \searrow A\epsilon X & \nearrow AXe \\ & AX & \end{array} \qquad \begin{array}{ccc} ATX & \xrightarrow{A\lambda} & AXT \\ \downarrow \alpha^* X & & \downarrow (\alpha X \cdot B\lambda)^* \\ BTX & \xrightarrow{B\lambda} & BXT \end{array}$$

commute, for all A and $\alpha: A \rightarrow BT$, in $\mathbf{K}$.

3. λ -algebras and implicit λ -machines

Now we recall the definitions and results, due to Arbib-Manes [3], which are necessary for further development.

DEFINITION 2. (i) Let λ be a distributive law of $X: \mathbf{K} \rightarrow \mathbf{K}$. A λ -algebra is a triple (Q, δ, ξ) where (Q, δ) is an X -dynamics and (Q, ξ) is a T -algebra in such a way that $\xi: (QT, Q\lambda \cdot \delta T) \rightarrow (Q, \delta)$ is an X -dynamorphism.
(ii) An λ -homomorphism $f: (Q, \delta, \xi) \rightarrow (Q', \delta', \xi')$ between λ -algebras is an X -dynamorphism $f: (Q, \delta) \rightarrow (Q', \delta')$ which is also a $\mathbf{K}^T$ -morphism $f: (Q, \xi) \rightarrow (Q', \xi')$.

Obviously, λ -algebras and λ -homomorphisms form a category, denoted by $\mathbf{K}^\lambda$. Especially, if T is the identity functor $1_{\mathbf{K}}: \mathbf{K} \rightarrow \mathbf{K}$, then a λ -algebra is an X -dynamics and a λ -homomorphism is an X -dynamorphism.

DEFINITION 3. Given a monad (T, e, m) , an implicit λ -machine M in $\mathbf{K}$ is a 9-tuple $M = (X, Q, \delta, \xi, I, \tau, Y, \theta, \beta)$, where $X: \mathbf{K} \rightarrow \mathbf{K}$ is an input process, (Q, δ, ξ) is a λ -algebra, $\tau: I \rightarrow Q$ is a $\mathbf{K}$ -morphism and $\beta: (Q, \xi) \rightarrow (Y, \theta)$ is a $\mathbf{K}^T$ -morphism.

If T is the identity functor, then an implicit λ -machine becomes a usual X -machine.

THEOREM 2. If $X: \mathbf{K} \rightarrow \mathbf{K}$ is an input process, then the forgetful functor $U: \mathbf{K}^\lambda \rightarrow \mathbf{K}$ has a left adjoint, that is, for each $\mathbf{K}$ -object I , there exists a λ -algebra $(IX^@T, IX^@\lambda \cdot I\mu_0 T, IX^@m)$ such that for any λ -algebra (Q, δ, ξ) and any $\mathbf{K}$ -morphism $f: I \rightarrow Q$, there exists a unique λ -homomorphism $f^*: = (f^*)^*: (IX^@T, IX^@\lambda \cdot I\mu_0 T, IX^@m) \rightarrow (Q, \delta, \xi)$ rendering the diagram in $\mathbf{K}$

commutative. (The morphism $f^*: IX^@ \rightarrow Q$ denotes the dynamorphic extension of $f: I \rightarrow Q$.)

DEFINITION 4. The *reachability morphism* of an implicit λ -machine $M = (X, Q, \delta, \xi, I, \tau, Y, \theta, \beta)$ is the unique λ -homomorphism $\tau^*: (IX^@T, IX^@ \cdot I\mu_0 T, IX^@m) \rightarrow (Q, \delta, \xi)$. The *response morphism* f_M of M is the response morphism of the X -machine $(X, Q, \delta, I, \tau, Y, \beta)$, that is, $f_M = \tau^* \cdot \beta$, where $\tau^*: IX^@ \rightarrow Q$.

4. Minimal realization of implicit λ -machine

In the sequel, we fix a $\mathbf{K}$ -object I and a T -algebra (Y, θ) and denote an implicit λ -machine $(X, Q, \delta, \xi, I, \tau, Y, \theta, \beta)$ by $(Q, \delta, \xi, \tau, \beta)$. Let $\mathbf{E}$ denotes the class of all coequalizers of $\mathbf{K}$.

DEFINITION 5. An implicit λ -machine M is called a *realization* of a response morphism $f: IX^@ \rightarrow Y$ if M has the response morphism f . M is *reachable* if the reachability morphism of M belongs to $\mathbf{E}$.

DEFINITION 6. Given two implicit λ -machines $M = (Q, \delta, \xi, \tau, \beta)$ and $M' = (Q', \delta', \xi', \tau', \beta')$, a *simulation* $\psi: M \rightarrow M'$ is a λ -homomorphism $\psi: (Q, \delta, \xi) \rightarrow (Q', \delta', \xi')$ such that the diagram

commutes.

For a response morphism $f: IX^@ \rightarrow Y$, $\mathbf{E}_f\text{-Mach}$ denotes the category whose objects are reachable realizations of f and whose morphisms are simulations.

The *minimal realization* of f is the terminal object of E_f -**Mach**.

DEFINITION 7 ([1]). (i) Given a response morphism $f: IX^@ \rightarrow Y$, a pair of morphism $\alpha, \gamma: E \rightarrow IX^@$ are *abstractly f -equivalent* if $\alpha^! \cdot f = \gamma^! \cdot f$.

(ii) $E_f \xrightarrow[\gamma]{\alpha} IX^@$ is the *abstract Nerode equivalence* of f if α and γ are abstractly f -equivalent and if whenever $\alpha', \gamma': E' \rightarrow IX^@$ are abstractly f -equivalent there exists a unique morphism $\psi: E' \rightarrow E_f$ such that the diagram

$$\begin{array}{ccc} E' & \begin{array}{c} \xrightarrow{\alpha'} \\ \xrightarrow{\gamma'} \end{array} & IX^@ \\ \downarrow \psi & & \uparrow \\ E_f & \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\gamma} \end{array} & \end{array}$$

commutes, that is, $\psi \cdot \alpha = \alpha'$ and $\psi \cdot \gamma = \gamma'$.

THEOREM 3. Let the response morphism $f: IX^@ \rightarrow Y$ satisfy the following four conditions:

- (1) f^* has an abstract Nerode equivalence $E_f \xrightarrow[\gamma]{\alpha} IX^@T$.
- (2) $s_f = \text{coeq}(\alpha, \gamma): IX^@T \rightarrow Q_f$ exists.
- (3) There exists a λ -algebra (Q_f, δ_f, ξ_f) with a λ -homomorphism $s_f: (IX^@T, IX^@ \cdot \lambda \cdot I\mu_0 T, IX^@m) \rightarrow (Q_f, \delta_f, \xi_f)$.
- (4) For a λ -homomorphism $s: (IX^@T, IX^@ \cdot \lambda \cdot I\mu_0 T, IX^@m) \rightarrow (Q, \delta, \xi)$, if $s: IX^@T \rightarrow Q$ is a coequalizer, then either sX and sT are epimorphisms or $sX^@T$ and sT are epimorphisms.

Then f has a minimal realization.

PROOF. From condition (1) we have the relation $\alpha^! \cdot f^* = \gamma^! \cdot f^*$. Also we have $\alpha \cdot f^* = E_f \eta \cdot \alpha^! \cdot f^* = E_f \eta \cdot \gamma^! \cdot f^* = \gamma \cdot f^*$. Since $s_f = \text{coeq}(\alpha, \gamma)$, there exists a unique morphism $\beta_f: Q_f \rightarrow Y$ such that the diagram

$$\begin{array}{ccccc} E_f & \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\gamma} \end{array} & IX^@T & \xrightarrow{s_f} & Q_f \\ & & & \searrow f^* & \downarrow \beta_f \\ & & & & Y \end{array}$$

commutes. Defining $\tau_f = I\eta \cdot IX^@e \cdot s_f: I \rightarrow Q_f$, we see that condition (3) ensures

s_f as the reachability morphism of $M_f = (Q_f, \delta_f, \xi_f, \tau_f, \beta_f)$, which is a reachable realization of f since $f_{M_f} = \tau_f^\dagger \cdot \beta_f = IX^{\otimes} e \cdot s_f \cdot \beta_f = IX^{\otimes} e \cdot f^\# = f$.

To show that M_f is minimal, let $M = (Q, \delta, \xi, \tau, \beta)$ be another reachable realization of f with reachability morphism $s = \tau^*: IX^{\otimes} T \rightarrow Q$. Since $s \in \mathbf{E}$, there exists a pair of morphism $a, b: A \rightarrow IX^{\otimes} T$ such that $s = \text{coeq}(a, b)$. The fact that $a^\dagger \cdot f^\# = a^\dagger \cdot s \cdot \beta = (a \cdot s)^\dagger \cdot \beta = (b \cdot s)^\dagger \cdot \beta = b^\dagger \cdot f^\#$, together with condition (1) induces a unique morphism $\psi: A \rightarrow E_f$ such that the diagram

commutes. Thus $a \cdot s_f = \psi \cdot \alpha \cdot s_f = \psi \cdot \gamma \cdot s_f = b \cdot s_f$. Hence there exists a unique morphism $\Gamma: Q \rightarrow Q_f$ such that the diagram

commutes. As the morphism s is an epimorphism, the relation $s \cdot \beta = f^\# = s_f \cdot \beta_f = s \cdot \Gamma \cdot \beta_f$ implies that $\beta = \Gamma \cdot \beta_f$. We have the following commutative diagram:

We have to check that the morphism Γ is a dynamorphism as well as a $\mathbf{K}^T$ -morphism. The following diagram commutes except the part (?).

Therefore we have

$$sT \cdot \xi \cdot \Gamma = sT \cdot \Gamma T \cdot \xi_f.$$

Since sT is an epimorphism, the part (?) of the diagram commutes. In the case when sX is an epimorphism, consider the following diagram:

The whole diagram commutes except the part (?). Since sX is an epimorphism,

the part (?) of the diagram commutes. In the case when $sX^@T$ is an epimorphism, consider the diagram (p. 142):

We note that the part (i) commutes by the definition of $sX^@$. Since e is a natural transformation, the part (ii) commutes. The parts (iii) and (iv) commute by Theorem 2. Therefore, we have the relation $sX^@T \cdot (1_Q)^* = (1_{IX^@T})^* \cdot s$ and, similarly, we have $s_f X^@T \cdot (1_{Q_f})^* = (1_{IX^@T}) \cdot s_f$. Hence the outer part of the diagram commutes. Since $sX^@T$ is an epimorphism, $sX^@T \cdot \Gamma X^@T \cdot (1_{Q_f})^* = sX^@T \cdot (1_Q)^* \cdot \Gamma$ implies $\Gamma X^@T \cdot (1_{Q_f})^* = (1_Q)^* \cdot \Gamma$. Also the part (v) of the diagram commutes by the definition of $\Gamma X^@$ and the functorial property of X , (vi) by the naturality of μ_0 , (vii) by the naturality of e , and (viii) by the fact that

$$\begin{aligned} Q\eta X \cdot Q\mu_0 \cdot QX^@e \cdot (1_Q)^* &= Q\eta X \cdot Q\mu_0 \cdot QX^@e \cdot (1_Q)^{*\sharp} \\ &= Q\eta X \cdot Q\mu_0 \cdot (1_Q)^{\sharp} = Q\eta X \cdot (1_Q)^{\sharp} X \cdot \delta = (Q\eta \cdot (1_Q)^{\sharp}) X \cdot \delta \\ &= 1_Q X \cdot \delta = \delta. \end{aligned}$$

The part (ix) of the diagram also commutes by the similar calculation. Thus the part (?) commutes. Hence there exists a simulation $\Gamma: M \rightarrow M_f$. The uniqueness of Γ is shown trivially.

References

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