ON AN APPROXIMATION OF BESSEL FUNCTION BY THE TAU METHOD

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1. Introduction

Many computational methods for Bessel functions have been developed hitherto and some of these methods are highly accurate, but there are only a few methods which use the numerical solutions of Bessel's differential equations. In this paper we shall apply the so-called tau method originated by C. Lanczos ([2]) and improved by E. L. Ortiz ([3]) for finding an approximate solution of Bessel's differential equation and show that $J_0(x)$ can be numerically calculated on a pretty large interval by applying the method successively. The error of the approximation can be also made small easily, and our method is applicable for calculating $J_\nu(x)$, where $\nu \neq 0$, by a slight modification of these arguments under the condition that initial values are given.

2. The tau method

In his memoir of 1969, E. L. Ortiz ([3]) gave a systematic account of Lanczos' tau method. Let D be a linear differential operator of order α with polynomial coefficients, and I an interval $[0, 1]$ on which the solution of the problem

$$(1) \quad Dy(x) = 0$$

is sought.

It is shown that an operator D of the type indicated above is uniquely associated with a sequence $\mathbf{Q} = \{Q_m(x)\}$, $m \in N_0 - S$, $\text{card}(S)$ finite, of canonical polynomials $Q_m(x)$ such that

$$(2) \quad DQ_m(x) = x^m + R_m(x),$$

where the residual polynomial $R_m(x)$ belongs to the subspace $\mathbf{R}_S = \text{span}_{s \in S} \{x^s\}$

and N_0 is the set of nonnegative integers.

When (1) has t polynomial solutions $y_1(x), y_2(x), \dots, y_t(x)$, we consider the perturbed problem of (1)

$$(3) \quad Dy_n^*(x) = H_n(x),$$

where $H_n(x)$ is a polynomial of degree n which depends on some free parameters τ_j , the number of which can be easily determined from the number of supplementary conditions to be satisfied and some intrinsic characteristics of the operator D . We set $r = \text{card}(S) + \alpha - 1$, then

$$(4) \quad H_n(x) = \sum_{i=0}^r \tau_i x^i T_{n-(r-i)}^*(x),$$

where $T_{n-(r-i)}^*(x)$ is a shifted Chebyshev polynomial of degree $n-(r-i)$. Thus (3) has a polynomial solution

$$(5) \quad y_n^*(x) = \sum_{m=0}^{n-(r-t)} c_m \sum_{\substack{i=0 \\ m+i \notin S}}^{r-t} \tau_i Q_{m+i}(x) + \sum_{j=1}^t \beta_j y_j(x),$$

where c_m are the coefficients of $T_{n-(r-i)}^*(x)$. Therefore we may construct the approximate solution $y_n^*(x)$ of (1).

The initial conditions provide α linear algebraic equations for determining the parameters τ_i, β_j . Let us denote $\text{card}(S)$ by s_1 . The fact that s_1 canonical polynomials are undefined introduces the other s_1 necessary conditions, since the coefficient of $x^k, k \in S$, of $H_n(x)$ must be cancelled out with the corresponding coefficient of x^k in the weighted sum of residuals.

The parameters τ_i and β_j are determined by solving the linear system of algebraic equations

$$(6) \quad \sum_{m=0}^{n-(r-t)} c_m \sum_{\substack{i=0 \\ m+i \notin S}}^{r-t} \tau_i Q_{m+i}^{(k)}(0) + \sum_{j=1}^t \beta_j y_j^{(k)}(0) = y^{(k)}(0), \quad k=0, 1, \dots, \alpha-1,$$

$$\sum_{m=0}^{n-(r-t)} c_m \sum_{\substack{i=0 \\ m+i \notin S}}^{r-t} \tau_i R_{m+i}(x) \equiv [H_n(x)]_S,$$

where $[H_n(x)]_S$ is a symbol for the terms of x^k ($k \in S$) in $H_n(x)$.

By these results, the problem of undefined canonical polynomials is solved and the uniqueness of $\mathbf{Q}$ is guaranteed by defining an equivalence relation on $\mathbf{Q}$. Moreover we can easily find the polynomial solutions of (1) if the equation has such solutions.

Introducing canonical polynomials brings the following advantages. To

produce a more accurate approximation to the solution $y(x)$ we need not repeat the whole computation, since in the next step of the computation we shall use the canonical polynomials already determined. It is also easy to change the range of the approximate solution since the canonical polynomials are independent of the interval on which the solution is sought. They are also independent of the initial conditions of the problem.

3. An application of the tau method to Bessel's differential equation

We consider the following Bessel's differential equation:

$$(7) \quad \ddot{y}(x) + \frac{1}{x} \dot{y}(x) + \left(1 - \frac{v^2}{x^2}\right)y(x) = 0,$$

with initial conditions $y(0) = J_v(0)$ and $\dot{y}(0) = J'_v(0)$. In the case of $v \neq 0$ the corresponding equation of (1) is

$$(8) \quad D_v y(x) = x^2 \ddot{y}(x) + x \dot{y}(x) + (x^2 - v^2)y(x) = 0,$$

and in the case of $v=0$ it is

$$(9) \quad D_0 y(x) = x \ddot{y}(x) + \dot{y}(x) + x y(x) = 0.$$

Since the generating polynomials $P_n(v, x)$ and $P_n(x)$ of order n associated with D_v and D_0 are

$$(10) \quad P_n(v, x) \equiv D_v x^n = x^{n+2} + (n^2 - v^2)x^n,$$

$$(11) \quad P_n(x) \equiv D_0 x^n = x^{n+1} + n^2 x^{n-1},$$

the canonical polynomial sequences are given by the following recursive relations:

$$(12) \quad Q_m(v, x) = x^{m-2} - [(m-2)^2 - v^2]Q_{m-2}(v, x), \quad m \in N_0 - S_v,$$

$$(13) \quad Q_m(x) = x^{m-1} - (m-1)^2 Q_{m-2}(x), \quad m \in N_0 - S_0.$$

Thus we obtain the following results:

FACT 1. *There are no polynomial solutions to (8) and (9).*

FACT 2. *The undefined index sets S_v and S_0 are $S_v = \{0, 1\}$ and $S_0 = \{0\}$, respectively.*

FACT 3. *For the canonical polynomials and the residual polynomials, if $m \in N_0 - S_0$, then*

$$Q_m(x) = Q_{m+1}(0, x),$$

$$R_m(x) = \frac{R_{m+1}(0, x)}{x},$$

therefore, if m is odd, $R_m(x) = 0$.

PROOF. Since for all $m \in N_0 - S_0$

$$D_v Q_{m+1}(0, x) = x^{m+1} + R_{m+1}(0, x),$$

by (8) and (9) we obtain the relation

$$D_0 Q_{m+1}(0, x) = x^m + \frac{R_{m+1}(0, x)}{x}.$$

This shows the fact.

Now in the case of $v=0$, by the above facts $Q_0(x)$ is undefined, and $\mathbf{Q} = \{Q_m(x)\}$ and $\mathbf{R} = \{R_m(x)\}$, $m \in N_0 - S_0$, are shown as follows:

$$\begin{array}{ll} Q_1(x) = 1 & ; \quad R_1(x) = 0, \\ Q_2(x) = x & ; \quad R_2(x) = 1, \\ Q_3(x) = x^2 - 4 & ; \quad R_3(x) = 0, \\ Q_4(x) = x^3 - 9x & ; \quad R_4(x) = -9 \\ \dots\dots & \dots\dots \end{array}$$

Since $r = \text{card}(S_0) + \alpha - 1 = 2$, we set

$$H_n(x) = \sum_{i=0}^2 \tau_i x^i \sum_{m=0}^{n-2} c_m x^m.$$

Then the solution of the perturbed equation $D_0 y_n^*(x) = H_n(x)$ is given by

$$(14) \quad y_n^*(x) = \sum_{m=0}^{n-2} c_m \sum_{\substack{i=0 \\ m+i \notin S_0}}^2 \tau_i Q_{m+i}(x),$$

because (9) has no polynomial solution. The corresponding linear equation of (6) is as follows:

$$\begin{aligned} & \tau_0 \left[\sum_{m=1}^{n-2} c_m Q_m^{(k)}(0) \right] + \tau_1 \left[\sum_{m=0}^{n-2} c_m Q_{m+1}^{(k)}(0) \right] + \tau_2 \left[\sum_{m=0}^{n-2} c_m Q_{m+2}^{(k)}(0) \right] \\ & = J_0^{(k)}(0), \quad k=0, 1, \\ (15) \quad & \tau_0 [c_0 - (c_1 R_1 + c_2 R_2 + \dots)] + \tau_1 [-(c_0 R_1 + c_1 R_2 + \dots)] \\ & + \tau_2 [-(c_0 R_2 + c_1 R_3 + \dots)] = 0, \end{aligned}$$

where R_m represents $R_m(x)$.

A selection of the numerical results of τ_0 , τ_1 and τ_2 is given in the table below.

Table 1.

n	τ_0	τ_1	τ_2
5	-0.4065×10^{-19}	0.6998×10^{-3}	0.4384×10^{-3}
10	0.5270×10^{-19}	0.1875×10^{-9}	-0.9779×10^{-11}
14	0.1219×10^{-16}	0.5036×10^{-16}	-0.1836×10^{-17}

4. An application of piecewise tau method to Bessel's differential equation

We consider a problem of finding an approximate solution of problem (9) over a finite interval $[0, A]$ by means of piecewise tau method, by that we mean a series of successive tau method.

In order to apply the tau method over $[a, a+1]$, where a is a nonnegative integer, we transform $T_{n-2}^*(x)$ into $T_{n-2}^a(x)$ for the interval $[a, a+1]$, and apply the tau method over $[a, a+1]$ with initial conditions $y_n^{a-1}(a)$, $\dot{y}_n^{a-1}(a)$, where $y_n^{a-1}(x)$ is the approximate solution of (9) found by this method over $[a-1, a]$. Since the canonical polynomial sequence $\mathbf{Q}$ is independent of the interval, we can use it for all steps. Moreover we use a perturbation term

$$(16) \quad H_n^a(x) = \sum_{i=0}^2 \tau_i^a x^i \sum_{m=0}^{n-2} c_m^a x^m,$$

where c_m^a are the coefficients of $T_{n-2}^a(x)$. We have solved the problem in the previous section when $a=0$.

Thus we may consider the following perturbed equation:

$$(17) \quad Dy_n^a(x) = H_n^a(x),$$

with initial conditions $y_n^a(a) = y_n^{a-1}(a)$ and $\dot{y}_n^a(a) = \dot{y}_n^{a-1}(a)$. Its solution is given by

$$(18) \quad y_n^a(x) = \sum_{m=0}^{n-2} c_m^a \sum_{\substack{i=0 \\ m+i \notin S_0}}^2 \tau_i^a Q_{m+i}(x).$$

To find τ_0^a , τ_1^a and τ_2^a , we solve the equation (15) after substituting τ_i^a for τ_i , c_m^a

for c_m and $y_n^{a-1(k)}(a)$ for $J_0^{(k)}(0)$, respectively. Proceeding in the same manner, we can find the approximate solution of the problem (9) over $[0, A]$ which is the approximate polynomial of $J_0(x)$.

Experimental results is presented in Table 2.

Table 2.

x	Ref. [1]		Values of approximate polynomials of $J_0(x)$		
			n = 5	n = 14	
0.5	0.938469	807241	0.938502	0.938469	807241
1.0	0.765197	686558	0.765206	0.765197	686558
1.5	0.511827	671736	0.511861	0.511827	671736
2.0	0.223890	779141	0.223912	0.223890	779141
3.0	-0.260051	954902	-0.260079	-0.260051	954902
4.0	-0.397149	809864	-0.397274	-0.397149	809864
5.0	-0.177596	771314	-0.177759	-0.177596	771314
6.0	0.150645	257251	0.150591	0.150645	257251
7.0	0.300079	270520	0.300231	0.300079	270520
8.0	0.171650	807138	0.171925	0.171650	807138
	...		...	...	
15.0	-0.014224	472827	...	-0.014224	472801

From the numerical results, when we use the Chebyshev polynomial of order $N=n-2$, we guess that the maximum error over $[0, 1]$ is about 10^{-N} . Moreover the error does not increase so much even if the length of the interval is increased.

All the computations were performed on the FACOM 230-75 at Computer Center, Kyushu University.

References

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