**STEINNESS OF A DOMAIN WITH A VANISHING
COHOMOLOGY SET IN A PRODUCT MANIFOLD
OF AN OPEN RIEMANN SURFACE AND A
COMPACT RIEMANN SURFACE
WITH GENUS $g \geq 1$**

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1. Introduction

By Cartan [2] and Behnke-Stein [1], any Cousin-I domain in $\mathbb{C}^2$ is a domain of holomorphy. By Thullen [14] a punctured polydisc in $\mathbb{C}^2$ is a Cousin-II domain which is not a domain of holomorphy. So the situation appears somewhat different between Cousin-I and Cousin-II problems. However, the above Cartan-Behnke-Stein's theorem can be extended to general Cousin problems as the following results which assert that a domain with a vanishing cohomology set in a good two dimensional complex manifold is a Stein manifold.

The author and Kazama [9] showed that a domain Ω in a two dimensional Stein manifold is a Stein manifold if there is a complex Lie group L with $H^1(\Omega, \mathcal{A}_L) = 0$ where $\mathcal{A}_L$ is the sheaf over Ω of all germs of holomorphic mappings in L . The author and Watanabe [10] obtained similar results for a proper subdomain Ω of a two dimensional complex projective space $\mathbb{P}_2$. By Weierstrass gap theorem, a compact Riemann surface with genus $g \geq 1$ does not equip a meromorphic function which is holomorphic in each point except in a point x_0 and which has a simple pole at x_0 . Generalizing lemmata in Kajiwar-Kazama [9] obtained for simple poles to poles of higher order, the author [8] proved that a domain Ω in a product manifold of an open Riemann surface and a one dimensional complex torus T is either a Stein manifold or is a product set $\Delta \times T$ if there is a complex Lie group L with $H^1(\Omega, \mathcal{A}_L) = 0$. Making use of the above lemmata, the author and Watanabe [11] proved pseudoconvexity of a domain Ω in a product manifold of two Riemann surfaces satisfying $H^1(\Omega, \mathcal{A}_L) = 0$ for a complex Lie group L .

Recently Brun [2] proved that a pseudoconvex domain Ω in a product

manifold of a Stein manifold X and a compact Riemann surface Y is a Stein manifold if $\{x\} \times Y$ is not contained in Ω for any point x of X . The main purpose of this paper is to prove Steinness of a domain Ω in a product manifold of an open Riemann surface R and a compact Riemann surface S with genus $g \geq 1$ satisfying $H^1(\Omega, \mathcal{A}_L) = 0$ for a complex Lie group L . In the proof we use the Weierstrass gap theorem in two opposite directions. Making use of existence of a gap value, we prove that $\{x\} \times Y$ is not contained in Ω for any point x of X . Making use of existence of a non gap value, we prove pseudoconvexity of Ω .

2. Non triviality of analxtic fiber bundles

We shall prove Lemmata 1, 2, 3 and 4 by the method of absurd without mention.

LEMMA 1. *Let y_0 be a point of a compact Riemann surface Y and m be a Weierstrass gap value at y_0 , that is, there is no meromorphic function in Y which is holomorphic in $Y - \{y_0\}$ and which has a pole of order $m \geq 1$ at y_0 . Let t be a biholomorphic mapping with $t(y_0) = 0$ of an open neighborhood W of y_0 onto an open disc $\{t \in \mathbb{C}; |t| < r\}$ in the complex plane $\mathbb{C}$, r_1 and r_2 be positive numbers with $r_1 < r_2$. Let L be a k -dimensional abelian complex Lie group and $(\mathbb{C}^k, \exp)$ be its universal covering group. Let X be any non zero vector in $\mathbb{C}^k$. We put $U = \{y \in Y; |t| < r_2\}$ and $V = Y - \{y \in Y; |t| \leq r_1\}$. Then, there is a positive integer p such that there are not $f \in H^0(U, \mathcal{A}_L)$ and $g \in H^0(V, \mathcal{A}_L)$ satisfying*

$$\exp\left(\frac{X}{pt^m}\right) = fg^{-1}$$

in $U \cap V$.

PROOF. Assume that, for any positive integer p , there are $f_p \in H^0(U, \mathcal{A}_L)$ and $g_p \in H^0(V, \mathcal{A}_L)$ satisfying

$$\exp\left(\frac{X}{pt^m}\right) = f_p g_p^{-1}$$

in $U \cap V$. Since U is simply connected, there is $F_p \in H^0(U, \mathcal{O}^k)$ with

$$f_p = \exp(F_p)$$

in U where $\mathcal{O}$ is the sheaf over Y of all germs of holomorphic functions. We put

$$G_p = F_p - \frac{X}{pt^m}$$

in $U \cap V = \{y \in Y; r_1 < |t| < r_2\}$. Since we have

$$\exp G_p = g_p \in H^0(V, \mathcal{A}_L)$$

G_p is holomorphically continued along any curve in V , in other words, G_p is a multiformly vector-valued holomorphic function in V . Let $\gamma = \{\gamma(t); 0 \leq t \leq 1\}$ be any closed curve in V . We put

$$n_p = G_p(\gamma(1)) - G_p(\gamma(0)) = \int_{\gamma} dG_p$$

for any $p \geq 1$. Each n_p is the variation of G_p along γ and is an element of the discrete subgroup $\Gamma = \ker \exp$ of $\mathbb{C}^k$. Assume that $n_q \neq 0$ for a positive integer q . For any positive integer v , we have

$$\exp\left(\frac{X}{vqt^m}\right) = \exp\left(\frac{F_q}{v}\right) \exp\left(\frac{G_q}{v}\right)^{-1}$$

in $U \cap V$. Since we have

$$\exp\left(\frac{\lambda}{vqt^m}\right) = \exp(F_{vq}) \exp(G_{vq})^{-1}$$

in $U \cap V$, we have

$$\exp\left(\frac{F_q}{v} - F_{vq}\right) = \exp\left(\frac{G_q}{v} - G_{vq}\right)$$

in $U \cap V$. Since Y is compact, the holomorphic mapping $\exp(G_q/v - G_{vq})$ of Y in L is a constant if L is linear. There is a k -vector $c_v \in \mathbb{C}^k$ such that

$$\frac{G_q}{v} - G_{vq} = c_v$$

in $U \cap V$ for any $v \geq 1$. Hence we have

$$n_{vq} = \frac{n_q}{v}$$

for any $v \geq 1$. Since $n_q \neq 0$, $|n_{vq}| \geq \delta$ for any $v \geq 1$ where $\delta = \inf_{c \in \Gamma - \{0\}} |c|$ for the discrete Γ . Then we have even if L is not linear

$$|n_q| = |\nu n_{\nu q}| \geq \nu \delta \longrightarrow \infty$$

as $\nu \rightarrow \infty$. But this is a contradiction. Hence we have $n_q = 0$ for any positive integer q . Thus we have proved that each G_p can be uniformly and holomorphically continued along any closed curve in V and gives a uniform holomorphic mapping G_p of V in L . Since we have

$$G_1 = F_1 - \frac{X}{t^m}$$

in $U \cap V$, some component of G_1 is a meromorphic function in Y which is holomorphic in $Y - \{y_0\}$ and which has a pole of order m at y_0 . Since m is a gap value at y_0 , this is a contradiction.

LEMMA 2. *Under the notations in Lemma 1, let B be a (k, k) matrix whose trace is not zero. Then, there is a positive integer p such that there are not $f \in H^0(U, A_{GL(k, \mathbb{C})})$ and $g \in H^0(V, A_{GL(k, \mathbb{C})})$ satisfying*

$$\exp\left(\frac{B}{pt^m}\right) = fg^{-1}$$

in $U \cap V$ where $GL(k, \mathbb{C})$ be the Lie group of all non singular (k, k) matrices.

PROOF. We put $\lambda = \text{tr } B \neq 0$. Taking determinants on the both sides, we have

$$\exp\left(\frac{\lambda}{pt^m}\right) = (\det f)(\det g)^{-1}$$

in $U \cap V$ for $\det f \in H^0(U, \mathcal{O}^*)$ and $\det g \in H^0(V, \mathcal{O}^*)$, where $\mathcal{O}^* = \mathcal{A}_{\mathbb{C} - \{0\}}$, what is a contradiction by Lemma 1.

LEMMA 3. *For a non-zero (k, k) diagonalizable matrix B whose trace is zero, Lemma 2 is valid.*

PROOF. There is a non singular matrix P such that $A = PBP^{-1}$ is a diagonal matrix whose diagonal elements are denoted in order by $\lambda_1, \lambda_2, \dots, \lambda_k$. We put $f' = Pf = (f_{ij})$ and $g' = Pg = (g_{ij})$. Then we have $f' \in H^0(U, A_{GL(k, \mathbb{C})})$, $g' \in H^0(V, A_{GL(k, \mathbb{C})})$ and

$$\exp\left(\frac{A}{pt^m}\right) = f'g'^{-1}$$

in $U \cap V$. $\exp(A/pt^m)$ is a diagonal matrix with diagonal elements $\exp(\lambda_1/pt^m)$,

$\exp(\lambda_2/pt^m), \dots, \exp(\lambda_k/pt^m)$. We have

$$g_{ij} \exp\left(\frac{\lambda_j}{pt^m}\right) = f_{ij}$$

in $U \cap V$. For any integer j with $1 \leq j \leq k$, the j -th column of f' is a non-zero k -vector at each point of V . Hence, there is an integer v_j with $1 \leq v_j \leq k$ such that $g_{v_j j}$ is not identically zero in V . Since $\lambda_1 + \lambda_2 + \dots + \lambda_k = 0$, we have

$$g_{v_1 1} g_{v_2 2} \dots g_{v_k k} = f_{v_1 1} f_{v_2 2} \dots f_{v_k k}$$

in $U \cap V$. $g_{v_1 1} g_{v_2 2} \dots g_{v_k k}$ defines a holomorphic function in the compact Riemann surface Y and is a non zero constant. Hence, for any j , $g_{v_j j}$ and $f_{v_j j}$ are, respectively, holomorphic functions in U and V which do not take the value zero anywhere. Since B is a non zero matrix, there is an integer j with $1 \leq j \leq k$ and $\lambda_j \neq 0$. Then we have

$$g_{v_j j} \exp\left(\frac{\lambda_j}{pt^m}\right) = f_{v_j j}$$

in $U \cap V$ for $g_{v_j j} \in H^0(U, \mathcal{O}^*)$ and $f_{v_j j} \in H^0(V, \mathcal{O}^*)$. This is a contradiction by Lemma 1.

LEMMA 4. For a non-zero (k, k) matrix B whose eigen-values are all zero, Lemma 2 is valid.

PROOF. There is a non-singular matrix P such that $A = PBP^{-1}$ is an upper-triangular matrix whose $(1, 2)$ element is one. Then $\exp(A/pt^m)$ is an upper-triangular matrix whose diagonal elements are all one and whose $(1, 2)$ elements is $1/pt^m$. Since each column of Pg is a non zero vector at each point of V , there is an integer v with $1 \leq v \leq k$ such that the $(v, 1)$ element of Pg , denoted by g_v , is not identically zero. The $(v, 1)$ element of Pf is denoted by f_v . The $(v, 2)$ elements of Pf and Pg are, respectively, denoted by f'_v and g'_v . Since we have

$$\exp\left(\frac{A}{pt^m}\right) = Pf(Pg)^{-1}$$

in $U \cap V$, we have

$$g_v = f_v, \quad \frac{g_v}{pt^m} + g'_v = f'_v$$

in $U \cap V$. Since g_v defines a holomorphic function in the compact Riemann

surface Y which is a non-zero constant, g'_v is a meromorphic function in Y which is holomorphic in $Y - \{y_0\}$ and which has a pole of order m at y_0 . Since m is a gap value at y_0 , this is a contradiction.

A complex linear Lie algebra is called *semi-simple* if its radical is $\{0\}$. If V is a complex linear space, a subset K of the End V is called *irreducible* if there is no nontrivial proper subspace S of V such that $fS \subset S$ for any $f \in K$.

PROPOSITION 1. *Let L be an n -dimensional complex linear Lie group acting on a k -dimensional complex linear space V , $\mathcal{L}$ be its infinitesimal algebra and $\exp: \mathcal{L} \rightarrow L$ be the exponential mapping. Under the notations of Lemma 1, assume that L acts irreducibly on V . Then, there are an element X of $\mathcal{L}$ and a positive integer p such that there are not $f \in H^0(U, \mathcal{A}_L)$ and $g \in H^0(V, \mathcal{A}_L)$ satisfying*

$$\exp\left(\frac{X}{pt^m}\right) = fg^{-1}$$

in $U \cap V$.

PROOF. We may assume that L is connected.

(1) *The case that L is abelian.* Proposition 1 holds for any non zero element X of $\mathcal{L}$ by Lemma 1.

(2-1) *The case that L is not abelian and that there is an element X of $\mathcal{L}$ with $\text{tr } X \neq 0$.* If Proposition 1 does not hold for this X , for any positive integer p , there are $f \in H^0(U, \mathcal{A}_L)$ and $g \in H^0(V, \mathcal{A}_L)$ with

$$\exp\left(\frac{X}{pt^m}\right) = fg^{-1}$$

in $U \cap V$. Since X is a non-zero (k, k) matrix whose trace is not zero and since $f \in H^0(U, \mathcal{A}_{GL(k, \mathbb{C})})$ and $g \in H^0(V, \mathcal{A}_{GL(k, \mathbb{C})})$, this is a contradiction by Lemma 2.

(2-2) *The case that L is not abelian and $\text{tr } X = 0$ for any $X \in \mathcal{L}$.* Let $\text{ad}: \mathcal{L} \rightarrow \mathcal{L}/(n, \mathbb{C})$ and $\text{Ad}: L \rightarrow GL(n, \mathbb{C})$ be the adjoint representations. Since L acts irreducibly on V and since $\text{tr } X = 0$ for any $X \in \mathcal{L}$, $\mathcal{L}$ is semisimple by the proposition at page 68 of Freudenthal-de Vries [5]. Since L is not abelian, there is an element X of $\mathcal{L}$ such that $\text{ad}X \neq 0$. Since $\mathcal{L}$ is semisimple, by the corollary at page 94 of Freudenthal-de Vries [5], there is an ad-cleaving of X , that is, there are elements Y and Z of $\mathcal{L}$ such that $X = Y + Z$, Y and Z are permutable, $\text{ad}Y$ is diagonalizable and that $\text{ad}Z$ is nilpotent. Since $\text{ad}X = \text{ad}Y + \text{ad}Z \neq 0$, we have $\text{ad}Y \neq 0$ or $\text{ad}Z \neq 0$.

(2-2-1) *The case that $\text{ad} Y \neq 0$.* If Proposition 1 does not hold for this Y , for any positive integer p there are $f \in H^0(U, \mathcal{A}_L)$ and $g \in H^0(V, \mathcal{A}_L)$ satisfying

$$\exp\left(\frac{Y}{pt^m}\right) = fg^{-1}$$

in $U \cap V$. Then we have

$$\exp\left(\frac{\text{ad} Y}{pt^m}\right) = \text{Ad} f (\text{Ad} g)^{-1}$$

in $U \cap V$ for a non zero (n, n) matrix $\text{ad} Y$ and $\text{Ad} f \in H^0(U, \mathcal{A}_{GL(n, \mathbb{C})})$ and $\text{Ad} g \in H^0(V, \mathcal{A}_{GL(n, \mathbb{C})})$. If $\text{tr} \text{ad} Y \neq 0$, this is a contradiction by Lemma 2. If $\text{tr} \text{ad} Y = 0$, this is a contradiction by Lemma 3.

(2-2-2) *The case that $\text{ad} Z \neq 0$.* Similarly, we arrive at a contradiction by Lemma 4.

PROPOSITION 2_k. *Proposition 1 holds in case that L acts reducibly on V .*

PROOF. We may assume that L is connected. We shall prove Proposition 2 by induction with respect to k . When $k=1$, L is abelian and we have Proposition 2₁ by Lemma 1. We assume the validity of Proposition 2_j for $1 \leq j \leq k-1$. Since L is reducible, there is a proper subspace S of V such that $fS \subset S$ for any $f \in L$. Let $e_1, e_2, \dots, e_k$ be basis of V . There are an integer s with $1 \leq s \leq k-1$ and $(k-s)$ linearly independent k -vectors $(a_{i1}, a_{i2}, \dots, a_{ik}) (s+1 \leq i \leq k)$ such that

$$S = \{z = \sum_{i=1}^s z_i e_i \in V; \sum_{j=1}^k a_{ij} z_j = 0, s+1 \leq i \leq k\}.$$

There are s k -vectors $(a_{i1}, a_{i2}, \dots, a_{ik}) (1 \leq i \leq s)$ such that $(a_{ij}) \in GL(k, \mathbb{C})$. We put

$$Az = \sum_{j=1}^k a_{ij} z_j e_i$$

for $z = \sum_{i=1}^k z_i e_i \in V$ and

$$L' = \{AXA^{-1} \in \text{End } V; X \in L\}$$

and

$$\mathcal{L}' = \{AfA^{-1} \in \text{End } V; f \in \mathcal{L}\}.$$

Then $\mathcal{L}'$ is the infinitesimal algebra of L' . We have

$$AS = \{z = \sum_{i=1}^s z_i e_i; z_i = 0, s+1 \leq i \leq k\}.$$

For the basis $e_1, e_2, \dots, e_k$, any element $X' \in \mathcal{L}'$ and any element $f' \in L'$ are regarded as (k, k) matrices. Since $X'AS \subset AS$ and $f'AS \subset AS$ for any $X' \in \mathcal{L}'$ and $f' \in L'$, these are of the forms

$$X' = \begin{bmatrix} X'_1 & X'_2 \\ 0 & X'_3 \end{bmatrix}, \quad f' = \begin{bmatrix} f'_1 & f'_2 \\ 0 & f'_3 \end{bmatrix}$$

where X'_1 and f'_1 are (s, s) matrices and X'_3 and f'_3 are $(k-s, k-s)$ matrices. We put $\pi_1(X) = X'_1$, $\pi_1(f) = f'_1$, $\pi_3(X) = X'_3$ and $\pi_3(f) = f'_3$ for $X \in \mathcal{L}'$ and $f' \in L'$. We put $L'_1 = \pi_1 L'$, $\mathcal{L}'_1 = \pi_1 \mathcal{L}'$, $L'_3 = \pi_3 L'$ and $\mathcal{L}'_3 = \pi_3 \mathcal{L}'$. Then $\mathcal{L}'_1$ and $\mathcal{L}'_3$ are, respectively, the infinitesimal algebras of L'_1 and L'_3 .

(1) *The case that there is an element X of $\mathcal{L}$ with $\pi_1 A X A^{-1} \neq 0$.* If Proposition 2 does not hold for this X , for any positive integers p , there are $f \in H^0(U, \mathcal{A}_L)$ and $g \in H^0(V, \mathcal{A}_L)$ satisfying

$$\exp\left(\frac{X}{pt^m}\right) = fg^{-1}$$

in $U \cap V$. Then we have

$$\exp\left(\frac{AXA^{-1}}{pt^m}\right) = (AfA^{-1})(AgA^{-1})^{-1}$$

in $U \cap V$. Then we have

$$\exp\left(\frac{\pi_1 A X A^{-1}}{pt^m}\right) = (\pi_1 A f A^{-1})(\pi_1 A g A^{-1})^{-1}$$

in $U \cap V$ for (s, s) matrix $\pi_1 A X A^{-1} \in \mathcal{L}'_1$, for $\pi_1 A f A^{-1} \in H^0(U, \mathcal{A}_{GL(s, \mathbb{C})})$ and for $\pi_1 A g A^{-1} \in H^0(V, \mathcal{A}_{GL(s, \mathbb{C})})$. This is a contradiction by Proposition 2_s because $1 \leq s \leq k-1$.

(2) *The case that there is an element X of $\mathcal{L}$ with $\pi_3 A X A^{-1} \neq 0$.* Similarly, we arrive at a contradiction by Proposition 2_{k-1} because $1 \leq k-s \leq k-1$.

(3) *The case that $\pi_1 A X A^{-1} = 0$ and $\pi_3 A X A^{-1} = 0$ for any $X \in \mathcal{L}$.* Let X be a non zero element of $\mathcal{L}$. Then AXA^{-1} and, therefore, X is a non zero nilpotent (k, k) matrix. Proposition 2 is valid for this X by Lemma 4.

THEOREM 1. *Let S be a compact Riemann surface with genus $g \geq 1$ and L*

be an n -dimensional complex Lie group with $n \geq 1$. Then we have $H^1(S, \mathcal{A}_L) \neq 0$.

PROOF. We may assume that L is connected. Let $\mathcal{L}$ be the Lie algebra of L . Let $\text{ad}: \mathcal{L} \rightarrow \mathfrak{gl}(n, \mathbb{C})$ and $\text{Ad}: L \rightarrow \text{GL}(n, \mathbb{C})$ be the adjoint representations. Let $\exp: \mathcal{L} \rightarrow L$ and $\exp: \mathfrak{gl}(n, \mathbb{C}) \rightarrow \text{GL}(n, \mathbb{C})$ be the exponential mappings. Let y_0 be a point of S . By the Weierstrass gap theorem, there is a gap value m at y_0 . There is a biholomorphic mapping t with $t(y_0)=0$ of an open neighborhood W of y_0 onto an open disc $\{t \in \mathbb{C}; |t| < r\}$ in the complex plane $\mathbb{C}$. We take U and V as in Lemma 1.

(1) *The case that L is abelian.* We have Theorem 1 by Lemma 1.

(2) *The case that L is not abelian.* There is an element X of $\mathcal{L}$ such that Proposition 1 or 2 is valid for the matrix $\text{ad}X$, U and V . If, for any positive integer p , there are $f \in H^0(U, \mathcal{A}_L)$ and $g \in H^0(V, \mathcal{A}_L)$ satisfying

$$\exp\left(\frac{X}{pt^m}\right) = fg^{-1}$$

in $U \cap V$, we have

$$\exp\left(\frac{\text{ad}X}{pt^m}\right) = \text{Ad}f(\text{Ad}g)^{-1}$$

in $U \cap V$ for $\text{Ad}f \in H^0(U, \mathcal{A}_{\text{GL}(n, \mathbb{C})})$ and $\text{Ad}g \in H^0(V, \mathcal{A}_{\text{GL}(n, \mathbb{C})})$. But this is a contradiction. Hence we have $H^1(\{U, V\}, \mathcal{A}_L) \neq 0$. This implies $H^1(S, \mathcal{A}_L) \neq 0$.

LEMMA 5. Let M be a complex manifold, Y be a compact Riemann surface with genus $g \geq 1$, Ω be a domain of the product manifold $M \times Y$. If there is a complex Lie group L with $H^1(\Omega, \mathcal{A}_L) = 0$, then there is no point x_0 of M such that $\{x_0\} \times Y \subset \Omega$.

PROOF. Let $\mathcal{U} = \{U_i; i \in I\}$ be an open covering of Y and $\{g_{ij}; i, j \in I\}$ be an element of $Z^1(\mathcal{U}, \mathcal{A}_L)$. For the projection $\pi: \Omega \rightarrow Y$,

$$\{g_{ij} \circ \pi; i, j \in I\} \in Z^1(\pi^{-1}(\mathcal{U}), \mathcal{A}_L)$$

where $\pi^{-1}(\mathcal{U}) = \{\pi^{-1}(U_i); i \in I\}$. Since $H^1(\Omega, \mathcal{A}_L) = 0$, there is an element $\{g_i; i \in I\} \in C^0(\pi^{-1}(\mathcal{U}), \mathcal{A}_L)$ such that $g_{ij} \circ \pi = g_i g_j^{-1}$ in $\pi^{-1}(U_i \cap U_j)$ for any $i, j \in I$. For the canonical injection $\eta: \{x_0\} \times Y \rightarrow \Omega$,

$$\{g_i \circ \eta; i \in I\} \in Z^1(\pi^{-1}(\mathcal{U}), \mathcal{A}_L)$$

satisfies

$$g_{ij} = (g_i \circ \eta)(g_j \circ \eta)^{-1}$$

in $U_i \cap U_j$ for any $i, j \in I$. This means that $H^1(Y, \mathcal{A}_L) = 0$. By Proposition 2, this is a contradiction.

3. Analytic vector bundles over the punctured polydisc

LEMMA 6. Let $H(x, y)$ be a continuous function in the product $[0, 1] \times [0, 1]$ of closed intervals in $\mathbf{R}$. We define a sequence $\{H_{pq}(x, y); p, q = 0, 1, 2, \dots\}$ of continuous functions in $[0, 1] \times [0, 1]$ by $H_{00}(x, y) = H(x, y)$ and

$$H_{p+1q}(x, y) = \int_0^x H_{pq}(s, y) ds$$

$$H_{pq+1}(x, y) = \int_0^y H_{pq}(x, t) dt$$

for $p, q \geq 0$. If $H_{pq}(1, 1) = 0$ for any $p, q \geq 1$, we have $H(x, y) = 0$ for any $(x, y) \in [0, 1] \times [0, 1]$.

PROOF. We have

$$H_{pq}(x, y) = \int_0^x \int_0^y \frac{(x-s)^{p-1}(y-t)^{q-1}}{(p-1)!(q-1)!} H(s, t) ds dt$$

for any $(x, y) \in [0, 1] \times [0, 1]$ and $p, q \geq 1$. Since $H_{pq}(1, 1) = 0$ for any $p, q \geq 1$, we have

$$\int_0^1 \int_0^1 (1-s)^{p-1}(1-t)^{q-1} H(s, t) ds dt = 0$$

for any $p, q \geq 1$. Since $\{(1-s)^p(1-t)^q; p, q \geq 0\}$ is a complete system in the space of all classes of square integrable functions in $[0, 1] \times [0, 1]$, we have $H(s, t) = 0$ for any $(s, t) \in [0, 1] \times [0, 1]$.

LEMMA 7. Let $\{b_{ij}; i, j = 0, 1, 2, \dots\}$ be a sequence of complex numbers such that, for positive numbers M and r , we have $|b_{ij}| \leq Mr^{i+j}$ for any $i, j \geq 0$. Let α and β be positive integers. If

$$\sum_{\substack{\alpha m \geq p \\ \beta n \geq p \\ m, n \geq 0}} \frac{b_{\alpha m - p, \beta n - p}}{m! n!} = 0$$

for any $p, q \geq 1$, we have $b_{ij} = 0$ for any $i, j \geq 0$.

PROOF. We put

$$H_{pq}(z_1, z_2) = \sum_{\substack{\alpha m \geq p \\ \beta n \geq q \\ m, n \geq 0}} \frac{b_{\alpha m - p, \beta n - q}}{m! n!} z_1^m z_2^n$$

for any $p, q \geq 0$. Each H_{pq} is an entire function in $\mathbb{C}^2$. We have

$$\int_0^{z_1} H_{pq}(\zeta, z_2) d\zeta = H_{p+\alpha, q}(z_1, z_2)$$

and

$$\int_0^{z_2} H_{pq}(z_1, \zeta) d\zeta = H_{p, q+\beta}(z_1, z_2)$$

for any $p, q \geq 0$. Since we have $H_{pq}(1, 1) = 0$ for any $p, q \geq 1$, Lemma 3 applies to $\alpha\beta$ functions $H_{pq}(0 \leq p \leq \alpha - 1, 0 \leq q \leq \beta - 1)$ and we have $b_{ij} = 0$ for any $i, j \geq 0$.

In the following Lemmata 8, 9, 10, 11 and 12, we shall put

$$U = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| < 1, 0 < |z_2| < 1\}$$

and

$$V = \{(z_1, z_2) \in \mathbb{C}^2; 0 < |z_1| < 1, |z_2| < 1\}.$$

and prove them by the method of absurd without mention of it.

LEMMA 8. For a non-zero complex number λ and positive integers α and β , there are not a holomorphic function f in U and a holomorphic function g in V such that

$$\frac{\lambda}{z_1^\alpha z_2^\beta} = f(z_1, z_2) - g(z_1, z_2)$$

in $U \cap V$.

PROOF. Let

$$f(z_1, z_2) = \sum_{v, \mu=-\infty}^{+\infty} a_{v\mu} z_1^v z_2^\mu, \quad g(z_1, z_2) = \sum_{v, \mu=-\infty}^{+\infty} b_{v\mu} z_1^v z_2^\mu$$

be Laurent series expansions in $U \cap V$. Then we have

$$\lambda = a_{-\alpha-\beta} - b_{-\alpha-\beta} = 0.$$

This is a contradiction.

LEMMA 9. Let L be an abelian complex Lie group and $\exp: \mathbb{C}^k \rightarrow L$ be the exponential mapping of the Lie algebra $\mathbb{C}^k$ of L in L . Let X be a non-zero vector in $\mathbb{C}^k$ and α, β be positive integers. There are not $f \in H^0(U, \mathcal{A}_L)$ and $g \in H^0(V, \mathcal{A}_L)$ such that

$$\exp\left(\frac{X}{z_1^\alpha z_2^\beta}\right) = fg^{-1}$$

in $U \cap V$.

PROOF. We may assume that L is connected. We put

$$U^* = \{(z_1, u_2) \in \mathbb{C}^2; |z_1| < 1, \operatorname{Re} u_2 < 0\},$$

$$V^* = \{(u_1, z_2) \in \mathbb{C}^2; \operatorname{Re} u_1 < 0, |z_2| < 1\}$$

and

$$W^* = \{(u_1, u_2) \in \mathbb{C}^2; \operatorname{Re} u_1 < 0, \operatorname{Re} u_2 < 0\}.$$

Since U^* and V^* are simply connected, there are $F \in H^0(U^*, \mathcal{O}^k)$ and $G \in H^0(V^*, \mathcal{O}^k)$ satisfying

$$f(z_1, \exp u_2) = \exp F(z_1, u_2), \quad g(\exp u_1, z_2) = \exp G(u_1, z_2)$$

in U^*, V^* respectively. Since W^* is connected, there is a vector b belonging to the kernel of $\exp$ such that

$$\exp(-\alpha u_1 - \beta u_2)X = F(\exp u_1, u_2) - G(u_1, \exp u_2) + b$$

in W^* . Hence we have

$$F(\exp u_1, u_2 + 2\pi i) = F(\exp u_1, u_2),$$

and

$$G(u_1 + 2\pi i, \exp u_2) = G(u_1, \exp u_2)$$

in W^* . There are $H \in H^0(U, \mathcal{O}^k)$ and $K \in H^0(V, \mathcal{O}^k)$ satisfying

$$H(z_1, \exp u_2) = F(z_1, u_2) + b$$

in U^* ,

$$K(\exp u_1, z_2) = G(u_1, z_2)$$

in V^* and

$$\frac{X}{z_1^\alpha z_2^\beta} = H(z_1, z_2) - K(z_1, z_2)$$

in $U \cap V$. This is a contradiction by Lemma 8.

LEMMA 10. Let B be a (k, k) -matrix with a non zero eigen-value and α, β be positive integers. There are not $F \in H^0(U, \mathcal{A}_{\text{GL}(k, \mathbb{C})})$ and $G \in H^0(V, \mathcal{A}_{\text{GL}(k, \mathbb{C})})$ such that

$$\exp\left(\frac{B}{z_1^\alpha z_2^\beta}\right) = FG^{-1}$$

in $U \cap V$.

PROOF. There is a non singular (k, k) matrix P such that $\lambda = PBP^{-1}$ is an upper triangular matrix whose $(1, 1)$ element λ is not zero. We have

$$(PF)^{-1} \exp\left(\frac{A}{z_1^\alpha z_2^\beta}\right) = (PG)^{-1}$$

in $U \cap V$. $\exp(A/z_1^\alpha z_2^\beta)$ is an upper triangular matrix whose $(1, 1)$ element is $\exp(\lambda/z_1^\alpha z_2^\beta)$. Since $(PF)^{-1}$ is a non singular (k, k) matrix at each point of V , the first column of $(PF)^{-1}$ is a non zero k -vector at each point of V . There is an integer v with $1 \leq v \leq k$ such that the $(v, 1)$ element f_v of $(PF)^{-1}$ is a holomorphic function in V which is not identically zero. Let g_v be the $(v, 1)$ element of the (k, k) matrix $(PG)^{-1}$. Then we have

$$f_v \exp\left(\frac{\lambda}{z_1^\alpha z_2^\beta}\right) = g_v$$

in $U \cap V$. Hence g_v is a holomorphic function in V which is not identically zero. Then $\{(U, f_v), (V, g_v)\}$ is a multiplicative Cousin distribution in the punctured polydisc $U \cup V = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| < 1, |z_2| < 1\} - \{(0, 0)\}$ which is of Cousin-II by Thullen [14]. There is a holomorphic function h in $U \cup V$ such that $f' = f_v/h \in H^0(U, \mathcal{O}^*)$, $g' = g_v/h \in H^0(V, \mathcal{O}^*)$ and

$$\exp\left(\frac{\lambda}{z_1^\alpha z_2^\beta}\right) = g' f'^{-1}$$

in $U \cap V$. This is a contradiction by Lemma 9.

The author can not prove Lemma 2 for a matrix B with a non zero eigen-value as in Lemma 9 because the compact Riemann surface Y is not of Cousin-II, $U \cup V$ being of Cousin-II by Thullen [14]. The compactness of Y , however, allows him to prove Lemma 3 which enables him to use the theory of semi-simple Lie groups in order to obtain Propositions 1 and 2.

LEMMA 11. *Let λ be a non-zero complex number and α, β be positive integers. There are not $h \in H^0(U \cup V, \mathcal{O})$, which is not identically zero, $f \in H^0(U, \mathcal{O})$ and $g \in H^0(V, \mathcal{O})$ such that*

$$h(z_1, z_2) \exp\left(\frac{\lambda}{z_1^\alpha} + \frac{\lambda}{z_2^\beta}\right) = f(z_1, z_2) - g(z_1, z_2)$$

in $U \cap V$.

PROOF. Let

$$h(z_1, z_2) = \sum_{i,j=0}^{\infty} b_{ij} z_1^i z_2^j$$

by Taylor expansion of H . Then, comparing the coefficients of $z_1^{-p} z_2^{-q}$, we have

$$\sum_{\substack{\alpha m \geq p \\ \beta n \geq q \\ m, n \geq 0}} \frac{b_{\alpha m - p, \beta n - q}}{m! n!} = 0$$

for any $p, q \geq 1$. By Lemma 7, h is identically zero.

LEMMA 12. *Let B be a non-zero (k, k) matrix, whose eigenvalues are all zero, and α, β be positive integers. There are not $F \in H^0(U, \mathcal{A}_{\text{GL}(k, \mathbb{C})})$, $G \in H^0(V, \mathcal{A}_{\text{GL}(k, \mathbb{C})})$ and $h \in H^0(U \cup V, \mathcal{O})$, which is not identically zero, such that*

$$\exp\left(h \exp\left(\frac{1}{z_1^\alpha} + \frac{1}{z_2^\beta}\right) B\right) = FG^{-1}$$

in $U \cap V$.

PROOF. There is a non singular matrix P such that $A = PBP^{-1}$ is an upper-triangular matrix whose diagonal elements are all zero and whose $(1, 2)$ element is one. We have

$$(PF)^{-1} \exp\left(h \exp\left(\frac{1}{z_1^\alpha} + \frac{1}{z_2^\beta}\right) A\right) = (PG)^{-1}$$

in $U \cap V$. The left hand side is an upper-triangular matrix whose diagonal elements are all one and whose $(1, 2)$ element is $h \exp(z_1^{-\alpha} + z_2^{-\beta})$. There is an integer v with $1 \leq v \leq k$ such that the $(v, 1)$ element f_{v1} of $(PF)^{-1}$ is not identically zero in U . The $(v, 1)$ element of $(PG)^{-1}$ is denoted by g_{v1} . The $(v, 2)$ elements of $(PF)^{-1}$ and $(PG)^{-1}$ are, respectively, denoted by f_{v2} and g_{v2} . Calculating the $(v, 1)$ and $(v, 2)$ elements of the both sides of the equation, we have

$$f_{v1} = g_{v1}, f_{v1} h \exp\left(\frac{1}{z_1^\alpha} + \frac{1}{z_2^\beta}\right) + f_{v2} = g_{v2}$$

in $U \cap V$. Hence $f_{v1} = g_{v1}$ defines a well-defined holomorphic function in $U \cup V$ which is not identically zero. This is a contradiction by Lemma 11.

4. Pseudoconvexity of a domain with a vanishing cohomology set

The following lemma is given in Kajiwara-Watanabe [10].

LEMMA 13. *Let Ω and Ω' be domains in a Stein manifold M with $\Omega \subset \Omega'$ such that any holomorphic function in Ω can be analytically continued in Ω' . Let L be a complex Lie group and S be a hypersurface in M . Then any holomorphic mapping of Ω - S in L can be continued to a holomorphic mapping of Ω' - S in L .*

A domain Ω in a complex manifold M is said to be *pseudoconvex* in M if, for any boundary point x of Ω in M , there is an open neighborhood W of x such that $\Omega \cap W$ is a Stein manifold.

THEOREM 2. *Let R and S be Riemann surfaces and Ω be a domain in the product manifold $R \times S$. If there is a complex Lie group L with $H^1(\Omega, \mathcal{A}_L) = 0$, then Ω is pseudoconvex in $R \times S$.*

PROOF. Assume that Ω is not pseudoconvex in $R \times S$. There are a positive number ε and a biholomorphic mapping $\psi = (\psi_1, \psi_2)$ of the polydisc

$$\tilde{D} = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| < 1 + \varepsilon, |z_2| < 1 + \varepsilon\}$$

in $R \times S$ such that for the subdomain

$$D = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| < \varepsilon, |z_2| < 1 + \varepsilon\} \\ \cup \{(z_1, z_2) \in \mathbb{C}^2; |z_1| < 1 + \varepsilon, 1 - \varepsilon < |z_2| < 1 + \varepsilon\}$$

of $\tilde{D}$ we have $\psi(D) \subset \Omega$ and that there is a point z^0 of $\tilde{D}$ with $(x_0, y_0) = \psi(z^0) \notin \Omega$.

Let g and g' be, respectively, the genus of R and S . Then $\alpha = g + 1$ and $\beta = g' + 1$ are, respectively, non gap values of R and S at x_0 and y_0 , that is, there are, respectively, meromorphic functions ξ and η in R and S which are holomorphic in $R - \{x_0\}$ and $S - \{y_0\}$ and which have poles of order α and β at x_0 and y_0 . We put

$$\Omega_1 = \{(x, y) \in \Omega; x \neq x_0\}$$

and

$$\Omega_2 = \{(x, y) \in \Omega; y \neq y_0\}.$$

Then $\mathfrak{U} = \{\Omega_1, \Omega_2\}$ is an open covering of Ω . We may assume that L is connected. Let $\mathscr{L}$ be the Lie algebra of L . Let $\text{ad}: \mathscr{L} \rightarrow \mathscr{L}(k, \mathbb{C})$ and $\text{Ad}: L \rightarrow \text{GL}(k, \mathbb{C})$ be the adjoint representations. Let $\exp: \mathscr{L} \rightarrow L$ and $\exp: \mathscr{L}(k, \mathbb{C}) \rightarrow \text{GL}(k, \mathbb{C})$ be the exponential mappings.

(1) *The case that L is abelian.* For a non zero element X of $\mathscr{L}$, we consider

$$\exp(\xi(x)\eta(y)X) \in z^1(\mathfrak{U}, \mathscr{A}_L).$$

Since $H^1(\Omega, \mathscr{A}_L) = 0$, there are $f \in H^0(\Omega_1, \mathscr{A}_L)$ and $g \in H^0(\Omega_2, \mathscr{A}_L)$ satisfying

$$\exp(\xi(x)\eta(y)X) = fg^{-1}$$

in $\Omega_1 \cap \Omega_2$. We put

$$S_1 = \{(z_1, z_2) \in \tilde{D}; \psi_1(z_1, z_2) = x_0\},$$

$$S_2 = \{(z_1, z_2) \in \tilde{D}; \psi_2(z_1, z_2) = y_0\},$$

$$\tilde{D}_1 = \{(z_1, z_2) \in \tilde{D}; \psi_1(z_1, z_2) \neq x_0\}$$

and

$$\tilde{D}_2 = \{(z_1, z_2) \in \tilde{D}; \psi_2(z_1, z_2) \neq y_0\}.$$

Then S_1 and S_2 are, hypersurfaces in $\tilde{D}$, that is, one-codimensional regular analytic sets in $\tilde{D}$. Any holomorphic function in D can be analytically continued in $\tilde{D}$. Since $f \in H^0(\Omega_1, \mathscr{A}_L)$ and $g \in H^0(\Omega_2, \mathscr{A}_L)$, we have $f \circ \psi \in H^0(D - S_1, \mathscr{A}_L)$ and $g \circ \psi \in H^0(D - S_2, \mathscr{A}_L)$. By Lemma 13, there are $f' \in H^0(\tilde{D}_1, \mathscr{A}_L)$ and $g' \in H^0(\tilde{D}_2, \mathscr{A}_L)$ such that $f' = f \circ \psi$ in $D - S_1$ and $g' = g \circ \psi$ in $D - S_2$. We put

$$W = \psi(\tilde{D}), W_1 = \psi(\tilde{D}_1), W_2 = \psi(\tilde{D}_2).$$

Then we have $f' \circ \psi^{-1} \in H^0(W_1, \mathcal{A}_L)$ and $g' \circ \psi^{-1} \in H^0(W_2, \mathcal{A}_L)$. $f' \circ \psi^{-1}$ and $g' \circ \psi^{-1}$ are, respectively, analytic prolongations of f and g to W_1 and W_2 . In other words, we can say that

$$f \in H^0(W_1, \mathcal{A}_L), g \in H^0(W_2, \mathcal{A}_L).$$

We have

$$\exp(\xi(x)\eta(y)X) = fg^{-1}$$

in $W_1 \cap W_2 = \{(x, y) \in \psi(\tilde{D}); x \neq x_0, y \neq y_0\}$. Since W is a neighborhood of (x_0, y_0) and since $W_1 \cup W_2 = W - \{(x_0, y_0)\}$, this is a contradiction by Lemma 9.

(2) *The case that L is not abelian.* There is an element X of $\mathcal{L}$ such that $B = \text{ad } X$ is a non zero (k, k) matrix.

(2-1) *The case that B has a non zero eigen-value.* We consider

$$\exp(\xi(x)\eta(y)X) \in z^1(\mathfrak{U}, \mathcal{A}_L).$$

Since $H^1(\Omega, \mathcal{A}_L) = 0$, there are $f \in H^0(\Omega_1, \mathcal{A}_L)$ and $g \in H^0(\Omega_2, \mathcal{A}_L)$ satisfying

$$\exp(\xi(x)\eta(y)X) = fg^{-1}$$

in $\Omega_1 \cap \Omega_2$. Similarly as in the case (1), we see that

$$f \in H^0(W_1, \mathcal{A}_L), g \in H^0(W_2, \mathcal{A}_L)$$

satisfy

$$\exp(\xi(x)\eta(y)X) = fg^{-1}$$

in $W_1 \cap W_2$. Then we have

$$\exp(\xi(x)\eta(y)B) = \text{Ad } f(\text{Ad } g)^{-1}$$

in $W_1 \cap W_2$ for $\text{Ad } f \in H^0(W_1, \mathcal{A}_{\text{GL}(k, \mathbb{C})})$ and $\text{Ad } g \in H^0(W_2, \mathcal{A}_L)$. This is a contradiction by Lemma 10.

(2-2) *The case that all eigen-values of B are zero.* We consider

$$\exp(\exp(\xi(x) + \eta(y))X) \in z^1(\mathfrak{U}, \mathcal{A}_L).$$

There are $f \in H^0(\Omega_1, \mathcal{A}_L)$ and $g \in H^0(\Omega_2, \mathcal{A}_L)$ satisfying

$$\exp(\exp(\xi(x) + \eta(y))X) = fg^{-1}$$

in $\Omega_1 \cap \Omega_2$. Then we have

$$\text{Ad } f \in H^0(W_1, \mathcal{A}_{\text{GL}(k, \mathbb{C})}), \text{Ad } g \in H^0(W_2, \mathcal{A}_{\text{GL}(k, \mathbb{C})})$$

and

$$\exp(\exp(\xi(x) + \eta(y))B) = \text{Ad } f(\text{Ad } g)^{-1}$$

in $W_1 \cap W_2$. This is a contradiction by Lemma 12.

5. Domain with a vanishing cohomology set

LEMMA 14. *Let X be a complex manifold and Y be an n -dimensional complex manifold which is a union of a family $\{Y_i; i \in I\}$ of everywhere dense Stein subsets Y_i whose universal covering manifolds are $(\mathbb{C}^n, \eta_i)$. Let Ω be a pseudoconvex domain in the product manifold $X \times Y$ such that there is a non empty open set U in X with $U \times Y \subset \Omega$. Let $\pi: X \times Y \rightarrow X$ be the projection. Then we have*

$$\Omega = \pi(\Omega) \times Y.$$

PROOF. Let E be the set of all points x of $\pi(\Omega)$ such that there is an open neighborhood $U(x)$ of x with $U(x) \times Y \subset \Omega$. By the definition of E and the assumption, E is a non empty open subset of the connected topological space $\pi(\Omega)$, Ω being connected. Let x_0 be a boundary point of E in $\pi(\Omega)$. Since $x_0 \in \pi(\Omega)$, there are an open neighborhood $U(x_0)$ of x_0 , a biholomorphic mapping τ of $U(x_0)$ onto a open polydisc $V(x_0)$ in a complex space $\mathbb{C}^m$ and a non empty open subset W of Y with $U(x_0) \times W \subset \Omega$. Since $E \cap U(x_0)$ is a non empty open set, there is a polydisc V' in $\mathbb{C}^m$ contained in $\tau(E) \cap V(x_0)$. Let i be any element of I . We put

$$\phi_i(z, w) = (\tau^{-1}(z), \eta_i(w))$$

for $(z, w) \in V(x_0) \times \mathbb{C}^n \subset \mathbb{C}^{m+n}$. Then $(V(x_0) \times \mathbb{C}^n, \phi_i)$ is a covering manifold of $U(x_0) \times Y_i$. Since Y_i is everywhere dense in Y , $W \cap Y_i$ is a non empty open set in Y . We have

$$(V' \times \mathbb{C}^n) \cup (V(x_0) \times \eta_i^{-1}(W \cap Y_i)) \subset \phi_i^{-1}(\Omega \cap (X \times Y_i)).$$

Since $W \cap Y_i$ is a non empty open set in Y , $\eta_i^{-1}(W \cap Y_i)$ is a non empty open set in $\mathbb{C}^n$. Since $X \times Y_i$ is pseudoconvex in $X \times Y$, $\phi_i^{-1}(\Omega \cap (X \times Y_i))$ is pseudoconvex in $\mathbb{C}^{m+n}$. Hence we have

$$V(x_0) \times \mathbb{C}^n \subset \phi_i^{-1}(\Omega \cap (X \times Y_i)).$$

This means

$$U(x_0) \times Y_i \subset \Omega.$$

Since i is arbitrary, we have

$$U(x_0) \times Y \subset \Omega.$$

Therefore, we have $U(x_0) \subset E$. Since E is a non empty closed and open set in the connected space $\pi(\Omega)$, we have $E = \pi(\Omega)$. Thus we have proved $\Omega = \pi(\Omega) \times Y$. Since Ω is a Stein manifold, $\pi(\Omega)$ is also a Stein manifold.

LEMMA 15. *Let Y be a product manifold of a finite family of complex projective spaces. Then, Lemma 14 holds for this Y .*

PROOF. Let $w_0 : w_1 : \dots : w_n$ be the homogeneous coordinates in the complex projective n -space $\mathbf{P}_n$. We put

$$Q_v = \{(w_0 : w_1 : \dots : w_n) \in \mathbf{P}_n; w_v \neq 0\}$$

for $0 \leq v \leq n$. Then each Q_v is an everywhere dense open subset of $\mathbf{P}_n$ which is biholomorphic with $\mathbf{C}^n$. Since $\{Q_v; v=0, 1, \dots, n\}$ is an open covering of $\mathbf{P}_n$, we have Lemma 15.

LEMMA 16. *Let Y be a compact Riemann surface with genus $g \leq 1$. Then, Lemma 15 holds for this Y .*

PROOF. The universal covering manifold of Y is the Riemann sphere $\mathbf{P}_1 = \mathbf{C} \cup \{\infty\}$ if $g=0$ and is the complex plane $\mathbf{C}$ if $g=1$.

THEOREM. *Let R be an open Riemann surface, S be a compact Riemann surface with genus g and Ω be a domain in the product manifold $R \times S$. Assume that there is a positive dimensional complex Lie group L with $H^1(\Omega, \mathcal{A}_L) = 0$. Then,*

(1) *in case that $g=0$, either Ω is a Stein manifold or there is a subdomain Δ of R with $\Omega = \Delta \times S$*

and

(2) *in case that $g \geq 1$, Ω is a Stein manifold.*

PROOF. (1) In this case, Ω is pseudoconvex by Theorem 2. If there is no point x of R with $\{x\} \times S \subset \Omega$, Ω is a Stein manifold by Brun [2]. If there is such a point, by the compactness of S , there is a non empty open subset U of R with

$U \times Y \subset \Omega$. Then we have $\Omega = A \times S$, where A is a subdomain of X , by Lemma 15.

(2) In this case, we have Theorem by Theorem 3.

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