ON AN AUTOMATON WHICH RECOGNIZES A FAMILY OF AUTOMATA

By

Satoru MIYANO

(Received April 11, 1977)

Introduction

In 1974 I. M. Havel [2] defined a finite branching automaton, motivated by the theory of state-space problem solving. The finite branching automaton accepts languages instead of strings. It seems, however, appropriate to say that a finite branching automaton accepts the minimal automata [1] of languages. From this point of view we shall define the notion that a finite branching automaton accepts an accessible automaton and define two distinct behaviors of a finite branching automaton. These are mentioned in Section 1.

In trying to calculate the behaviors, we encounter the following difficulties. There is no method by which we can easily verify whether a language is accepted by the given finite branching automaton. The main purpose of Section 2 is to overcome this difficulty. The necessary and sufficient condition for an accessible automaton to be accepted by the finite branching automaton is obtained. The difficulties are, however, not yet over. In the ordinary automata theory, there is a correspondence between a finite automaton and a regular grammar. In the theory of finite branching automata, there has not been such a grammar that corresponds to a finite branching automaton. In Section 3 we shall construct tree productions from a finite branching automaton and convert the trees generated by the tree productions into automata. Our main theorem asserts that the family of languages accepted by the finite branching automaton in the sense of Havel [2] coincides with the behaviors of trim automata [1] generated by its tree productions.

1. Basic Definitions

In the paper we shall use the notations introduced by S. Eilenberg [1].

Let Σ be a finite alphabet. A finite branching Σ -automaton is a triple $\mathcal{B} = (Q, q_0, B)$, where (Q, q_0) is a finite deterministic Σ -automaton without terminal states and $B = \langle B_0, B_1 \rangle$ is a pair of subsets of $Q \times 2^\Sigma$. We denote by $\mathcal{L}(\Sigma)$ the set of all non-empty languages over Σ . A language $L \in \mathcal{L}(\Sigma)$ is said to be strictly accepted by a finite branching Σ -automaton $\mathcal{B} = (Q, q_0, B)$ if the following conditions (1.1) and (1.2) hold:

$$(q_0 w, \{\sigma \in \Sigma | \sigma^{-1} w^{-1} L \neq \phi\}) \in B_0.$$
$$(q_0w, \{\sigma \in \Sigma | \sigma^{-1}w^{-1}L \not\equiv \phi\}) \in B_1.$$

EXAMPLE 1.1. Let $\Sigma = \{\sigma, \tau\}$. Consider the branching Σ -automaton:

$$B = \langle B_0, B_1 \rangle : B_0 = B_1 = Q \times 2^\Sigma.$$

EXAMPLE 1.2. Let $\Sigma = \{\sigma, \tau\}$. Consider the branching Σ -automaton:

$$B_1 = \{(q_0, \{\tau\}), (q_1, \{\sigma, \tau\}), (q_2, \{\tau\}), (q_2, \phi)\}.$$

Next we shall define the notion that a finite branching Σ -automaton accepts

Let $\mathcal{B}=(Q, q_0, B)$ be a finite branching Σ -automaton. An accessible Σ -automaton (not necessarily finite) $\mathcal{A}=(Q', q'_0, T')$ is accepted by $\mathcal{B}$ if the following conditions (1.3) and (1.4) hold:

$$(q'_0 w, q_0 w, q'^{-1} Q' \cap \Sigma) \in B_0 \quad \text{for each } w \in q_0'^{-1} q'.$$
$$(q'_0 w, q_0 w, t'^{-1} Q' \cap \Sigma) \in B_1 \quad \text{for each } w \in q_0'^{-1} t'.$$

We now define another behavior of a finite branching Σ -automaton. A language L is accepted by $\mathcal{B}$ if there exists an accessible Σ -automaton $\mathcal{A}$ with $|\mathcal{A}|=L$ which $\mathcal{B}$ accepts [1]. The languages accepted by $\mathcal{B}$ form a subset $|\mathcal{B}|$ of $\mathcal{L}(\Sigma)$, called the *behavior* of $\mathcal{B}$. A subset X of $\mathcal{L}(\Sigma)$ is said to be *recognizable* if there exists a finite branching Σ -automaton $\mathcal{B}$ such that $|\mathcal{B}|=X$.

$$(Q, q_0): \begin{array}{c} \xrightarrow{\tau} \\ \text{ } \\ \xrightarrow{\sigma} \end{array} q_0 \xrightarrow{\sigma} q_1 \xrightarrow{\tau} q_2 \circlearrowright_{\sigma, \tau}$$

$$B_1 = \{(q_2, \{\sigma, \tau\})\}.$$

$$\tau \circ \tau \rightarrow q'_0 \xrightarrow{\sigma} q'_1 \xrightarrow{\tau} q'_2 \xrightleftharpoons[\tau]{\sigma} q'_3 \xrightarrow{\sigma} \sigma$$

We call the method the *block diagram method*. It is very useful to see that an automaton is accepted. It seems, however, difficult to calculate the behavior.

2. An Automaton Accepted by a Finite Branching Automaton

In this section we shall establish a necessary and sufficient condition for an accessible Σ -automaton to be accepted by a finite branching Σ -automaton $\mathcal{B}$. The condition is especially useful in the case of finite accessible Σ -automata.

Let $\mathcal{B} = (Q, q_0, B)$ be a finite branching Σ -automaton and let $\mathcal{A} = (Q', q'_0, T')$ be an accessible Σ -automaton (not necessarily finite). We adjoin to Q a new state denoted by $\square$ and convert $Q \cup \square$ into a right Σ -module by setting

$$\begin{aligned} q \cdot \sigma &= q\sigma & \text{if } q\sigma \neq \phi, \\ q \cdot \sigma &= \square & \text{if } q\sigma = \phi, \\ \square \cdot \sigma &= \square \end{aligned}$$

for each $\sigma \in \Sigma$.

We denote by $Q''_{\mathcal{B}}$ the set of all vectors

$$(q', q, q'^{-1}Q' \cap \Sigma),$$

where $q' \in Q'$ and $q \in Q$ or $q = \square$. We now convert the set $Q''_{\mathcal{B}}$ into a right Σ -module by setting

$$\begin{aligned} & (q', q, q'^{-1}Q' \cap \Sigma) \cdot \sigma \\ &= \begin{cases} (q'\sigma, q\sigma, \sigma^{-1}q'^{-1}Q' \cap \Sigma) & \text{if } (q'\sigma, q\sigma, \sigma^{-1}q'^{-1}Q' \cap \Sigma) \in Q''_{\mathcal{B}}, \\ \text{undefined,} & \text{otherwise} \end{cases} \end{aligned}$$

for each $\sigma \in \Sigma$.

By setting

$$\begin{aligned} q''_{0\mathcal{B}} &= (q'_0, q_0, q_0^{-1}Q' \cap \Sigma), \\ T''_{\mathcal{B}} &= \{(t', q, t'^{-1}Q' \cap \Sigma) \in Q''_{\mathcal{B}} \mid t' \in T'\}, \end{aligned}$$

we obtain a Σ -automaton $\mathcal{A}''_{\mathcal{B}} = (Q''_{\mathcal{B}}, q''_{0\mathcal{B}}, T''_{\mathcal{B}})$. Then we define $\mathcal{A}'_{\mathcal{B}} = (Q'_{\mathcal{B}}, q'_{0\mathcal{B}}, T'_{\mathcal{B}})$ by the accessible part of $\mathcal{A}''_{\mathcal{B}}$.

Next we convert the set $Q''_{\mathcal{B}}$ into another right Σ -module by setting

$$(q', q, q'^{-1}Q' \cap \Sigma) \cdot \sigma$$

$$= \begin{cases} (q'\sigma, q\sigma, \sigma^{-1}q'^{-1}Q' \cap \Sigma) & \text{if } \begin{cases} q'\sigma \in Q' - T' \text{ and} \\ (q\sigma, \sigma^{-1}q'^{-1}Q' \cap \Sigma) \in B_0, \\ \text{or} \\ q'\sigma \in T' \text{ and} \\ (q\sigma, \sigma^{-1}q'^{-1}Q' \cap \Sigma) \in B_1, \end{cases} \\ \text{undefined,} & \text{otherwise} \end{cases}$$

for each $\sigma \in \Sigma$.

The subset $q_{0\mathcal{A}}$ of $Q''_{\mathcal{A}}$ is defined by

$$q_{0\mathcal{A}} = \begin{cases} \{(q'_0, q_0, q_0'^{-1}Q' \cap \Sigma)\} & \text{if } \begin{cases} q'_0 \in Q' - T' \text{ and } (q_0, q_0'^{-1}Q' \cap \Sigma) \in B_0, \\ \text{or} \\ q'_0 \in T' \text{ and } (q_0, q_0'^{-1}Q' \cap \Sigma) \in B_1, \end{cases} \\ \phi, & \text{otherwise.} \end{cases}$$

Thus we obtain a Σ -automaton $\mathcal{A}''' = (Q''_{\mathcal{A}}, q_{0\mathcal{A}}, T''_{\mathcal{A}})$. Then we define $\mathcal{A}_{\mathcal{A}} = (Q_{\mathcal{A}}, q_{0\mathcal{A}}, T_{\mathcal{A}})$ by the accessible part of $\mathcal{A}'''$.

PROPOSITION 2.1. *If a finite branching Σ -automaton $\mathcal{B} = (Q, q_0, B)$ accepts an accessible Σ -automaton $\mathcal{A} = (Q', q'_0, T')$, then the following (2.1) and (2.2) hold:*

- (2.1) *Let q' be any element of $Q' - T'$. Then we have*
 $(q'_0, q_0, q_0'^{-1}Q' \cap \Sigma) \cdot w = (q'_0w, q_0w, w^{-1}q_0'^{-1}Q' \cap \Sigma) \in Q_{\mathcal{A}} - T_{\mathcal{A}}$
for each $w \in q_0'^{-1}q'$.
- (2.2) *Let t' be any element of T' . Then we have*
 $(q'_0, q_0, q_0'^{-1}Q' \cap \Sigma) \cdot w = (q'_0w, q_0w, w^{-1}Q' \cap \Sigma) \in T_{\mathcal{A}}$
for each $w \in q_0'^{-1}t'$.

PROOF. We prove by induction on the length of word. We first prove the statement for the empty word $1 \in \Sigma^*$. Assume that $q'_0 \in Q' - T'$. Since $\mathcal{A}$ is accepted by $\mathcal{B}$, we have $(q_0w, w^{-1}q_0'^{-1}Q' \cap \Sigma) \in B_0$ for each $w \in q_0'^{-1}q'_0$. Clearly $1 \in q_0'^{-1}q'_0$. Thus we have $(q_0, q_0'^{-1}Q' \cap \Sigma) \in B_0$. This proves that $q'_0 \in Q' - T'$ and $(q_0, q_0'^{-1}Q' \cap \Sigma) \in B_0$. By the definition of $\mathcal{A}_{\mathcal{A}}$ we have $(q'_0, q_0, q_0'^{-1}Q' \cap \Sigma) \in Q_{\mathcal{A}} - T_{\mathcal{A}}$. In the case when $q'_0 \in T'$, we can show in the same way that $(q'_0, q_0, q_0'^{-1}Q' \cap \Sigma) \in T_{\mathcal{A}}$.

Now assume that (2.1) and (2.2) hold for any word of length equal to $n \geq 0$. Let q' be any element of $Q' - T'$. Assume that $w\sigma \in q_0'^{-1}q'$, where $|w|=n$ and $\sigma \in \Sigma$. Clearly $q_0'w \neq \phi$. Then it follows from the assumption of induction that $(q_0', q_0, q_0'^{-1}Q' \cap \Sigma) \cdot w = (q_0'w, q_0w, w^{-1}q_0'^{-1}Q' \cap \Sigma) \in Q_{\mathcal{A}}^{\mathcal{A}}$. Since $\mathcal{A}$ is accepted by $\mathcal{B}$ and since $q' \in Q' - T'$, we have $(q_0w\sigma, \sigma^{-1}w^{-1}q_0'^{-1}Q' \cap \Sigma) \in B_0$. This proves that $q' = (q_0'w)\sigma \in Q' - T'$ and that $((q_0'w)\sigma, \sigma^{-1}(w^{-1}q_0'^{-1})Q' \cap \Sigma) \in B_0$. By the definition of $\mathcal{A}_{\mathcal{A}}$ we have

$$(q_0'w, q_0w, w^{-1}q_0'^{-1}Q' \cap \Sigma) \cdot \sigma = (q_0'w\sigma, q_0w\sigma, \sigma^{-1}w^{-1}q_0'^{-1}Q' \cap \Sigma) \in Q_{\mathcal{A}}^{\mathcal{A}} - T_{\mathcal{A}}^{\mathcal{A}}.$$

This proves that the statement (2.1) holds for each $w \in \Sigma^*$ such that $|w|=n+1$.

In the case when $q' \in T'$, we can also show that the statement (2.2) holds for each $w \in \Sigma^*$ such that $|w|=n+1$. This proves (2.1) and (2.2).

PROPOSITION 2.2. *A finite branching Σ -automaton $\mathcal{B}=(Q, q_0, B)$ accepts an accessible Σ -automaton $\mathcal{A}=(Q', q_0', T')$ if and only if $\mathcal{A}_{\mathcal{A}}=\mathcal{A}_{\mathcal{A}}$.*

PROOF. Assume that $\mathcal{A}$ is accepted by $\mathcal{B}$. We first show that $Q_{\mathcal{A}}^{\mathcal{A}}=Q_{\mathcal{A}}^{\mathcal{A}}$. By the definition it is trivial that $Q_{\mathcal{A}}^{\mathcal{A}} \subset Q_{\mathcal{A}}^{\mathcal{A}}$. Let $q_{\mathcal{A}}^{\mathcal{A}}=(q_0', q_0, q_0'^{-1}Q' \cap \Sigma) \cdot w$ be any element of $Q_{\mathcal{A}}^{\mathcal{A}}$. Then $q_0'w \neq \phi$. Thus we have $w \in q_0'^{-1}(q_0'w)$. By Proposition 2.1 we have

$$(q_0', q_0, q_0'^{-1}Q' \cap \Sigma) \cdot w = (q_0'w, q_0w, w^{-1}q_0'^{-1}Q' \cap \Sigma) \in Q_{\mathcal{A}}^{\mathcal{A}}.$$

Consequently, $Q_{\mathcal{A}}^{\mathcal{A}}=Q_{\mathcal{A}}^{\mathcal{A}}$. Proposition 2.1 shows that $\mathcal{A}_{\mathcal{A}}$ and $\mathcal{A}'_{\mathcal{A}}$ have the same transitions. It can be also verified that $q_{0\mathcal{A}}^{\mathcal{A}}=q_{0\mathcal{A}}^{\mathcal{A}}$ and that $T_{\mathcal{A}}^{\mathcal{A}}=T_{\mathcal{A}}^{\mathcal{A}}$. Thus $\mathcal{A}_{\mathcal{A}}=\mathcal{A}'_{\mathcal{A}}$.

Conversely assume that $\mathcal{A}_{\mathcal{A}}=\mathcal{A}'_{\mathcal{A}}$. Let q' be any element of $Q' - T'$. Then we have $(q_0'w, q_0w, w^{-1}q_0'^{-1}Q' \cap \Sigma) \in Q_{\mathcal{A}}^{\mathcal{A}}$ for each $w \in q_0'^{-1}q'$. Since $Q_{\mathcal{A}}^{\mathcal{A}}=Q_{\mathcal{A}}^{\mathcal{A}}$, we have $(q_0'w, q_0w, w^{-1}q_0'^{-1}Q' \cap \Sigma) \in Q_{\mathcal{A}}^{\mathcal{A}} - T_{\mathcal{A}}^{\mathcal{A}}$. By the definition of $\mathcal{A}_{\mathcal{A}}$ we have $(q_0w, q_0'^{-1}Q' \cap \Sigma) \in B_0$ for each $w \in q_0'^{-1}q'$.

In the case when $t' \in T'$, we can show in the same way that $(q_0w, t'^{-1}Q' \cap \Sigma) \in B_1$ for each $w \in q_0'^{-1}t'$. This proves that $\mathcal{A}$ is accepted by $\mathcal{B}$.

PROPOSITION 2.3. *If an accessible Σ -automaton $\mathcal{A}=(Q', q_0', T')$ is accepted by a finite branching Σ -automaton $\mathcal{B}=(Q, q_0, B)$, then $|\mathcal{A}|=|\mathcal{A}_{\mathcal{A}}|$, where $|\mathcal{A}|$ means the behavior of $\mathcal{A}$ in the ordinary automata theory.*

PROOF. By Proposition 2.2 we have $\mathcal{A}_{\mathcal{A}}=\mathcal{A}'_{\mathcal{A}}$. Therefore

$$|\mathcal{A}_{\mathcal{A}}|=|\mathcal{A}'_{\mathcal{A}}|=q_{0\mathcal{A}}^{\mathcal{A}}{}^{-1}T_{\mathcal{A}}^{\mathcal{A}}$$

$$\begin{aligned}
&= \{w \in \Sigma^* | (q'_0 w, q_0 w, w^{-1} q'^{-1} Q' \cap \Sigma) \in T'_{\mathcal{A}}\} \\
&= \{w \in \Sigma^* | q'_0 w \in T'\} \\
&= |\mathcal{A}|.
\end{aligned}$$

COROLLARY 2.4. *Let $\mathcal{B}$ be a finite branching Σ -automaton. Then*

$$(2.3) \quad |\mathcal{B}| = \{|\mathcal{A}_{\mathcal{B}}| \mid \mathcal{A} \text{ is accepted by } \mathcal{B}\},$$

$$(2.4) \quad \|\mathcal{B}\| = \{|\mathcal{A}_{L_{\mathcal{B}}}| \mid \mathcal{A}_L \text{ is the minimal automaton of a language } L \text{ which is strictly accepted by } \mathcal{B}\}.$$

PROPOSITION 2.5. *Let $\mathcal{B} = (Q, q_0, B)$ be a finite branching Σ -automaton. If $\mathcal{A} = (Q', q'_0, T')$ is a trim Σ -automaton accepted by $\mathcal{B}$, then so is $\mathcal{A}_{\mathcal{B}}$.*

PROOF. First we prove that $\mathcal{A}_{\mathcal{B}}$ is trim. Let $(q', q, q'^{-1} Q' \cap \Sigma)$ be any element of $Q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}}$. Since $\mathcal{A}_{\mathcal{B}}$ is accessible, there exists $w \in \Sigma^*$ such that $q'_0 w = q'$ and that $q_0 w = q$. Since $\mathcal{A}$ is trim, there exists $v \in \Sigma^*$ such that $q'_0 w v \in T'$. It follows from Proposition 2.1 that

$$\begin{aligned}
&(q'_0 w, q_0 w, w^{-1} q'^{-1} Q' \cap \Sigma) \cdot v \\
&= (q'_0 w v, q_0 w v, v^{-1} w^{-1} q'^{-1} Q' \cap \Sigma) \in T'_{\mathcal{A}}.
\end{aligned}$$

Thus $\mathcal{A}_{\mathcal{B}}$ is coaccessible. Therefore $\mathcal{A}_{\mathcal{B}}$ is trim.

Let $q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}} = (q', q, q'^{-1} Q' \cap \Sigma)$ be any element of $Q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}}$. If $q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}} \in Q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}} - T'_{\mathcal{A}_{\mathcal{B}}}$, then $q' \notin T'$. It follows from Proposition 2.1 that $(q', q, q'^{-1} Q' \cap \Sigma) = (q'_0 w, q_0 w, w^{-1} q'^{-1} Q' \cap \Sigma)$ for each $w \in q_0^{-1} q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}}$. Since $q' = q'_0 w \notin T'$, we have $(q_0 w, q'^{-1} Q' \cap \Sigma) \in B_0$. It follows from Proposition 2.2 that

$$\begin{aligned}
q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}-1} Q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}} \cap \Sigma &= \{\sigma \in \Sigma \mid (q', q, q'^{-1} Q' \cap \Sigma) \cdot \sigma \in Q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}}\} \\
&= \{\sigma \in \Sigma \mid (q', q, q'^{-1} Q' \cap \Sigma) \cdot \sigma \in Q'_{\mathcal{A}_{\mathcal{B}}}\} \\
&= \{\sigma \in \Sigma \mid (q' \sigma, q \sigma, \sigma^{-1} q'^{-1} Q' \cap \Sigma) \in Q'_{\mathcal{A}_{\mathcal{B}}}\} \\
&= \{\sigma \in \Sigma \mid q' \sigma \in \phi\} \\
&= q'^{-1} Q' \cap \Sigma.
\end{aligned}$$

Thus we have

$$(q_0 w, q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}-1} Q_{\mathcal{A}_{\mathcal{B}}}^{\mathcal{A}} \cap \Sigma) \in B_0.$$

In the case when $q_{\mathcal{A}} \in T_{\mathcal{A}}$, we can also show in the same way that $(q_0 w, q_{\mathcal{A}}^{-1} Q_{\mathcal{A}} \cap \Sigma) \in B_1$ for each $w \in q_0 \mathcal{A}^{-1} q_{\mathcal{A}}$. This proves that $\mathcal{A}_{\mathcal{A}}$ is accepted by $\mathcal{B}$.

Note that $\|\mathcal{B}\| \subseteq |\mathcal{B}|$ for any finite branching Σ -automaton $\mathcal{B}$. The equality, however, does not necessarily hold.

EXAMPLE 2.6. With $\Sigma = \{\sigma, \tau\}$ let $\mathcal{B} = (Q, q_0, B)$ be the following finite branching Σ -automaton:

$$B = \langle B_0, B_1 \rangle : B_0 = \{(q_0, \{\sigma, \tau\}), (q_1, \{\sigma, \tau\}), (q_2, \phi)\},$$

$$B_1 = \{(q, \phi)\}.$$

Consider the language $L = \{\sigma\tau, \tau\}$. The minimal automaton of L is as follows:

We use the blocks described in Example 1.3 instead of vectors. Then $\mathcal{A}'_{L, \mathcal{B}}$ and $\mathcal{A}_{L, \mathcal{B}}$ are as follows:

It is manifest that $\mathcal{A}'_{L, \mathcal{B}} \cong \mathcal{A}_{L, \mathcal{B}}$. Thus $L \in \|\mathcal{B}\|$. On the other hand consider the Σ -automaton $\mathcal{A}'$ with $|\mathcal{A}'| = L$:

Using the block diagram method, it can be verified that $\mathcal{A}'$ is accepted by $\mathcal{B}$.

3. Tree Productions

Let $\Omega = (\Omega_V, \Omega_E)$ be a pair of (finite) sets called labelling alphabet. An Ω -labelled directed graph G consists of two sets (not necessarily finite) G_E and G_V with four mappings $\partial_0: G_E \rightarrow G_V$, $\partial_1: G_E \rightarrow G_V$, $m_E: G_E \rightarrow \Omega_E$ and $m_V: G_V \rightarrow \Omega_V$. In pictorial representations of labelled directed graphs, nodes will be indicated by their labels. An edge e with $\partial_0(e) = v_0$, $\partial_1(e) = v_1$, $m_V(v_0) = A_0$, $m_V(v_1) = A_1$ and $m_E(e) = \sigma$ will be indicated by an arrow pointing from A_0 to A_1 with label σ .

Let $\mathcal{B} = (Q, q_0, B)$ be a finite branching Σ -automaton. We define Γ_0 and Γ_1 by

$$\Gamma_0 = \{(q, A, 0) | (q, A) \in B_0\} \text{ and}$$

$$\Gamma_1 = \{(q, A, 1) | (q, A) \in B_1\}, \text{ respectively.}$$

We define an alphabet $\Omega_{\mathcal{B}} = (\Omega_V, \Omega_E)$, called the labelling alphabet of $\mathcal{B}$, by

$$\Omega_V = \Gamma_0 \cup \Gamma_1,$$

$$\Omega_E = \Sigma.$$

Next we define a set $\mathcal{BGS}$ as follows: The elements of $\mathcal{BGS}$ are $\Omega_{\mathcal{B}}$ -labelled directed graphs, called tree productions,

such that $A = \{\sigma_1, \dots, \sigma_n\}$ and $(q, A, k_0), (q \sigma_i, A_i, k_i) \in \Omega_V$ for $i = 1, \dots, n$. Note that if $A = \phi$, then the $\Omega_{\mathcal{B}}$ -labelled directed graph (q, A, k_0) without edge belongs to $\mathcal{BGS}$.

Given two tree productions T_1 and T_2 of $\mathcal{BGS}$,

T_2 is said to be composable to T_1 if there exists a node of T_1 with a label $(q', A'$,

$k')$ and the composition is written as follows:

Note that we can not compose any tree production to a tree production without edge. Using these tree productions we may construct an $\Omega_{\mathcal{B}}$ -labelled directed tree as follows: Let G_0 be an element of $\mathcal{BGS}$. Compose an element of $\mathcal{BGS}$ to G_0 if possible. Thus we obtain an $\Omega_{\mathcal{B}}$ -labelled directed tree G_1 . Next compose an element of $\mathcal{BGS}$ to G_1 in the same way. Proceeding the procedure as possible as we can, there results the required $\Omega_{\mathcal{B}}$ -labelled directed graph (possibly infinite). We call the tree the *pseudoprimitive tree* generated by $\mathcal{BGS}$. We denote by $I_{\mathcal{B}}$ and $T_{\mathcal{B}}$ the sets

$$I_{\mathcal{B}} = \{(q_0, A, k) \in \Omega_V \mid q_0 \text{ is the initial state of } \mathcal{B}\} \text{ and}$$

$$T_{\mathcal{B}} = \{(q, A, k) \in \Omega_V \mid k = 1\}, \text{ respectively.}$$

Elements of $I_{\mathcal{B}}$ and $T_{\mathcal{B}}$ are called the initial variables and the terminal variables, respectively. A pseudoprimitive tree is said to be *primitive* if it has a root labelled by an initial variable. A *forest* generated by $\mathcal{BGS}$ is a disjoint union of pseudoprimitive trees.

Let G be a forest generated by $\mathcal{BGS}$. A congruence R on G is an equivalence relation on the set of nodes of G satisfying the following conditions (3.1), (3.2) and (3.3):

(3.1) If vRv' , then the label of v coincides with that of v' .

(3.2) If $v_1Rv'_1$ and $v_1 \xrightarrow{\sigma} v_2$ is an edge of G , then there exists an edge $v'_1 \xrightarrow{\sigma} v'_2$.

(3.3) If $v_1Rv'_1$, $v_1 \xrightarrow{\sigma} v_2$ and $v'_1 \xrightarrow{\sigma} v'_2$, then we have $v_2Rv'_2$.

We may construct an $\Omega_{\mathcal{B}}$ -labelled directed graph from a forest $G = (G_E, G_V, \partial_0, \partial_1, m_E, m_V)$ with a congruence R . We denote by $\tilde{G}_V$ the quotient of G_V mod R and define $\tilde{G}_E, \tilde{\partial}_0, \tilde{\partial}_1, \tilde{m}_V$, and $\tilde{m}_E$ as follows:

$$\tilde{G}_E = \{\tilde{e}: [v] \longrightarrow [v'] \mid e: v \longrightarrow v' \text{ is an edge of } G\},$$

$$\tilde{\delta}_0([v] \longrightarrow [v']) = [v],$$

$$\tilde{\delta}_1([v] \longrightarrow [v']) = [v'],$$

$$\tilde{m}_E([v] \longrightarrow [v']) = m_E(v \longrightarrow v'),$$

$$\tilde{m}_V([v]) = m_V(v),$$

where $[v]$ denotes the equivalence class of $v \bmod R$ and $v \rightarrow v'$ denotes an edge of G . Note that these mappings are well-defined since R is a congruence. We denote the graph by G/R .

If G is a primitive tree, then G/R may be converted into a Σ -automaton as follows: The states of the automaton consist of the nodes of G/R . The initial state is the equivalence class of the root of G and the terminal states consist of the nodes whose labels belong to Γ_1 . The transition is defined by the labelled arrows. The resulting automaton is called the Σ -automaton primitively generated by $\mathcal{BGS}$.

If G is a forest, then we may also convert G/R into a Σ -automaton by designating for the initial state a node whose label is an initial variable. The states, the terminal states and the transition are defined in the same way. The resulting Σ -automaton is called a Σ -automaton generated by $\mathcal{BGS}$.

PROPOSITION 3.1. *Let G be a forest generated by $\mathcal{BGS}$ and let R be a congruence on G . We denote by $\mathcal{A}' = (Q', q'_0, T')$ the Σ -automaton converted from G/R . Let $\mathcal{A}'^a = (Q'^a, q'_0, T'^a)$ be the accessible part of $\mathcal{A}'$. Then there exists a primitive tree T and a congruence R_0 on T such that $\mathcal{A}'^a$ coincides with the Σ -automaton converted from T/R_0 .*

PROOF. Let $q'_0 = [v_0]$ be the initial state of $\mathcal{B}$ and let (q_0, A, k) be the label of v_0 . By the definition there exists a pseudoprimitive tree T' , which is a connected component of G , such that T' has a node v'_0 which is equivalent to $v_0 \bmod R$. Removing the nodes of T' which are not accessible from v'_0 , we obtain a primitive tree T . Let R_0 be the restriction of R to T . It is trivial that R_0 is an equivalence relation and that (3.1) and (3.3) hold. Let $v_1 R_0 v'_1$ and let $v_1 \xrightarrow{\sigma} v_2$ be an edge of T . Since R is a congruence on G , there exists an edge $v'_1 \xrightarrow{\sigma} v'_2$ of G . Since v'_1 is accessible from v'_0 , it follows that v'_2 is also accessible. Thus the edge $v'_1 \xrightarrow{\sigma} v'_2$ belongs to T . This proves that (3.2) holds. Thus R_0 is a congruence on T . Let $\mathcal{A}'' = (Q'', q''_0, T'')$ be the Σ -automaton converted from

T/R_0 . Consider the mapping

$$\varphi: Q'' \longrightarrow Q'^a$$

defined by $\varphi([v]_0) = [v]$, where $[v]_0$ is the equivalence class of $v \bmod R_0$. We assert that φ is an isomorphism [1] of $\mathcal{A}''$ and $\mathcal{A}'^a$. By the definition of R_0 φ is injective. Let $[v]$ be any state of $\mathcal{A}'^a$. Since $\mathcal{A}'^a$ is accessible, there exists a path $c: v_0 \rightarrow v$ in G . Since G is a disjoint union of pseudoprimitive trees, there exists a pseudoprimitive tree which has a path $c'': v_0'' \rightarrow v''$ such that $|c''| = |c|$ ($|c|$ means the label of the path c [1]), $[v_0''] = [v_0]$ and $[v] = [v'']$. Since $v_0 R v_0'$, we obtain inductively a path $c': v_0' \rightarrow v'$ in T such that $|c'| = |c|$ and that $[v''] = [v'] = [v]$. This proves that φ is surjective. To see that φ is a proper state-mapping [1], it suffices to show the followings:

$$(3.4) \quad \varphi([v_0']_0) = [v_0'],$$

$$(3.5) \quad \varphi([v]_0) \cdot \sigma = \varphi([v]_0 \cdot \sigma) \quad \text{if } [v]_0 \cdot \sigma \neq \phi \text{ and } \varphi([v]_0) \cdot \sigma \neq \phi,$$

$$(3.6) \quad \varphi(T'') = T'^a.$$

These can be proved in the same way.

Thus φ is a bijective proper state-mapping. This shows that φ is an isomorphism.

PROPOSITION 3.2. *Let $\mathcal{B} = (Q, q_0, B)$ be a finite branching Σ -automaton and let $\mathcal{A} = (Q', q'_0, T')$ be an accessible Σ -automaton accepted by $\mathcal{B}$. Then the following (3.7) and (3.8) hold:*

$$(3.7) \quad \mathcal{A}_{\mathcal{B}} \text{ is generated by } \mathcal{B}GS.$$

$$(3.8) \quad \text{If } \mathcal{A} \text{ coincides with the accessible } \Sigma\text{-automaton constructed from a primitive tree } T \text{ generated by } \mathcal{B}GS \text{ and a congruence } R \text{ on it, then } \mathcal{A} \text{ is accepted by } \mathcal{B}.$$

PROOF. We denote by GS the set of all tree productions of the following forms:

where $q_{\mathcal{A}}^{\sigma(i)} = q_{\mathcal{A}}^{\sigma} \cdot \sigma_i$ and $\{\sigma_1, \dots, \sigma_n\} = \{\sigma \in \Sigma \mid q_{\mathcal{A}}^{\sigma} \cdot \sigma \neq \phi\}$. Note that if $q_{\mathcal{A}}^{\sigma} \cdot \sigma = \phi$

for each $\sigma \in \Sigma$, then the graph $q_{\mathcal{A}}^{\sigma}$ without edge belongs to GS. The initial variable of GS is defined to be $q_0^{\mathcal{A}}$. With GS and $q_0^{\mathcal{A}}$ we obtain only one primitive tree T . Since the state of $\mathcal{A}_{\mathcal{A}}$ is written in the form $(q', q, q'^{-1}Q' \cap \Sigma)$, we may consider that T is generated by $\mathcal{B}GS$. Let v and v' be the nodes of T with labels $q_{\mathcal{A}}^{\sigma}$ and $q'_{\mathcal{A}}^{\sigma}$, respectively. Then we define a congruence R as follows: vRv' if and only if $q_{\mathcal{A}}^{\sigma} = q'_{\mathcal{A}}^{\sigma}$. These are the required primitive tree and the congruence. It can be seen that the Σ -automaton converted from T/R coincides with $\mathcal{A}_{\mathcal{A}}$.

Next let q' be any element of Q' . For each $w \in q_0'^{-1}q'$, there exists a path $c: v_0 \rightarrow v$ in T , where $|c|=w$, $[v_0]=q_0'$ and $[v]=q'$. Let (q, A, k) be the label of v . Then we have $q'^{-1}Q' \cap \Sigma = A$ and $q = q_0'w$. This shows that $\mathcal{A}$ is accepted by $\mathcal{B}$.

PROPOSITION 3.3. *Let $\mathcal{B}=(Q, q_0, B)$ be a finite branching Σ -automaton and let $\mathcal{A}'=(Q', q_0', T')$ and $\mathcal{A}''=(Q'', q_0'', T'')$ be two trim Σ -automata. If $\mathcal{A}'$ is accepted by $\mathcal{B}$ and $|\mathcal{A}'|=|\mathcal{A}''|$, then $\mathcal{A}''$ is also accepted by $\mathcal{B}$.*

PROOF. It suffices to show that the following (3.9) and (3.10) hold:

(3.9) *Let q'' be any element of $Q'' - T''$. Then we have*

$$(q_0''w, q_0''w, q''^{-1}Q'' \cap \Sigma) \in B_0 \text{ for each } w \in q_0''^{-1}q''.$$

(3.10) *Let t'' be any element of T'' . Then we have*

$$(q''w, q_0''w, t''^{-1}Q'' \cap \Sigma) \in B_1 \text{ for each } w \in q''^{-1}t''.$$

We first show that (3.9) holds. Let q'' be any element of $Q'' - T''$ and let w be any element of $q_0''^{-1}q''$. We shall show that $w^{-1}q_0''^{-1}Q' \cap \Sigma = q'^{-1}Q'' \cap \Sigma$. Let σ be any element of $q''^{-1}Q'' \cap \Sigma$. Then $q'' = q_0''w\sigma \in Q''$. Since $\mathcal{A}''$ is coaccessible, there exists $u \in \Sigma^*$ such that $q_0''w\sigma u \in T''$. Since $|\mathcal{A}'|=|\mathcal{A}''|$, we have $w\sigma u \in |\mathcal{A}'|$. Therefore $q_0''w\sigma u \in \phi$, and thus $\sigma \in w^{-1}q_0''^{-1}Q' \cap \Sigma$. This proves that $q''^{-1}Q'' \cap \Sigma \subset w^{-1}q_0''^{-1}Q' \cap \Sigma$. Conversely let σ be any element of $w^{-1}q_0''^{-1}Q' \cap \Sigma$. Since $\mathcal{A}'$ is coaccessible, there exists $u' \in \Sigma^*$ such that $q_0''w\sigma u' \in T'$. Since $|\mathcal{A}'|=|\mathcal{A}''|$, we have $w\sigma u' \in |\mathcal{A}''|$. Therefore $q_0''w\sigma u' \in \phi$, and thus $\sigma \in w^{-1}q_0''^{-1}Q'' \cap \Sigma$. Since $w^{-1}q_0''^{-1}Q'' \cap \Sigma = q''^{-1}Q'' \cap \Sigma$, it follows that $q''^{-1}Q'' \cap \Sigma = w^{-1}q_0''^{-1}Q' \cap \Sigma$.

Since $q_0''w = q'' \in Q'' - T''$, we have $w \notin |\mathcal{A}'|=|\mathcal{A}''|$. Since $\mathcal{A}''$ is coaccessible, there exists $w' \in \Sigma^*$ such that $q''ww' \in T''$. This implies $q'ww' \in T'$. Thus we have $q_0''w \in Q' - T'$. Since $\mathcal{A}'$ is accepted by $\mathcal{B}$, we have $(q_0''w, w^{-1}q_0''^{-1}Q' \cap \Sigma) \in B_0$. Therefore $(q_0''w, q''^{-1}Q'' \cap \Sigma) \in B_0$. This proves (3.9).

In the same way we can prove (3.10).

COROLLARY 3.4. *If a finite branching Σ -automaton $\mathcal{B}$ accepts a trim Σ -automaton $\mathcal{A}$, then the minimal automaton of the language $|\mathcal{A}|$ is also accepted by $\mathcal{B}$.*

Now we can prove our main theorem.

THEOREM 3.5. *Let $\mathcal{B}$ be a finite branching Σ -automaton. Then*

$$(3.11) \quad \|\mathcal{B}\| = \{|\mathcal{A}| \mid \mathcal{A} \text{ is a trim } \Sigma\text{-automaton generated by } \mathcal{B}GS\},$$

$$(3.12) \quad |\mathcal{B}| = \{|\mathcal{A}| \mid \mathcal{A} \text{ is an accessible } \Sigma\text{-automaton generated by } \mathcal{B}GS\}.$$

PROOF. We define the sets B_1 , B_2 , B_3 and B_4 as follows:

$$B_1 = \{|\mathcal{A}_{L\mathcal{B}}| \mid \mathcal{A}_{L\mathcal{B}} \text{ is the minimal automaton of a language } L \text{ strictly accepted by } \mathcal{B}\},$$

$$B_2 = \{|\mathcal{A}_{\mathcal{B}}| \mid \mathcal{A} \text{ is a trim } \Sigma\text{-automaton accepted by } \mathcal{B}\},$$

$$B_3 = \{|\mathcal{A}| \mid \mathcal{A} \text{ is a trim } \Sigma\text{-automaton primitively generated by } \mathcal{B}GS\},$$

$$B_4 = \{|\mathcal{A}| \mid \mathcal{A} \text{ is a trim } \Sigma\text{-automaton generated by } \mathcal{B}GS\}.$$

It follows from Corollary 3.4 that $\|\mathcal{B}\| = B_1$. Obviously $B_1 \subset B_2$. We shall show that $B_2 \subset B_1$. Let $|\mathcal{A}_{\mathcal{B}}|$ be any element of B_2 . Since $\mathcal{A}$ is a trim Σ -automaton, it follows from Corollary 3.4 that the minimal automaton of $|\mathcal{A}|$ is accepted by $\mathcal{B}$. By Proposition 2.3 we have $|\mathcal{A}| = |\mathcal{A}_{\mathcal{B}}|$. Thus $|\mathcal{A}_{\mathcal{B}}| \in B_1$. This proves that $B_1 = \|\mathcal{B}\| = B_2$.

Next we shall show that $B_2 = B_3$. Let $\mathcal{A}$ be any trim Σ -automaton accepted by $\mathcal{B}$. Then it follows from Proposition 2.5 that $\mathcal{A}_{\mathcal{B}}$ is also a trim Σ -automaton accepted by $\mathcal{B}$. By (3.7) $\mathcal{A}_{\mathcal{B}}$ is a trim Σ -automaton primitively generated by $\mathcal{B}GS$. Thus $|\mathcal{A}_{\mathcal{B}}| \in B_3$ and thus $B_2 \subset B_3$. Conversely it follows from (3.8) and Proposition 2.3 that $B_3 \subset B_2$. This proves that $B_2 = B_3$.

It follows from Proposition 3.1 that $B_3 = B_4$. This proves (3.11).

We can also prove (3.12) in the same way.

4. Acknowledgement

The author wishes to express his gratitude for the guidance and encouragement received from Professor S. Huzino.

References

- [1] S. EILENBERG, *Automata, Languages, and Machines*, Vol. A, Academic Press, 1974.
- [2] I. M. HAVEL, *Finite Branching Automata*, KYBERNETIKA 10 (1974), 281–302.