DOMAIN WITH A VANISHING COHOMOLOGY SET IN A PRODUCT MANIFOLD OF AN OPEN RIEMANN SURFACE AND A ONE DIMENSIONAL COMPLEX TORUS

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(Received May 20, 1977)

1. Introduction

By Cartan [3] and Behnke-Stein [2], any Cousin-I domain in $\mathbb{C}^2$ is a domain of holomorphy. By Thullen [11], Cousin-II domains in $\mathbb{C}^2$ are not necessarily domains of holomorphy. The author and Kazama [8] proved, however, that a domain Ω in a two dimensional Stein manifold is a Stein manifold if there exists a complex Lie group L with $H^1(\Omega, \mathcal{A}_L) = 0$ where $\mathcal{A}_L$ is the sheaf of all germs of holomorphic mappings in L . Recently, inspired by a work [4] of Dloussky, the author and Watanabe [9] proved that a proper subdomain Ω of the complex projective 2-space $\mathbb{P}_2$ is a Stein manifold if there exists a complex Lie group L with $H^1(\Omega, \mathcal{A}_L) = 0$. The aim of the present paper is to prove that a domain Ω of a product manifold of an open Riemann surface R and a one dimensional complex torus T is either a Stein manifold or satisfies $\Omega = \Delta \times T$ for a subdomain Δ of R if there exists a complex Lie group L with $H^1(\Omega, \mathcal{A}_L) = 0$. In the proof we use some generalizations of lemmata in Kajiwara-Kazama [8] and the affirmative solution of the Levi problem in $R \times T$ obtained by Matsugu [10].

2. Pseudoconvex domain in $R \times T$

Let M be a complex manifold. An open subset Ω of M is called a pseudoconvex domain in M if, for any Stein subdomain D of M , $D \cap \Omega$ is a Stein manifold. In this paper, we denote exclusively by ω_1, ω_2 two complex numbers which are linearly independent over the real number field $\mathbb{R}$. The numbers ω_1, ω_2 generate a subgroup Γ of the complex number field $\mathbb{C}$, namely

$$\Gamma = \{m_1\omega_1 + m_2\omega_2; m_1, m_2 \in \mathbf{Z}\}$$

where $\mathbf{Z}$ is the additive group of integers. We denote exclusively by T the complex torus $\mathbf{C}/\Gamma$. The one dimensional complex torus T has a natural complex structure so that the canonical mapping $\tau: \mathbf{C} \rightarrow T$ is a covering mapping. We denote by R a connected open Riemann surface and by $\pi: R \times T \rightarrow R$ the projection. Let ψ_1 and ψ_2 be, respectively, mappings of sets X_1 and X_2 in sets Y_1 and Y_2 . We denote by $\psi_1 \times \psi_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ the mapping defined by $(\psi_1 \times \psi_2) \cdot (x_1, x_2) = (\psi_1(x_1), \psi_2(x_2))$ for $(x_1, x_2) \in X_1 \times X_2$.

LEMMA 1. *Let Ω be a pseudoconvex subdomain of $R \times T$. If there exists a point x_0 of R such that $\pi^{-1}(\{x_0\}) = \{x_0\} \times T$, then there exists a subdomain Δ of R such that $\Omega = \Delta \times T$.*

PROOF. Let S be the set of all points x of $\pi(\Omega)$ such that $\pi^{-1}(x) = \{x\} \times T$. Since T is compact, S is an open subset of $\pi(\Omega)$. Let x be an adherent point of S in $\pi(\Omega)$. There exists $t \in T$ such that $(x, t) \in \Omega$. There exist an open connected neighborhood U of x , an open connected neighborhood V of t and a biholomorphic mapping ψ of the disc $D = \{z \in \mathbf{C}; |z| < 1\}$ onto U with $\psi(0) = y$ such that $U \times V \subset \Omega$. Since $U \cap S \neq \emptyset$, there exists an open disc $D' = \{z \in \mathbf{C}; |z - z_0| < \varepsilon\}$ contained in D with $\psi(D') \subset S$. For the locally biholomorphic mapping $\psi \times \tau: D \times \mathbf{C} \rightarrow U \times T$, $(\psi \times \tau)^{-1}(\Omega \cap (U \times T))$ is a pseudoconvex domain in $\mathbf{C}^2$ containing $(D' \times \mathbf{C}) \cup (D \times \tau^{-1}(V))$. Therefore, we have $(\psi \times \tau)^{-1}(\Omega \cap (U \times T)) \supset D \times \mathbf{C}$. Hence we have $U \subset S$. Since S is a non-empty closed and open subset of the connected topological space $\pi(\Omega)$, we have $\pi(\Omega) = S$.

LEMMA 2. *Let Ω be a pseudoconvex subdomain of $R \times T$. Then, either Ω is a Stein manifold or there exists an open subset Δ of R with $\Omega = \Delta \times T$.*

PROOF. By Lemma 1, we may assume that $\pi^{-1}(\{x\}) \not\subseteq \{x\} \times T$ for any $x \in R$. Then, by Matsugu [10], Ω is a Stein manifold.

3. Dual of the theorem of identity

LEMMA 3. *Let $H(x, y)$ be a continuous function in the product $[0, 1] \times [0, 1]$ of closed intervals in $\mathbf{R}$. We define a sequence $\{H_{pq}(x, y); p, q = 0, 1, 2, \dots\}$ of continuous functions in $[0, 1] \times [0, 1]$ by $H_{00}(x, y) = H(x, y)$ and*

$$H_{p+1q}(x, y) = \int_0^x H_{pq}(s, y) ds$$

$$H_{p+q+1}(x, y) = \int_0^y H_{pq}(x, t) dt$$

for $p, q \geq 0$. If $H_{pq}(1, 1) = 0$ for any $p, q \geq 1$, we have $H(x, y) = 0$ for any $(x, y) \in [0, 1] \times [0, 1]$.

PROOF. We have

$$H_{pq}(x, y) = \int_0^x \int_0^y \frac{(x-s)^{p-1} (y-t)^{q-1}}{(p-1)!(q-1)!} H(s, t) ds dt$$

for any $(x, y) \in [0, 1] \times [0, 1]$ and $p, q \geq 1$. Since $H_{pq}(1, 1) = 0$ for any $p, q \geq 1$, we have

$$\int_0^1 \int_0^1 (1-s)^{p-1} (1-t)^{q-1} H(s, t) ds dt = 0$$

for any $p, q \geq 1$. Since $\{(1-s)^p (1-t)^q; p, q \geq 0\}$ is a complete system in the space of all classes of square integrable functions in $[0, 1] \times [0, 1]$, we have $H(s, t) = 0$ for any $(s, t) \in [0, 1] \times [0, 1]$.

LEMMA 4. Let $\{b_{ij}; i, j = 0, 1, 2, \dots\}$ be a sequence of complex numbers such that, for positive numbers M and r , we have $|b_{ij}| \leq M r^{i+j}$ for any $i, j \geq 0$. Let α and β be positive integers. If

$$\sum_{\substack{\alpha m \geq p \\ \beta n \geq q \\ m, n \geq 0}} \frac{b_{\alpha m - p, \beta n - q}}{m! n!} = 0$$

for any $p, q \geq 1$, we have $b_{ij} = 0$ for any $i, j \geq 0$.

PROOF. We put

$$H_{pq}(z_1, z_2) = \sum_{\substack{\alpha m \geq p \\ \beta n \geq q \\ m, n \geq 0}} \frac{b_{\alpha m - p, \beta n - q}}{m! n!} z_1^m z_2^n$$

for any $p, q \geq 0$. Each H_{pq} is an entire function in $\mathbb{C}^2$. We have

$$\int_0^{z_1} H_{pq}(\zeta, z_2) d\zeta = H_{p+\alpha, q}(z_1, z_2)$$

and

$$\int_0^{z_2} H_{pq}(z_1, \zeta) d\zeta = H_{p, q+\beta}(z_1, z_2)$$

for any $p, q \geq 0$. Since we have $H_{pq}(1, 1) = 0$ for any $p, q \geq 1$, Lemma 3 applies to $\alpha\beta$ functions H_{pq} ($0 \leq p \leq \alpha - 1, 0 \leq q \leq \beta - 1$) and we have $b_{ij} = 0$ for any $i, j \geq 0$.

4. Non-vanishing cohomology set of Thullen's domain

In this paragraph, we put

$$U = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| < 1, 0 < |z_2| < 1\}$$

and

$$V = \{(z_1, z_2) \in \mathbb{C}^2; 0 < |z_1| < 1, |z_2| < 1\}.$$

LEMMA 5. *For a non-zero complex number λ and positive integers α and β , there are not a holomorphic function f in U and a holomorphic function g in V such that*

$$\frac{\lambda}{z_1^\alpha z_2^\beta} = f(z_1, z_2) - g(z_1, z_2)$$

in $U \cap V$.

PROOF. Laurent expansion of f and g gives Lemma 5.

LEMMA 6. *Let L be an abelian complex Lie group and $\exp: \mathbb{C}^m \rightarrow L$ be the exponential mapping of the Lie algebra $\mathbb{C}^m$ of L in L . Let X be a non-zero vector in $\mathbb{C}^m$ and α, β be positive integers. There are not $f \in H^0(U, \mathcal{A}_L)$ and $g \in H^0(V, \mathcal{A}_L)$ such that*

$$\exp\left(\frac{X}{z_1^\alpha z_2^\beta}\right) = fg^{-1}$$

in $U \cap V$.

PROOF. Using the universal covering manifolds of U and V , we can reduce Lemma 6 to Lemma 5.

LEMMA 7. *Let B be an (m, m) -matrix with a non-zero eigenvalue and α, β be positive integers. There are not $F \in H^0(U, \mathcal{A}_{GL(m, \mathbb{C})})$ and $G \in H^0(V, \mathcal{A}_{GL(m, \mathbb{C})})$ such that*

$$\exp\left(\frac{B}{z_1^\alpha z_2^\beta}\right) = FG^{-1}$$

in $U \cap V$.

PROOF. Using triangulation of the matrix B and Thullen's result [11] that $U \cup V$ is a Cousin-II domain, we can reduce Lemma 7 to Lemma 6.

LEMMA 8. Let λ be a non-zero complex number and α, β be positive integers. There are not $h \in H^0(U \cup V, \mathcal{O})$, which is not identically zero, $f \in H^0(U, \mathcal{O})$ and $g \in H^0(V, \mathcal{O})$ such that

$$h(z_1, z_2) \exp\left(\frac{\lambda}{z_1^\alpha} + \frac{\lambda}{z_2^\beta}\right) = f(z_1, z_2) - g(z_1, z_2)$$

in $U \cap V$ where $\mathcal{O}$ is the sheaf of all germs of holomorphic functions.

PROOF. Let

$$h(z_1, z_2) = \sum_{i,j=0}^{\infty} b_{ij} z_1^i z_2^j$$

be Taylor expansion of h . Then, comparing the coefficients of $z_1^{-p} z_2^{-q}$, we have

$$\sum_{\substack{\alpha m \geq p \\ \beta n \geq q \\ m, n \geq 0}} \frac{b_{\alpha m - p, \beta n - q}}{m! n!} = 0$$

for any $p, q \geq 1$. By Lemma 4, h is identically zero.

LEMMA 9. Let B be a non-zero (m, m) matrix, whose eigenvalues are all zero, and α, β be positive integers. There are not $F \in H^0(U, \mathcal{A}_{GL(m, \mathbb{C})})$, $G \in H^0(V, \mathcal{A}_{GL(m, \mathbb{C})})$ and $h \in H^0(U \cup V, \mathcal{O})$, which is not identically zero, such that

$$\exp\left(h \exp\left(\frac{1}{z_1^\alpha} + \frac{1}{z_2^\beta}\right) B\right) = FG^{-1}$$

in $U \cap V$.

PROOF. Making use of the Jordan's normal form of B , we can reduce Lemma 9 to Lemma 8.

5. Main result

LEMMA 10. Let Ω be a domain in $R \times T$ such that there exists a positive dimensional complex Lie group L with $H^1(\Omega, \mathcal{A}_L) = 0$. Then Ω is a pseudo-

convex domain in $R \times T$.

PROOF. At first, we treat the case that L is not abelian. Then there exists an element X of the Lie algebra $\mathcal{L}$ of L such that, for the adjoint representation $\text{ad}: \mathcal{L} \rightarrow L$, $B = \text{ad } X$ is a non-zero (m, m) matrix. Assume that Ω is not pseudo-convex. There exist a positive number ε , a biholomorphic mapping ψ of the set

$$\tilde{D} = \{(w_1, w_2) \in \mathbb{C}^2; |w_1| < 1 + \varepsilon, |w_2| < 1 + \varepsilon\}$$

in $R \times T$ and a point (w_1^0, w_2^0) of $\tilde{D}$ such that, for the set

$$D = \{(w_1, w_2) \in \mathbb{C}^2; |w_1| < \varepsilon, |w_2| < 1 + \varepsilon\}$$

$$\cup \{(w_1, w_2) \in \mathbb{C}^2; |w_1| < 1 + \varepsilon, 1 - \varepsilon < |w_2| < 1 + \varepsilon\},$$

we have $\psi(D) \subset \Omega$ but $(x_0, t_0) = \psi(w_1^0, w_2^0) \notin \Omega$. Since the open Riemann surface is a Cousin-I manifold by Florack [6], there exists a meromorphic function $\xi(x)$ in R which is holomorphic in $R - \{x_0\}$ and which has a simple pole at x_0 . There exists a point $z^0 \in \mathbb{C}^2$ with $\tau(z^0) = t_0$. We put

$$\zeta(z) = \sum_{\gamma \in \Gamma} \frac{1}{(z - z^0 - \gamma)^2}.$$

Then ζ is a meromorphic function in $\mathbb{C}$ which is invariant under Γ and which has a double pole at $z^0 + \gamma$ for any $\gamma \in \Gamma$. There exists a meromorphic function $\eta(t)$ in T such that $\zeta = \eta \circ \tau$. η is holomorphic in $T - \{t_0\}$ and has a double pole at t_0 . In case that B has a non-zero eigen-value, we put

$$h(x, t) = \xi(x)\eta(t)$$

for $(x, t) \in (R - \{x_0\}) \times (T - \{t_0\})$. In case that all eigen-values of B are zero, we put

$$h(x, t) = \exp(\zeta(x) + \eta(t))$$

for $(x, t) \in (R - \{x_0\}) \times (T - \{t_0\})$. We put $\Omega_1 = \{(x, t) \in \Omega; x \neq x_0\}$ and $\Omega_2 = \{(x, t) \in \Omega; t = t_0\}$. Then $\mathcal{U} = \{\Omega_1, \Omega_2\}$ is an open covering of Ω and $\exp(h(x, t)B) \in H^0(\Omega_1 \cap \Omega_2, \mathcal{A}_L)$ defines an element of $Z^1(\mathcal{U}, \mathcal{A}_L)$. Since $Z^1(\mathcal{U}, \mathcal{A}_L) = B^1(\mathcal{U}, \mathcal{A}_L)$, there exist $G_1 \in H^0(\Omega_1, \mathcal{A}_L)$ and $G_2 \in H^0(\Omega_2, \mathcal{A}_L)$ such that

$$\exp(h(x, t)B) = G_1 G_2^{-1}$$

in $\Omega_1 \cap \Omega_2$. We put $V = \psi(D)$, $V_1 = \{(x, t) \in V; x \neq x_0\}$, $V_2 = \{(x, t) \in V; t \neq t_0\}$, $\tilde{V} = \psi(\tilde{D})$, $\tilde{V}_1 = \{(x, t) \in \tilde{V}; x \neq x_0\}$ and $\tilde{V}_2 = \{(x, t) \in \tilde{V}; t \neq t_0\}$. Then $\tilde{D}, \tilde{V}$,

$\tilde{V}_1$ and $\tilde{V}_2$ are, respectively, envelopes of holomorphy of D , V , V_1 and V_2 . By Proposition of Kajiwar-Watanabe [9], $G_1 \in H^0(V_1, \mathcal{A}_L)$ and $G_2 \in H^0(V_2, \mathcal{A}_L)$ can, respectively, be extendable to elements of $H^0(\tilde{V}_1, \mathcal{A}_L)$ and $H^0(\tilde{V}_2, \mathcal{A}_L)$. $\tilde{V}_1 \cup \tilde{V}_2 = \tilde{V} - \{(x_0, t_0)\}$ contains a punctured neighborhood of (x_0, t_0) of Thullen's type. For the adjoint representation $\text{Ad}: \mathcal{G}(m, \mathbb{C}) \rightarrow GL(m, \mathbb{C})$, we have $\exp(hB) = (\text{Ad } G_1)(\text{Ad } G_2)^{-1}$ for $\text{Ad } G_i \in H^0(\tilde{V}_i, \mathcal{A}_{GL(m, \mathbb{C})})$. This contradicts to Lemmata 7 and 9 for $\alpha=1$ and $\beta=2$. In case that L is abelian, we make use of Lemma 6 instead of Lemmata 7 and 9.

THEOREM. *Let R be an open Riemann surface and T be a one dimensional complex torus. Let Ω be a domain in $R \times T$ such that there exists a positive dimensional complex Lie group L with $H^1(\Omega, \mathcal{A}_L) = 0$. Then, either there exists a subdomain Δ of R with $\Omega = \Delta \times T$ or Ω is a Stein manifold.*

PROOF. Since Ω is pseudo-convex by Lemma 10, we have Theorem by Lemma 2.

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