FUNCTORIAL RELATIONS BETWEEN SYSTEMATICS

By

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In [1] and [2], H. Ehrig et al. presented the general concept "Systematic" which can represent a lot of systems and automata by its ability of describing the external behaviors of classes of systems. In this report, we shall discuss on functors expressing simulations in a sense.

§1. Systematics

Consider the class of non-initialized automata of Mealy type. We denote an automaton of the class by $A=(S, d, l)$, where input and output sets are fixed. These automata and automata homomorphisms form a category of automata, which we denote by **Aut**. A behavior of an automaton A is defined by $E(A) = \{l^+(s, -): I^+ \rightarrow O\} \subseteq \langle I^+, O \rangle$, where $l^+: S \times I^+ \rightarrow O$ is the extension of the output function $l: S \times I \rightarrow O$. It is easy to see that E can be extended to a functor $E: \mathbf{Aut} \rightarrow \mathbf{B}$, where **B** is the partially ordered category whose objects are behaviors for some automaton and morphisms are inclusions. Thus we have the following triple (or a special functor)

$$\mathfrak{S} = (\mathbf{Aut}, \mathbf{B}, E).$$

We want to call such a triple "Systematic".

DEFINITION 1.1. A systematic is a triple $\mathfrak{S}=(\mathbf{S}, \mathbf{B}, E)$, where **S** is a category (called a system category), **B** is a partially ordered class (called a behavior category), E is a functor $E: \mathbf{S} \rightarrow \mathbf{B}$ (called a behavior functor).

In many cases, classes of automata and systems and their external behaviors can be described by such systematics.

Now we define "*Minimal Realizing Principle*" on this general setting.

An **S**-object of the system category **S** is called minimal, if for every **S**-

object S' such that $E(S') \subseteq E(S)$, there exists a unique $\mathbf{S}$ -morphism $\phi: S' \rightarrow S$. A functor $M^*: \mathbf{B} \rightarrow \mathbf{S}$ is called a minimal realizing functor (m.r.f. for short) if M^* is a right inverse and a right adjoint to the behavior functor E . And, moreover, if such a m.r.f. exists, we say that the systematic $\mathfrak{S}$ satisfies a minimal realizing principle (M. R. P. for short).

Remark that we can obtain a minimal realization for a given behavior $b \in \mathbf{B}$ by using a m.r.f., if $\mathfrak{S}$ satisfies the M. R. P..

§2. Examples from automata theory

In order to discuss a functorial relation between systematics, we examine a few systematics of automata. Let $\mathbf{ND-Aut}$ be the category of non-deterministic automata with initial states.

Strictly speaking, objects of $\mathbf{ND-Aut}$ are $A = (S, d, l, a)$, where, d, l, a are non-deterministic functions (n.d.f. for short), and automata homomorphisms are n.d.f.

Of course, input and output sets are fixed.

A behavior of an object of an $\mathbf{ND-Aut}$ A is defined as

$$E^{nd}(A) := (I^+ \simeq \{1\} \times I^+ \xrightarrow{a \times I^+} S \times I^+ \xrightarrow{l^+} O).$$

Let $\mathbf{B}^{nd}$ be the discrete category whose objects are n.d.f. with domains I^+ and codomains O .

Since $\mathbf{B}^{nd}$ is discrete, it is obvious that E^{nd} forms a functor $E^{nd}: \mathbf{ND-Aut} \rightarrow \mathbf{B}^{nd}$. In §1, we have introduced the systematic of non-initialized (deterministic) automata $\mathfrak{Aut}$. In a similar way, we have a systematic $\mathfrak{Aut}^{ini} = (\mathbf{Aut}^{ini}, \mathbf{B}^{ini}, E^{ini})$ of initialized (deterministic) automata, whose behavior is $E^{ini}((S, d, l, a)) = l^+(a(1), -): I^+ \rightarrow O$. Remark that $\mathbf{B}^{ini}$ is the discrete category whose objects are functions $I^+ \rightarrow O$.

The systematics $\mathfrak{Aut}^{ini}$ and $\mathfrak{ND-Aut}$ are the first examples of our discussions. It is a well known fact that we can construct an equivalent initial deterministic automaton for a given initial NDA. That is, for each object $A = (S, d, l, a) \in \mathbf{ND-Aut}$, an object in $\mathbf{Aut}^{ini}$ $PA = (PS, \bar{d}, \bar{l}, \bar{a})$ is equivalent to A , where $\bar{a}(1) := a(1)$, $\bar{d}(M, x) := \bigcup_{s \in M} d(s, x)$, $\bar{l}(M, x) := \bigcup_{s \in M} l(s, x)$ for a non-empty subset M of S . Moreover, for each morphism $(f: A \rightarrow A') \in \mathbf{ND-Aut}$, we define $Pf: PA \rightarrow PA'$ as $Pf(M) := f(M)$ for $M \in PS$. By PS we mean a power set of S which does not include the empty set. We can see that P is a functor from $\mathbf{ND-Aut}$ to $\mathbf{Aut}^{ini}(PO)$. Equivalences are described by $E^{nd}(A) = V \circ E^{ini}(PA)$ in this case,

where $V: PO \rightarrow O$ is defined by $V(M) = M$ for $M \in PO$ (V is a n.d.f.: $PO \rightarrow O$).

Connection of two behavior categories $\mathbf{B}^{ini}(PO)$ and $\mathbf{B}^{nd}$ is the functor $V: \mathbf{B}^{ini}(PO) \rightarrow \mathbf{B}^{nd}$, where $V(f: I^+ \rightarrow PO) := (V \circ f: I^+ \rightarrow O) \in \mathbf{B}^{nd}$. (See (2.1).) Thus we have the following situation as displayed by (2.2):

$$(2.1) \quad \begin{array}{ccc} I^+ & \xrightarrow{f} & PO \\ & \searrow \scriptstyle v(f) & \downarrow \scriptstyle V \\ & & O \end{array} \quad \begin{array}{c} \\ \\ \scriptstyle := \end{array}$$

The “*Simulator*” P defined above becomes a right adjoint functor to J , where J

$$(2.2) \quad \begin{array}{ccc} \mathbf{Aut}^{ini}(PO) & \xrightarrow{E^{ini}} & \mathbf{B}^{ini}(PO) \\ \uparrow \scriptstyle P & \curvearrowright & \downarrow \scriptstyle V \\ \mathbf{ND-Aut} & \xrightarrow{E} & \mathbf{B} \end{array}$$

is the inclusion functor $J: \mathbf{Aut}^{ini}(PO) \rightarrow \mathbf{ND-Aut}$ defined by $J(S, d, l, a) := (S, d, l, a)$, ($l = V \circ l: S \times I \rightarrow O$) $\in \mathbf{ND}$. In other word, a power automaton PA is cofree over A with respect to J . In general, adjunctions frequently appears when we correspond a class of machines or systems to another one. This property is typical for the categorical machine theory.

Now we give a general idea of this report, using the equation given by (2.2). The functor P , which assigns a power automaton for each initial non-deterministic automaton, is a simulator, comparing behaviors of A and PA by the functor $V: \mathbf{B}^{ini}(PO) \rightarrow \mathbf{B}$. Thus PA simulates A “with respect to V ”, that is, in a broad sense, the behavior of $A \in \mathbf{ND-Aut}$ can be obtained by using P and V , and $E^{ini}(PA)$ represents the external behavior of A .

Our generalizations or viewpoints are as follows:

- (1) functors between system categories are obtained by “effective” constructions.
- (2) functors between behavior categories describe the relationship of external behaviors of different two classes of systems.

§ 3. Simulations of systematics

Now we give formal definitions.

Let $\mathfrak{S}_i = (\mathbf{S}_i, \mathbf{B}_i, E_i)$ be systematics. Unless otherwise stated, all systematics are assumed to be realizing: that is, for each behavior $B \in \mathbf{B}$, there exists a system which realizes B .

DEFINITION 3.1. A functor $F: \mathbf{S}_1 \rightarrow \mathbf{S}_2$ is called a (type 1) simulator with respect to $G: \mathbf{B}_1 \rightarrow \mathbf{B}_2$ if, for all object $S \in \mathbf{S}$, $GE_1S \subseteq E_2FS$ holds. (refer to (3.1)).

Similarly, a (type 2) simulator $F: \mathbf{S}_1 \rightarrow \mathbf{S}_2$ with respect to $G: \mathbf{B}_2 \rightarrow \mathbf{B}_1$ is defined (refer to (3.2)).

$$\begin{array}{ccc} \mathbf{S}_2 & \xrightarrow{E_2} & \mathbf{B}_2 \\ F \uparrow & & \uparrow G \\ \mathbf{S}_1 & \xrightarrow{E_1} & \mathbf{B}_1 \end{array}$$

(type 1)

$$\begin{array}{ccc} \mathbf{S}_2 & \xrightarrow{E_2} & \mathbf{B}_2 \\ F \uparrow & & \downarrow G \\ \mathbf{S}_1 & \xrightarrow{E_1} & \mathbf{B}_1 \\ E_1S \subseteq GE_2FS. \end{array}$$

(type 2)

In addition, a type 1 [type 2] simulator is called strict if (3.1) [(3.2)] is commutative. Remark that so called “behavior preserving functor” is the case that $\mathbf{B}_1 \cong \mathbf{B}_2$ and the simulator is strict.

PROPOSITION 3.1. A type 1 (2) simulation is closed under compositions, where compositions of simulations are defined by compositions of each functors.

PROPOSITION 3.2. If $\mathfrak{S}_2$ satisfies M.R.P. and there exists a functor $G: \mathbf{B}_1 \rightarrow \mathbf{B}_2$, then we have a type 1 strict simulator with respect to G .

PROOF.

$$\begin{array}{ccc} \mathbf{S}_2 & \xrightarrow{E_2} & \mathbf{B}_2 \\ F \uparrow & \xleftarrow{M_2^*} & \uparrow G \\ \mathbf{B}_1 & \xrightarrow{E_1} & \mathbf{S}_1 \end{array}$$

Take $F := M_2^*GE_1$, then
 $E_2F = E_2M_2^*GE_1 = GE_1$
 where M_2^* is a m.r.f. of $\mathbf{S}_2$.

EXAMPLE. The systematic of initial reachable automata of Mealy type satisfies M.R.P.. Let $\mathbf{B}_1$ be a behavior category of initial reachable automata of Moore type which is similarly defined, where the objects are morphisms from I^* to O . The restriction $G(f: I^* \rightarrow O) = (f': I^+ \rightarrow O)$ becomes a functor which connects behavior categories. Thus we know the existence of a simulator F .

PROPOSITION 3.3. If $\mathfrak{S}_1$ satisfies M.R.P. and $F: \mathbf{S}_1 \rightarrow \mathbf{S}_2$ is a strict

simulator with respect to $G: \mathbf{B}_2 \rightarrow \mathbf{B}_1$ which preserves minimal objects, and G satisfies the property that $Gb = Gb'$ implies $b = b'$ for any $b, b' \in \mathbf{B}_2$, then $\mathfrak{S}_2$ also satisfies M.R.P.

PROOF.

$$\begin{array}{ccc}
 \mathbf{S}_2 & \xrightleftharpoons[E_2]{E_2} & \mathbf{B}_2 \\
 \uparrow F & \begin{array}{c} M_2^* \\ M_1^* \end{array} & \downarrow G \\
 \mathbf{S}_1 & \xrightleftharpoons[E_1]{E_1} & \mathbf{B}_1
 \end{array}$$

Take $M_2^* := FM_1^*G$, then
 $GE_2M_2^* = GE_2FM_1^*G$
 $= E_1M_1^*G = G.$

By the assumption, $E_2M_2^*b = b$ for each $b \in \mathbf{B}_2$, and since $\mathbf{B}_2$ is partially ordered, $E_2M_2^* = 1_{\mathbf{B}_2}$. Moreover M_2^*b is minimal for each object $b \in \mathbf{B}_2$. Thus, M_2^* is a m.r.f. for $\mathfrak{S}_2$.

PROPOSITION 3.4. If F is a strict simulator (type 1) with respect to $G: \mathbf{B}_1 \rightarrow \mathbf{B}_2$ which preserves minimal systems, and $\mathfrak{S}_1$ satisfies M.R.P. and G satisfies the property that for each $b \in \mathbf{B}_2$ there exists a $\mathbf{B}_1$ -object b' such that $Gb' = b$, then $\mathfrak{S}_2$ also satisfies M.R.P.

PROOF. For each $\mathbf{B}_2$ -object b , there exists a $\mathbf{B}_1$ -object b' such that $Gb' = b$. The object FM_1^*b' is the minimal $\mathbf{S}_2$ -object, and $E_2FM_1^*b' = GE_1M_1^*b'$. Thus FM_1^*b' realizes b . We define a subsystematic of $\mathfrak{S}_2$ as follows:

Collect such minimal objects FM_1^*b' which realize b for each $\mathbf{B}_2$ -object b .

This subsystematic is minimal and realizing and hence $\mathfrak{S}_2$ satisfies M.R.P.

In Propositions 3.3 and 3.4, the functor, which preserves minimal objects, is essential. Hence we examine the condition for functors to preserve minimal objects.

§4. Minimality preserving functors

In the following, we restrict type 1 (2) simulator to the case of $\mathbf{B}_1 \cong \mathbf{B}_2$ (or $\mathbf{B}_1 = \mathbf{B}_2$).

PROPOSITION 4.1. Let $R: \mathbf{S}_2 \rightarrow \mathbf{S}_1$ be a right adjoint functor to $L: \mathbf{S}_1 \rightarrow \mathbf{S}_2$ and L is a strict simulator and R is a simulator: that is, $E_1RB \cong E_2B$. Then if $B \in \mathbf{S}_2$ is minimal, then RB is also minimal.

PROOF.

We first establish that, if $E_1 A \subseteq E_1 R B$, then $E_2 L A \subseteq E_2 B$. Let $\eta: LR \rightarrow 1_{S_2}$ be the counit of the adjunction. The morphism $\eta_B: LRB \rightarrow B$ is a S_2 morphism, hence $E_1 R B = E_2 L R B \subseteq E_2 B$. Since $E_2 B \subseteq E_1 R B$, $E_1 R B = E_2 B$. Finally we have $E_2 L A = E_1 A \subseteq E_1 R B = E_2 B$.

Since B is minimal, $\text{Mor}_{S_1}(A, RB) \cong \text{Mor}_{S_2}(LA, B)$ has exactly one morphism. That is, RB is also minimal.

REMARK. Under the same assumptions of Proposition 4.1, a left adjoint functor $L: S_1 \rightarrow S_2$ does not preserve minimal systems in general. Consider the next example. Let $\mathbf{Aut}_e^{ini}$ be a full subcategory of $\mathbf{Aut}^{ini}$ whose objects are reachable automata. A functor $C: \mathbf{Aut}^{ini} \rightarrow \mathbf{Aut}_e^{ini}$, which takes the reachable part of $\mathbf{Aut}^{ini}$ -object, is right adjoint to an inclusion functor $J: \mathbf{Aut}_e^{ini} \rightarrow \mathbf{Aut}^{ini}$, which preserves behavior. The automaton A defined by (4.1) is minimal in $\mathbf{Aut}_e^{ini}$, but not minimal in $\mathbf{Aut}^{ini}$, because there is no homomorphism whose domain is an automaton B defined by (4.2).

$$(4.1) \quad A: \longrightarrow s_1 \circ x \setminus y_1$$

$$(4.2) \quad B:$$

THEOREM 4.2. If $\mathfrak{S}_1$ satisfies M.R.P., the following conditions are equivalent:

- (1) $\mathfrak{S}_2$ satisfies M.R.P.,
- (2) there exists a strict simulator $F: S_1 \rightarrow S_2$ which preserves minimal systems.

Proof of the theorem depends on the Propositions 3.2 and 3.3. And, moreover, the following sufficient condition holds.

PROPOSITION 4.3. If $F: S_1 \rightarrow S_2$ preserves behaviors and, for each $B \in S_2$, there exists an object A such that $FA = B$, and moreover, for every S_1 -objects A and A' , $\text{Mor}_{S_1}(A, A') \cong \text{Mor}_{S_2}(FA, FA')$, then F preserves minimal objects.

References

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