A DISTRIBUTIVE LAW OVER FUZZY THEORY AND ITS APPLICATION

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1. Introduction

In [1], M. A. Arbib and E. G. Manes gave a definition of an X -machine in an arbitrary category $\mathbf{K}$ and have shown the existence of a minimal realization of the machine. By introducing a distributive law $\lambda: TX \rightarrow XT$, in [2], they have also shown that a Kleisli machine in $\mathbf{K}$ is an X_λ -machine in a Kleisli category for a suitable functor X_λ .

In this paper we extend the distributive law $\lambda: TX \rightarrow XT$ to $\lambda^\circ: TX^\circ \rightarrow X^\circ T$ and apply it to max-product automaton defined in [4].

2. Preliminaries

Now we recall the definitions and results, due to Arbib-Manes [2], which are necessary for further development. In the following the symbol $\mathbf{K}$ denotes an arbitrary category.

DEFINITION 1. Let $T: |\mathbf{K}| \rightarrow |\mathbf{K}|$ be a mapping between $\mathbf{K}$ -objects. We denote a $\mathbf{K}$ -morphism $\alpha: A \rightarrow BT$ by $\alpha: A \rightarrow B$ and call it a fuzzy relation. Let $\circ$ be a composition of fuzzy relations. If the composition $\circ$ satisfies the associativity law and $Ae: A \rightarrow AT$ for each A in $\mathbf{K}$ satisfies the relations $f^A \circ \beta = f \cdot \beta$ (where $f^A = f \cdot Be$) as well as $\beta \circ Ce = \beta$, for each $f: A \rightarrow B$ and $\beta: B \rightarrow C$, then we call the triple $T = (T, \circ, e)$ a *fuzzy theory* over $\mathbf{K}$.

From the above definition it follows easily that $Ae \circ \alpha = \alpha$ for each $\alpha: A \rightarrow B$. We form a category whose objects are the same as that of $\mathbf{K}$ and morphisms the fuzzy relations; $\circ$ is the composition and $Ae: A \rightarrow A$ the identities for this category. We call it a Kleisli category and denote it by $\mathbf{K}_T$.

Each morphism $\alpha: A \rightarrow BT$ has a canonical extension $\alpha^*: AT \rightarrow BT$ defined by

$\alpha^* = 1_{AT} \circ \alpha$ and this yields that $Ae \cdot \alpha^* = \alpha$. For each $\alpha: A \rightarrow BT$ and $\beta: B \rightarrow CT$ we get $\alpha \circ \beta = \alpha \cdot \beta^*$. $T: \mathbf{K} \rightarrow \mathbf{K}$ becomes a functor when we define $fT: AT \rightarrow BT$ by $fT = (f^4)^*$ for $f: A \rightarrow B$.

3. Kleisli machines and distributive laws

In the following development we fix a fuzzy theory $\mathbf{T} = (T, \circ, e)$.

DEFINITION 2. A *Kleisli machine* in $\mathbf{K}$ is a 7-tuple $(X, Q, \delta, I, \tau, Y, \beta)$ where $X: \mathbf{K} \rightarrow \mathbf{K}$ is an input process, and $\delta: QX \rightarrow QT$, $\tau: I \rightarrow QT$ and $\beta: Q \rightarrow YT$ are $\mathbf{K}$ -morphisms.

DEFINITION 3. By a *lifting* of a functor $X: \mathbf{K} \rightarrow \mathbf{K}$ to $\mathbf{K}_T$ we mean a functor $\bar{X}: \mathbf{K}_T \rightarrow \mathbf{K}_T$ such that the diagram of functors

$$\begin{array}{ccc} \mathbf{K}_T & \xrightarrow{X} & \mathbf{K}_T \\ (-)^4 \uparrow & & \uparrow (-)^4 \\ \mathbf{K} & \xrightarrow{X} & \mathbf{K} \end{array}$$

commutes where $(-)^4$ is an embedding from $\mathbf{K}$ to $\mathbf{K}^T$ by $f \mapsto f^4$.

DEFINITION 4. A *distributive law* of X over $\mathbf{T}$ is an assignment to each $\mathbf{K}$ -object A a $\mathbf{K}$ -morphism $A\lambda: ATX \rightarrow AXT$ such that the diagrams

$$\begin{array}{ccc} ATX & \xrightarrow{A\lambda} & AXT \\ \swarrow A\circ X & & \searrow AX\circ \\ & AX & \end{array} \qquad \begin{array}{ccc} ATX & \xrightarrow{A\lambda} & AXT \\ \alpha^4 X \downarrow & & \downarrow (\alpha X \cdot B\lambda)^4 \\ BTX & \xrightarrow{B\lambda} & BXT \end{array}$$

commutes for all A and $\alpha: A \rightarrow BT$ in $\mathbf{K}$.

There exists a bijective correspondence between liftings $\bar{X}$ and distributive laws of X . We write X_λ rather than $\bar{X}$ for the lifting corresponding to λ . It can be easily deduced that $AX_\lambda = AX$ for each $\mathbf{K}_T$ -object A , $\alpha X_\lambda = \alpha X \cdot B\lambda$ for each fuzzy relation $\alpha: A \rightarrow B$ and $\lambda: TX \rightarrow XT$ is a natural transformation.

Now we state a result, the proof of which can be found in [2].

THEOREM 1. If $X: \mathbf{K} \rightarrow \mathbf{K}$ is an input process, so is $X_\lambda: \mathbf{K}_T \rightarrow \mathbf{K}_T$, that is, if $((AX^\circ, A\mu_0), A\eta)$ is a free pair over A in $\mathbf{K}$, then $((AX^\circ, (A\mu_0)^4), (A\eta)^4)$ is a free pair over A in $\mathbf{K}_T$.

DEFINITION 5. Let $\lambda: TX \rightarrow XT$ be a distributive law. In accordance with the above theorem, for each B , we can define a unique fuzzy relation $B\lambda^\circ: BTX^\circ \rightarrow BX^\circ$ rendering the diagrams in $\mathbf{K}_T$

$$\begin{array}{ccc}
 BT & \xrightarrow{(BT\eta)^A} & BTX^\circ \\
 & \searrow B\eta T & \downarrow B\lambda^\circ \\
 & & BX^\circ
 \end{array}
 \qquad
 \begin{array}{ccc}
 BTX^\circ X & \xrightarrow{(BT\mu_0)^A} & BTX^\circ \\
 \downarrow B\lambda^\circ X_\lambda & & \downarrow B\lambda^\circ \\
 BX^\circ X & \xrightarrow{(B\mu_0)^A} & BX^\circ
 \end{array}$$

commutative.

PROPOSITION 2. For each A , $AX_\lambda^\circ = AX^\circ$ and $\alpha X_\lambda^\circ = \alpha X^\circ \cdot B\lambda^\circ$ for each fuzzy relation $\alpha: A \rightarrow B$.

PROOF: We consider the following diagrams:

$$\begin{array}{ccc}
 A & \xrightarrow{A\eta} & AX^\circ \\
 \downarrow \alpha & & \downarrow \alpha X^\circ \\
 BT & \xrightarrow{B\eta T} & BTX^\circ \\
 & \nearrow B\eta T & \downarrow B\lambda^\circ \\
 & & BX^\circ T
 \end{array}
 \quad
 \begin{array}{ccc}
 AX^\circ X & \xrightarrow{A\mu_0} & AX^\circ \\
 \downarrow \alpha X^\circ X & & \downarrow \alpha X^\circ \\
 BTX^\circ X & \xrightarrow{BT\mu_0} & BTX^\circ \\
 \downarrow B\lambda^\circ X & & \downarrow B\lambda^\circ \\
 BX^\circ TX & \xrightarrow{BX^\circ \lambda} & BX^\circ XT \xrightarrow{B\mu_0 T} BX^\circ T
 \end{array}$$

(i) (ii) (iii) (iv)

We note that (i) and (ii) commute by the definition of αX° . The definition of λ° gives the relations $B\eta T = (BT\eta)^A \circ B\lambda^\circ = BT\eta \cdot B\lambda^\circ$ and $BT\mu_0 \cdot B\lambda^\circ = (BT\mu_0)^A \circ B\lambda^\circ = B\lambda^\circ X_\lambda (B\mu_0)^A = (B\lambda^\circ X \cdot BX^\circ \lambda) \cdot (B\mu_0)^A = B\lambda^\circ X \cdot BX^\circ \lambda \cdot B\mu_0 T$. Thus we have the commutativity of the parts (iii) and (iv) of the diagrams. Hence the above diagrams commute. From the commutativity of the left hand part of this diagram, we have the relation $(A\eta)^A \circ (\alpha X^\circ \cdot B\lambda^\circ) = A\eta \cdot \alpha X^\circ \cdot B\lambda^\circ = \alpha \cdot B\eta T = \alpha \circ (B\eta)^A$, and from the commutativity of the right hand part we get $(A\mu_0)^A \circ (\alpha X^\circ \cdot B\lambda^\circ) = A\mu_0 \cdot \alpha X^\circ \cdot B\lambda^\circ = \alpha X^\circ X \cdot B\lambda^\circ X \cdot BX^\circ \lambda \cdot B\mu_0 T = (\alpha X^\circ \cdot B\lambda^\circ) X \cdot BX^\circ \lambda \cdot (B\mu_0)^A = (\alpha X^\circ \cdot B\lambda^\circ) X_\lambda \circ (B\mu_0)^A$. By the uniqueness of the morphism $\alpha X_\lambda^\circ: AX^\circ \rightarrow BX^\circ$ in $\mathbf{K}_T$, we have $\alpha X_\lambda^\circ = \alpha X^\circ \cdot B\lambda^\circ$.

Let $T = (T, \circ, e)$ be a fuzzy theory over **Set**, where **Set** is the category of sets and functions.

PROPOSITION 3. For the functor $X = - \times X_0: \mathbf{Set} \rightarrow \mathbf{Set}$,

- (1) $Q\lambda: QT \times X_0 \rightarrow (Q \times X_0)T$, $(p, x) \mapsto (p)(in_x T)$ is a distributive law of X , where $in_x: Q \rightarrow Q \times X_0$ is defined by $(q)in_x = (q, x)$;
 (2) $Q\lambda^@: QT \times X_0^* \rightarrow (Q \times X_0^*)T$, $(p, w) \mapsto (p)in_w T$ satisfies definition 5, where $in_w: Q \rightarrow Q \times X_0^*$ is defined by $(q)in_w = (q, w)$.

PROOF: The proof of (1) can be found in [2].

From the fact that $(q)in_A = (q, A) = (q)Q\eta$ for each $q \in Q$, we have $in_A = Q\eta$ and hence $in_A T = Q\eta T$. We note that $(p)QT\eta \cdot Q\lambda^@ = (p, A)Q\lambda^@ = (p)in_A T = (p)Q\eta T$ for each $p \in QT$. Hence the part (iii) in the diagram constructed in the proof of proposition 2 is commutative. For $(p, w, x) \in QT \times X_0^* \times X_0$, we have

$$\begin{aligned}
 & (p, w, x)(Q\lambda^@ \times X_0) \cdot (Q \times X_0^*)\lambda \cdot Q\mu_0 T \\
 &= ((p)in_w T, x)(Q \times X_0^*)\lambda \cdot Q\mu_0 T \\
 &= ((p)in_w T)in_x T \cdot Q\mu_0 T = (p)(in_w \cdot in_x \cdot Q\mu_0)T \\
 &= (p)in_{wx} T = (p, wx)Q\lambda^@ = (p, w, x)QT\mu_0 \cdot Q\lambda^@.
 \end{aligned}$$

Therefore the part (iv) in the above-mentioned diagram commutes. Thus (2) holds.

4. An application of $\lambda^@$

Let R be the set of all nonnegative real numbers. We define T by $AT = \{f | f: A \rightarrow R, \text{supp}(f) \text{ is finite}\}$ and define $Ae: A \rightarrow AT$ by

$$(a')[(a)Ae] = \begin{cases} 1 & \text{if } a = a', \\ 0 & \text{if } a \neq a' \end{cases}$$

for each $a, a' \in A$, and the composition $\circ$ of $\alpha: A \rightarrow B$ and $\beta: B \rightarrow C$ by

$$(c)[(a)(\alpha \circ \beta)] = \max_{b \in B} ((b)[(a)\alpha] \cdot (c)[(b)\beta])$$

for each $a \in A$ and $c \in C$. Then we can easily verify that $\mathbf{T} = (T, \circ, e)$ is a fuzzy theory over **Set**. Let I and Y be one-element sets. Since $X = - \times X_0: \mathbf{Set} \rightarrow \mathbf{Set}$ is an input process, a Kleisli machine $M = (X, Q, \delta, I, \tau, Y, \beta)$ can be viewed as an X_λ -machine $\bar{M} = (X_\lambda, Q, \delta, I, \tau, Y, \beta)$. Thus there exists a unique morphism $\delta^*: Q \times X_0^* \rightarrow Q$ such that the diagrams

commute in $\mathbf{Set}_T$. From the commutativity of diagram (i), it can be easily verified that

$$(q')[(q, \lambda)\delta^*] = \begin{cases} 1 & \text{if } q = q', \\ 0 & \text{if } q \neq q' \end{cases}$$

for each $q, q' \in Q$. Also it follows from the diagram (ii) that

$$\begin{aligned} (q')[(q, wx)\delta^*] &= (q')[(q, w, x)Q\mu_0 \cdot \delta^*] \\ &= (q')[(q, w, x)(Q\mu_0)^{\delta} \circ \delta^*] \\ &= (q')[(q, w, x)(\delta^* X_{\lambda} \circ \delta)] \\ &= \max_{(q'', x') \in Q \times X_0} ((q'', x')[(q, w, x)\delta^* X_{\lambda}] \cdot (q')[(q'', x')\delta]), \end{aligned}$$

where

$$\begin{aligned} (q'', x')[(q, w, x)\delta^* X_{\lambda}] &= (q'', x')[(q, w, x)((\delta^* \times X_0) \cdot Q\lambda)] \\ &= (q'', x')[((q, w)\delta^*, x)Q\lambda] \\ &= (q'', x')[((q, w)\delta^*)in_x T] \\ &= (q'', x')[((q, w)\delta^*)in_x^4] \\ &= (q'', x')[((q, w)\delta^*)(1_{QT} \circ in_x^4)] \\ &= \max_{\bar{q} \in Q} ((\bar{q})[((q, w)\delta^*)1_{QT}] \cdot (q'', x')[(\bar{q})(in_x \cdot (Q \times X_0)e)]) \\ &= \max_{\bar{q} \in Q} ((\bar{q})[(q, w)\delta^*] \cdot (q'', x')[(\bar{q}, x)(Q \times X_0)e]) \\ &= (q'')[(q, w)\delta^*] \quad (\text{for } q'' = \bar{q}, x' = x) \end{aligned}$$

for each $q, q' \in Q, w \in X_0^*$ and $x \in X_0$.

Hence, we have

$$(q')[(q, wx)\delta^*] = \max_{q'' \in Q} ((q'')[(q, w)\delta^*] \cdot (q')[(q'', x)\delta]).$$

The definitions of δ^* and $\tau X_\lambda^\otimes$ yield the commutativity of the following diagrams in $\mathbf{Set}_T$.

$$\begin{array}{ccc} I & \xrightarrow{(I\eta)^d} & IX^\otimes \\ \tau \downarrow & & \downarrow \tau X_\lambda^\otimes \\ Q & \xrightarrow{(Q\eta)^d} & QX^\otimes \\ & \searrow Qe & \downarrow \delta^* \\ & & Q \end{array} \quad \begin{array}{ccc} IX^\otimes X & \xrightarrow{(I\mu o)^d} & IX^\otimes \\ \tau X_\lambda^\otimes X_\lambda \downarrow & & \downarrow \tau X_\lambda^\otimes \\ QX^\otimes X & \xrightarrow{(Q\mu o)^d} & QX^\otimes \\ \delta^* X_\lambda \downarrow & & \downarrow \delta^* \\ QX & \xrightarrow{\beta} & Q \end{array}$$

Therefore, if we denote the response morphism of $\bar{M}$ by f_M , we have $f_M = \tau X_\lambda^\otimes \circ \delta^* \circ \beta$.

Let $I = Y = \{0\}$. For $w \in X_0^*$, we have

$$\begin{aligned} (0)[(0, w)f_M] &= (0)[(0, w)(\tau X_\lambda^\otimes \circ \delta^* \circ \beta)] \\ &= (0)[(0, w)((\tau X^\otimes \cdot Q\lambda^\otimes) \circ \delta^* \circ \beta)] \\ &= \max_{(q_1, w') \in Q \times X_0^*} \max_{q_2 \in Q} ((q_1, w')[(0, w)\tau X^\otimes \cdot Q\lambda^\otimes] \cdot (q_2) \\ &\quad [(q, w')\delta^*] \cdot (0)[(q_2)\beta]), \end{aligned}$$

where $(q_1, w')[(0, w)\tau X^\otimes \cdot Q\lambda^\otimes] = (q_1, w')[(0, w)(\tau \times X_0^*) \cdot Q\lambda^\otimes]$

$$\begin{aligned} &= (q_1, w')[((0)\tau)in_w T] \\ &= (q_1, w')[((0)\tau)in_w^{A*}] \\ &= (q_1, w')[((0)\tau)(1_{QT} \circ in_w^A)] \\ &= \max_{q' \in Q} ((q')[((0)\tau)1_{QT}] \cdot (q_1, w')[(q')(in_w \cdot (Q \times X_0^*)e)]) \\ &= \max_{q' \in Q} ((q')[(0)\tau] \cdot (q_1, w')[(q', w)(Q \times X_0^*)e]) \\ &= (q_1)[(0)\tau] \quad (\text{for } q_1 = q', w' = w). \end{aligned}$$

Hence, we have

$$(0)[(0, w)f_M] = \max_{q_1 \in Q} \max_{q_2 \in Q} ((q_1)[(0)\tau] \cdot (q_2)[(q_1, w)\delta^*] \cdot (0)[(q_2)\beta]).$$

These results coincide with the definition of max-product automaton given by E. S. Santos [4].

References

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