SPECTRA OF FLAG MANIFOLDS

By

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1. Introduction

Let M be a compact connected Riemannian manifold with a Riemannian metric tensor g and Δ the Laplace-Beltrami operator acting on the space of differentiable functions on M . Then it is well known that Δ has discrete eigenvalues with finite multiplicity. $Sp(M, g)$, the set of all eigenvalues (with multiplicity) of Δ , is an important invariant in Riemannian geometry. So it is very interesting to investigate $Sp(M, g)$ and related problems. For example, to know how much information about M can be extracted from $Sp(M, g)$. It seems to be impossible to determine explicitly $Sp(M, g)$ for general Riemannian manifolds. So it may be natural to investigate these for standard manifolds as model spaces and determine them explicitly. In relatively few cases such as flat tori, Klein bottles and compact symmetric spaces containing compact group manifolds, $Sp(M, g)$ are explicitly determined and related problems, for examples, eigenfunctions, Zeta functions, coefficients of the Minakshisundaram-Pleijel's expansion and so on are determined explicitly (e.g., [1], [3], [14], [15]). Recently $Sp(M, g)$ for differential forms and eigenforms of complex projective spaces and spheres are determined [7].

The aims of this paper are to specify $Sp(M, g)$ of a flag manifold $M=G/H$, determine the minimal eigenvalue, and describe ζ -function and the fundamental solution of the heat equation on M in group-theoretical terms. Our manifold under consideration is not the so-called generalized flag manifold $M=G/H$ where H is the centralizer $C(T_1)$ of an arbitrary torus T_1 of G (e.g., [17], [18]) but the one such that H is a maximal torus. The determination of $Sp(M, g)$ essentially depends on a Theorem due to H. Freudenthal [5]. Our method is based on the representation theory of compact Lie groups.

2. Preliminaries

In this section, we mention the relation between the Laplace-Beltrami operator Δ_B and the invariant differential operator γ_B on $M=G/H$ associated with an invariant bilinear form B on $\mathfrak{G}_0$, the Lie algebra of G which is not assumed to be compact.

Let G be a connected Lie group, H a closed subgroup and $\mathfrak{G}_0$ (resp. $\mathfrak{H}_0$) the Lie algebra of G (resp. H). A homogeneous space $M=G/H$ with a G -invariant (indefinite) Riemannian metric g is said to be *naturally reductive* if it admits an $\text{Ad}(H)$ -invariant decomposition $\mathfrak{G}_0 = \mathfrak{H}_0 + \mathfrak{M}_0$ satisfying the condition

$$B(X, [Z, Y]_{\mathfrak{M}_0}) + B([Z, X]_{\mathfrak{M}_0}, Y) = 0,$$

where B is an $\text{Ad}(H)$ -invariant non-degenerate symmetric bilinear form on $\mathfrak{G}_0$ corresponding to g . Assume that $\mathfrak{G}_0$ admits an $\text{Ad}(G)$ -invariant non-degenerate symmetric bilinear form B such that its restriction $B_{\mathfrak{H}_0}$ of B to $\mathfrak{H}_0$ is non-degenerate. Then the direct sum decomposition $\mathfrak{G}_0 = \mathfrak{H}_0 + \mathfrak{M}_0$ defined by

$$\mathfrak{M}_0 = \{X \in \mathfrak{G}_0 : B(X, Y) = 0 \quad \text{for all } Y \in \mathfrak{H}_0\}$$

is $\text{Ad}(H)$ -invariant and the restriction $B_{\mathfrak{M}_0}$ of B to $\mathfrak{M}_0$ is also non-degenerate and $\text{Ad}(H)$ -invariant. Then the homogeneous space $M=G/H$ is naturally reductive with respect to the decomposition $\mathfrak{G}_0 = \mathfrak{H}_0 + \mathfrak{M}_0$ defined above and the G -invariant Riemannian metric g defined by B (See [9]). In what follows, we assume, in addition, $B_{\mathfrak{M}_0}$ is positive definite, i.e., M is Riemannian.

Let $U_{\mathfrak{M}_0}$ and $U_{\mathfrak{H}_0}$ be open neighbourhoods of 0 in $\mathfrak{M}_0$ and $\mathfrak{H}_0$, respectively, such that the mapping $(X, Y) \rightarrow \exp X \cdot \exp Y$ is a diffeomorphism of $U_{\mathfrak{M}_0} \times U_{\mathfrak{H}_0}$ onto an open neighbourhood of e in G and such that the natural projection $\pi: G \rightarrow G/H$ is one-to-one on the set $\exp U_{\mathfrak{M}_0}$. Let $X_1, \dots, X_r$ be a basis of $\mathfrak{M}_0$, $X_{r+1}, \dots, X_n$ a basis of $\mathfrak{H}_0$. Then the mapping

$$\exp(x_1 X_1 + \dots + x_r X_r) \cdot \exp(x_{r+1} X_{r+1} + \dots + x_n X_n) \longrightarrow (x_1, \dots, x_n)$$

is a coordinate system on the neighbourhood $\exp U_{\mathfrak{M}_0} \cdot \exp U_{\mathfrak{H}_0}$ of e in G and the mapping

$$\pi(\exp(x_1 X_1 + \dots + x_r X_r)) \longrightarrow (x_1, \dots, x_r)$$

is a coordinate system on the neighbourhood $\pi(\exp U_{\mathfrak{M}_0})$ of $0 = \{H\}$ in G/H .

Note that in our case this coordinate system $(x_1, \dots, x_r)$ is *normal* by virtue of the natural reductivity.

Let $\mathbf{D}(G)$, $\mathbf{D}_0(G)$ and $\mathbf{D}(G/H)$ be the set of left G -invariant differential operators on G , the set of operators in $\mathbf{D}(G)$ which are invariant under all right translations from H and the set of all G -invariant differential operators on G/H , respectively. Let $C_0^\infty(G)$ denote the set of all C^∞ -functions on G which are constant on each coset gH . Then the mapping $f \rightarrow \tilde{f}$ where $\tilde{f} = f \circ \pi$ is an isomorphism of the algebra $C^\infty(G/H)$ onto $C_0^\infty(G)$. Each $D \in \mathbf{D}_0(G)$ leaves the subspace $C_0^\infty(G)$ invariant; let D_0 denote its restriction to $C_0^\infty(G)$. Then $\mathbf{D}(G/H)$ is canonically isomorphic to the algebra of restrictions $\{D_0 : D \in \mathbf{D}_0(G)\}$. The isomorphism $\mu : \mathbf{D}_0(G) \rightarrow \mathbf{D}(G/H)$ is given by $[\mu(D_0)f]^\sim = D_0 \tilde{f}$ for any $f \in C^\infty(G/H)$ (e.g., [6]).

Under the above situation, put $a_{ij} = B(X_i, X_j)$ ($1 \leq i, j \leq n$) and let (a^{ij}) denote the inverse of the matrix (a_{ij}) . Then the element $\gamma_B = \sum_{i,j=1}^n a^{ij} X_i X_j$ of $U(\mathfrak{G}_0)$, the universal enveloping algebra of $\mathfrak{G}_0$, is independent of the choice of a basis (X_i) and $\text{Ad}(G)$ -invariant. So γ_B is an element of $\mathbf{D}_0(G)$ as a differential operator, where $X \in \mathfrak{G}_0$ is regarded as a left-invariant vector field on G ;

$$(Xf)(g) = \left[\frac{d}{dt} f(g \cdot \exp tX) \right] (0) \quad (1)$$

for $g \in G$ and $f \in C^\infty(G)$.

Let Δ_B be the Laplace-Beltrami operator associated with B . Then we have the following

PROPOSITION 1. $\Delta_B = \mu(\gamma_B)$, or equivalently $\tilde{\Delta}_B = \gamma_B$.

PROOF. Since γ_B is independent of the choice of a basis, we choose a basis $X_1, \dots, X_r$ of $\mathfrak{M}_0$ which are orthonormal with respect to $B_{\mathfrak{M}_0}$ and a basis $X_{r+1}, \dots, X_n$ of $\mathfrak{H}_0$ such that

$$\gamma_B = X_1^2 + \dots + X_r^2 + Q(X_{r+1}, \dots, X_n),$$

where $Q(X_{r+1}, \dots, X_n) = \sum_{j=1}^{n-r} \varepsilon_{r+j} X_j^2$ ($\varepsilon_{r+j} = \pm 1$, $1 \leq j \leq n-r$).

Then $(x_1, \dots, x_r)$ is a normal coordinate system with respect to an orthonormal basis (X_i) in $T_0(M)$. So we have

$$(\Delta_B f)(0) = \sum_{i=1}^r \frac{\partial^2 f}{\partial x_i^2}(0) \quad (f \in C^\infty(M)).$$

Put $P(\partial_1, \dots, \partial_r) = \sum_{i=1}^r \left(\frac{\partial}{\partial x_i} \right)^2$. Then we have

$$(\Delta_B f)(p) = [(P(\partial_1, \dots, \partial_r) \tilde{f})(g \cdot \exp(x_1 X_1 + \dots + x_r X_r))](0) \quad (2)$$

for $p = g \cdot 0$ and $f \in C^\infty(M)$. On the other hand, using (1), an easy calculation shows that $(\gamma_B \tilde{f})(g)$ is equal to the right hand side of (2). This completes the proof. q. e. d.

COROLLARY 1 (e.g., [6] p. 451). *Let G be a connected non-compact semi-simple Lie group with finite center, H a maximal compact subgroup of G and B the Killing form of $\mathfrak{G}_0$. Then $\Delta_B = \mu(\Omega)$, where $\gamma_B = \Omega$ is the Casimir element in $U(\mathfrak{G}_0)$.*

PROOF. The Cartan decomposition of $\mathfrak{G}_0$ satisfies our condition. q. e. d.

Let G be a compact connected Lie group. Then $\mathfrak{G}_0 = \mathfrak{C}_0 \oplus \mathfrak{G}'_0$, where $\mathfrak{C}_0$ is the center of $\mathfrak{G}_0$ and $\mathfrak{G}'_0 = [\mathfrak{G}_0, \mathfrak{G}_0]$ is semi-simple and compact. Put $B = B_0 + (-B_1)$, where B_0 is an arbitrary positive definite symmetric bilinear form on $\mathfrak{C}_0$ and B_1 is the Killing form of $\mathfrak{G}'_0$. Let H be a closed subgroup of G .

COROLLARY 2 (e.g., [16] p. 20). *Under the above situation, we have $\Delta_B = \mu(\gamma_B)$.*

We apply Corollary 2 to compact simple Lie groups. In this case, we have $\Delta = \mu(\Omega)$.

3. Zeta Functions and Fundamental Solutions of Heat Equations on G/H

In this section, we describe the ζ -function associated with $Sp(M, g)$ and the fundamental solution of the heat equation on M in group-theoretical terms. Let G be a compact connected simple Lie group. Let dg , dh , and $d\mu(x)$ denote the invariant measures on G , H , and $M = G/H$, respectively, normalized by

$$\int_G dg = \int_H dh = 1 \quad \text{and} \quad \int_G f(g) dg = \int_{G/H} \left\{ \int_H f(gh) dh \right\} d\mu(x),$$

where $x = g \cdot 0 \in M$ and $f \in C(G)$.

Let $\hat{G}$ be the set of all equivalence classes of irreducible unitary representations of G . Furthermore we regard $\hat{G}$ as the complete set of the representatives and denote a class $[\rho] \in \hat{G}$ by ρ if there is no confusion.

An element $\rho \in \hat{G}$ is said to be of class 1 relative to H if there exists a non-zero

vector v in V_ρ , the representation space of ρ , such that $\rho(h)v = v$ for all $h \in H$. We denote by $\hat{G}(H)$ the set of all elements of $\hat{G}$ which are of class 1 relative to H . Let $L_2(M)$ be the usual Hilbert space over $\mathbb{C}$ with respect to $d\mu$. G acts on $L_2(M)$ through its left regular representation;

$$(\pi(g)f)(x) = f(g^{-1}x)$$

for $f \in L_2(M)$, $g \in G$ and $x \in M$. $(\pi, L_2(M))$ is nothing but the unitary representation of G which is unitarily induced from the trivial representation 1_H of H . Then as a special case of the well known Frobenius Reciprocity Law on induced representations of a compact group, $(\pi, L_2(M))$ decomposes as follows:

THEOREM 1. (A. Weil [20]) *We have*

$$L_2(M) = \sum_{\rho \in \hat{G}(H)} m(\rho) V_\rho$$

as a unitary left G -module, where V_ρ is a representation space of $\rho \in \hat{G}$ and $m(\rho)$ a multiplicity of ρ in $L_2(M)$.

PROOF. See [19], Chap. 5, 5.3.3.

q. e. d.

REMARK. $m(\rho)$ = the multiplicity of 1_H in ρ

$$= \dim \{v \in V_\rho : \rho(h)v = v \text{ for all } h \in H\}.$$

Since $\pi(g) \cdot \Delta = \Delta \cdot \pi(g)$ for all $g \in G$, by Schur's lemma Δ is a scalar operator on each V_ρ for $\rho \in \hat{G}(H)$. If we canonically identify (ρ, V_ρ) with a subrepresentation of $(\pi, L_2(M))$, then we have the following

PROPOSITION 2. *We have*

$$\gamma|_{V_\rho} = \sum_{i,j=1}^{d(\rho)} a^{ij} \cdot d\rho(X_i) \cdot d\rho(X_j),$$

or equivalently

$$d\pi(\Omega)|_{V_\rho} = \sum_{i,j=1}^{d(\rho)} a^{ij} d\pi(X_i)|_{V_\rho} \cdot d\pi(X_j)|_{V_\rho},$$

where $d\pi$ is the infinitesimal representation of $U(\mathfrak{G}_0)$ on $C^\infty(M)$ and $d(\rho) = \dim V_\rho$.

PROOF. This follows immediately from the definition of γ and a simple calculation.

q. e. d.

So the eigenvalue of Δ on V_ρ is that of $\sum_{i,j} a^{ij} d\rho(X_i) \cdot d\rho(X_j)$. We denote this value by $q(\Delta)$ or $q(\gamma)$. Define

$$\zeta_M(t) = \sum_{\lambda} e^{-\lambda t},$$

where λ ranges over the eigenvalues (with multiplicity) of Δ . Then we have the following

PROPOSITION 3. *If the Riemannian metric on M is defined by the Killing form B , then $M=G/H$ has ζ -function given by*

$$\zeta_M(t) = \sum_{\rho \in \hat{G}(H)} m(\rho) \cdot d(\rho) \cdot e^{-q(\gamma)t}. \quad (3)$$

PROOF. This is a direct consequence of Theorem 1 and Proposition 2.

q. e. d.

REMARK. The above formula is formal. To specify $\zeta_M(t)$, we must determine $\hat{G}(H)$, $m(\rho)$, $d(\rho)$ and $q(\gamma)$. These are done later when H is a maximal torus.

To describe the fundamental solution of the heat equation on M , we need the Peter-Weyl Theorem for homogeneous spaces of compact groups. Let $v_1, \dots, v_{n_\rho}$ is an orthonormal basis of V_ρ ($\rho \in \hat{G}(H)$) such that $v_1, \dots, v_{m_\rho}$ is a basis of $V_\rho^0 = \{v \in V_\rho : \rho(h)v = v \text{ for all } h \in H\}$. Let $\rho_{j\alpha}(g) = (\rho(g)v_\alpha, v_j)$ ($1 \leq j \leq n_\rho$, $1 \leq \alpha \leq m_\rho$) and $S = \{\sqrt{n_\rho} \cdot \rho_{j\alpha} : \rho \in \hat{G}(H), 1 \leq j \leq n_\rho \text{ and } 1 \leq \alpha \leq m_\rho\}$. Then we have

THEOREM 2. (E. Cartan [4]) *S is a complete orthonormal system and any continuous function on M is uniformly approximated by finite linear combinations of elements of S .*

We put

$$\Psi_\rho = \sum_{\alpha=1}^{m_\rho} \rho_{\alpha\alpha} \quad (4)$$

and

$$E(t, x) = \sum_{\rho \in \hat{G}(H)} d(\rho) \cdot e^{-q(\gamma)t} \cdot \Psi_\rho(x) \quad (x = g \cdot 0 \in M). \quad (5)$$

Note that ψ_ρ is independent of the choice of an orthonormal basis of V_ρ^0 , ψ_ρ

is regarded as a function on M and coincides with the character of ρ if $H = \{e\}$. Then we have

$$\Delta\psi_\rho = q(\Delta) \cdot \psi_\rho. \quad (6)$$

PROPOSITION 4. *The right hand side of (5) converges absolutely and uniformly on $[\varepsilon, \infty) \times M$ for $\varepsilon > 0$.*

PROOF. We have the following estimate of (5);

$$\begin{aligned} \left| \sum_{\rho \in \hat{G}(H)} d(\rho) e^{-q(\Delta)t} \cdot \psi_\rho(x) \right| &\leq \sum_{\rho} n_{\rho} \cdot e^{-q(\Delta)t} \cdot \left(\sum_{\alpha=1}^{m_{\rho}} |\rho_{\alpha\alpha}(x)| \right) \\ &\leq \sum_{\rho} n_{\rho} \cdot e^{-q(\Delta)t} \cdot m_{\rho} = \sum_{\rho} m_{\rho} \cdot n_{\rho} \cdot e^{-q(\Delta)t} \\ &= \sum m(\rho) \cdot d(\rho) e^{-q(\Delta)t} = \zeta_M(t). \end{aligned}$$

The convergence of $\zeta_M(t)$ is assured in general compact Riemannian manifolds (e.g., [1]). q. e. d.

We put

$$\varepsilon(t, x, y) = E(t, g'^{-1}g) \quad (x = g \cdot 0 \text{ and } y = g' \cdot 0) \quad (7)$$

It is easy to see that $E(t, g'^{-1}g)$ depends only on cosets $x = g \cdot 0$ and $y = g' \cdot 0$. So the above ε has a meaning. Then we have the following

THEOREM 3 (cf. [15]). *$\varepsilon(t, x, y)$ is the fundamental solution of the heat equation $\frac{\partial}{\partial t} u(t, x) = \Delta_x u(t, x)$, that is, $\varepsilon(t, x, y)$ has the following properties;*

- (1) $0 < \varepsilon(t, x, y) \in C^\infty((0, \infty) \times M \times M)$,
- (2) $\frac{\partial}{\partial t} \varepsilon = \Delta_x \varepsilon = \Delta_y \varepsilon$,
- (3) $\lim_{t \rightarrow 0+} \int_M \varepsilon(t, x, y) f(y) d\mu(y) = f(x) \quad \text{for } f \in C(M).$

PROOF. Since the existence and the uniqueness are assured in general compact Riemannian manifolds (e.g., [1]), we will only mention that our ε is the one in the general case.

$$E(t, y^{-1}x) = \sum_{\rho \in \hat{G}(H)} d(\rho) e^{-q(\Delta)t} \cdot \psi_\rho(y^{-1}x)$$

$$\begin{aligned}
&= \sum_{\rho} n_{\rho} \cdot e^{-q(\Delta)t} \left(\sum_{\alpha=1}^{m_{\rho}} \rho_{\alpha\alpha}(y^{-1}x) \right) \\
&= \sum_{\rho} n_{\rho} \cdot e^{-q(\Delta)t} \left(\sum_{\alpha=1}^{m_{\rho}} \sum_{j=1}^{n_{\rho}} \rho_{\alpha j}(y^{-1}) \rho_{j\alpha}(x) \right) \\
&= \sum_{\rho} \sum_{\alpha, j} e^{-q(\Delta)t} (\sqrt{n_{\rho}} \rho_{j\alpha}(x)) \cdot (\sqrt{n_{\rho}} \cdot \rho_{j\alpha}(y)).
\end{aligned}$$

From Theorem 2 and the fact that $\sqrt{n_{\rho}} \rho_{j\alpha}(x)$ is the normalized eigenfunction of Δ corresponding to an eigenvalue $q(\Delta)$, it follows that the last term is nothing but the sum $\sum_i e^{-\lambda_i t} \varphi_i(x) \overline{\varphi_i(y)}$ in the general case, where φ_i is the normalized eigenfunction corresponding to λ_i . So our ε is nothing but a version of it in group-theoretical terms. By the way, (1) easily follows from Proposition 4 and (2) follows from (6). q. e. d.

4. Spectra of Flag Manifolds

As mentioned in the introduction, we treat a flag manifold $M = G/H$ such that H is a maximal torus. Without loss of generality, we may assume G is a *simply connected* compact simple Lie group. Therefore algebraic properties of $\hat{G}$ are reduced to those of $\mathfrak{G}_0$ and *vice versa*. Furthermore, they are reduced to the properties of the corresponding complex simple Lie algebra $\mathfrak{G}$.

Let $\mathfrak{G}$ be a complex simple Lie algebra, $\mathfrak{H}$ a Cartan subalgebra, Δ the root system of $(\mathfrak{G}, \mathfrak{H})$ and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ the simple root system with respect to some fixed lexicographic order. Let $(\ , \)$ denote the bilinear form in $\mathfrak{H}$ and $\mathfrak{H}^*$ induced from the Killing form B of $\mathfrak{G}$. If we put $\alpha_i^* = 2\alpha_i/(\alpha_i, \alpha_i)$ ($i = 1, 2, \dots, l$), then the set $\{\omega_j \in \mathfrak{H}^* : (\alpha_i^*, \omega_j) = \delta_{ij}, 1 \leq i, j \leq l\}$ forms a fundamental weight system. Let $\hat{\mathfrak{G}}$ denote the set of all dominant integral forms:

$$\hat{\mathfrak{G}} = \{ \Lambda = \sum_{i=1}^l m_i \omega_i : m_i \geq 0 \ (1 \leq i \leq l) \text{ are integers} \}.$$

Also $\hat{\mathfrak{G}}(\mathfrak{H})$ denotes the set of all elements in $\hat{\mathfrak{G}}$ which are of class 1 relative to $\mathfrak{H}$. This means that the irreducible representation $(\xi_{\Lambda}, V_{\Lambda})$ with the highest weight $\Lambda \in \hat{\mathfrak{G}}$ has a non-zero $\xi_{\Lambda}(\mathfrak{H})$ -invariant vector in the representation space V_{Λ} . Then the following result on $\hat{\mathfrak{G}}(\mathfrak{H})$ is known and plays an important role in our later arguments. The original paper states the condition under which an irreducible representation has 0 as a weight.

THEOREM 4. (H. Freudenthal [5]) *Let $(\xi_{\Lambda}, V_{\Lambda})$ be the irreducible repre-*

sensation with the highest weight $\lambda \in \hat{\mathfrak{G}}$. Then λ belongs to $\hat{\mathfrak{G}}(\mathfrak{H})$ if and only if all the weights of (ξ_λ, V_λ) are linear combinations of roots with integral coefficients.

If we investigate with some care the relation between $\{\omega_1, \dots, \omega_l\}$ and $\{\alpha_1, \dots, \alpha_l\}$, we have the following version of the Theorem.

THEOREM 5. $\hat{\mathfrak{G}} \ni \lambda (\neq 0)$ belongs to $\hat{\mathfrak{G}}(\mathfrak{H})$ if and only if $\lambda = \sum_{i=1}^l p_i \alpha_i$, where $p_i \geq 1$ ($1 \leq i \leq l$) are integers.

PROOF. Only if part is obvious. Because every weight λ of (ξ_λ, V_λ) is of the form $\lambda = \lambda - \sum_{i=1}^l n_i \alpha_i$, where n_i ($1 \leq i \leq l$) are non-negative integers. Therefore if $\lambda = \sum_{i=1}^l p_i \alpha_i$, then every weight λ is of the form $\lambda = \sum_{i=1}^l (p_i - n_i) \alpha_i$ and the result follows from Theorem 4.

Though if part is proved case by case by making use of the classification, for later use, we prove directly without using the classification. Let $\lambda \in \hat{\mathfrak{G}}(\mathfrak{H})$. Then 0 is a weight of (ξ_λ, V_λ) and $0 = \lambda - \sum_{i=1}^l p_i \alpha_i$, where p_i ($1 \leq i \leq l$) are non-negative integers. On one hand, λ is written as $\lambda = \sum_{i=1}^l m_i \omega_i$, where m_i ($1 \leq i \leq l$) are non-negative integers. So we have $\lambda = \sum_{i=1}^l p_i \alpha_i = \sum_{i=1}^l m_i \alpha_i$. We note that integers (m_i) and (p_i) are related by the Cartan matrix $A = (a_{ij})$ as follows;

$$(m_1, \dots, m_l) = (p_1, \dots, p_l) \cdot A. \quad (8)$$

So we have a system $\mathfrak{M}(\mathfrak{G})$ of linear inequalities;

$$\mathfrak{M}(\mathfrak{G}): \begin{cases} m_1 = p_1 a_{11} + \dots + p_l a_{l1} \geq 0 \\ \dots\dots\dots \\ m_i = p_1 a_{1i} + \dots + p_l a_{li} \geq 0 \\ \dots\dots\dots \\ m_l = p_1 a_{1l} + \dots + p_l a_{ll} \geq 0 \end{cases}$$

Suppose $p_i = 0$ for some i . Then $m_i = p_1 a_{1i} + \dots + p_{i-1} a_{i-1i} + p_{i+1} a_{i+1i} + \dots + p_l a_{li} \geq 0$. Since $\Pi = \{\alpha_1, \dots, \alpha_l\}$ is the fundamental root system and $\mathfrak{G}$ is simple, there exists at least an index $j (\neq i)$ such that $a_{ji} = 2(\alpha_j, \alpha_i) / (\alpha_i, \alpha_i) < 0$. Then we have $p_j = 0$ because $m_i \geq 0$, $p_k \geq 0$ ($1 \leq k \leq l$), $a_{ki} \leq 0$ ($k \neq i$) and $a_{ji} < 0$. Thus the set $I_i = \{j: a_{ji} < 0\}$ is not empty and $p_j = 0$ for any $j \in I_i$. If α_i is an edge point of the Dynkin diagram of $\mathfrak{G}$, then $I_i = \{j\}$ and there exists an index $k (\neq i, \neq j)$

such that $a_{kj} < 0$. In this case, we see $p_k = 0$ by considering m_j in $\mathfrak{M}(\mathfrak{G})$. If α_i is not an edge point, $\mathbf{I}_i$ contains at least two indices. For each $j \in \mathbf{I}_i$, we can find an index k ($\neq i, \neq j$) such that $a_{kj} < 0$. Then we see $p_k = 0$ by the same reason. Whichever case occurs, we consider the set $\mathbf{I}_j$ and continue the same process. Since the Dynkin diagram is connected, it follows finally that $p_1 = \dots = p_l = 0$. This contradicts to the assumption that $\Lambda = \sum_{i=1}^l p_i \alpha_i \neq 0$. q. e. d.

Now we can specify $\hat{\mathfrak{G}}(\mathfrak{H})$ by using $p_1, \dots, p_l$ as parameters.

THEOREM 5'.

$$\hat{\mathfrak{G}}(\mathfrak{H}) = \left\{ \Lambda = \sum_{i=1}^l p_i \alpha_i \in \mathfrak{H}^*: \begin{array}{ll} (1) \ p_i \in \mathbf{Z}, \ p_i \geq 1 & (1 \leq i \leq l), \\ (2) \ (p_1, \dots, p_l) \text{ satisfies } \mathfrak{M}(\mathfrak{G}). \end{array} \right\}.$$

COROLLARY 1. $\hat{\mathfrak{G}} = \hat{\mathfrak{G}}(\mathfrak{H})$ if and only if $\mathfrak{G}$ is of type $\mathbf{E}_8, \mathbf{F}_4$ or $\mathbf{G}_2$.

So, in these cases every irreducible representation appears as a direct summand of $(\pi, L_2(M))$ and $\hat{\mathfrak{G}}(\mathfrak{H}) = \hat{\mathfrak{G}}$ has $\{\omega_1, \dots, \omega_l\}$ as a free basis.

COROLLARY 2. $\hat{\mathfrak{G}}(\mathfrak{H})$ is closed under the addition.

PROOF. $\mathfrak{M}(\mathfrak{G})$ is a system of linear inequalities. q. e. d.

So $\hat{\mathfrak{G}}(\mathfrak{H})$ becomes a semi-group under the addition. But in contrast with symmetric Riemannian spaces [13], there exist no free bases for $\hat{\mathfrak{G}}(\mathfrak{H})$ except $\mathfrak{G} = \mathbf{E}_8, \mathbf{F}_4$ and $\mathbf{G}_2$ (See Proposition 7).

In case of a flag manifold, ζ -function is expressed as follows. Factors appeared in ζ_M are expressible by the well known formulae in the theory of semi-simple Lie algebras.

THEOREM 6. We have

$$\zeta_M(t) = \sum_{\Lambda \in \hat{\mathfrak{G}}(\mathfrak{H})} m(\Lambda) \cdot d(\Lambda) \cdot e^{-q(\Lambda)t}, \quad (9)$$

where $m(\Lambda)$ is the multiplicity of the weight 0 of (ξ_Λ, V_Λ) , $d(\Lambda) = \dim V_\Lambda$ and $q(\Lambda)$ is the eigenvalue of the Casimir operator on V_Λ .

As for $d(\Lambda)$, we have the famous Weyl Dimension Formula (e.g., [8], [19] and [12], [17] for compact groups):

$$d(\Lambda) = \prod_{\alpha > 0} \frac{(\Lambda + \delta, \alpha)}{(\delta, \alpha)} \quad \left(\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha \right).$$

This formula is very useful because of its easy computability. As for $m(\lambda)$, the present author only knows two formulae which are very well in theory but work not so easily. One is the Kostant Multiplicity Formula (e.g., [8], [10], [19]) and the other is the Freudenthal Recurrence Formula [18]. We quote here the first one:

$$m(\lambda) = \sum_{s \in \mathbf{W}} \text{sgn}(s) \cdot \mathbf{P}(s(\lambda + \delta) - \delta), \quad (10)$$

where $\mathbf{P}(\mu)$ is the number of ways of writing μ as a sum of positive roots for an integral form μ and $\mathbf{W}$ is the Weyl group.

On the second formula, there is a programme for computer given by Krusemeyer (BIT=Nordisk Tidskrift for Informationsbehandling 11 (1971), 310-316) for determining the multiplicities of the dominant weights. There is a useful table of the multiplicities of weights to use a routine of him for finding the multiplicity of a general weight (See G. B. Seligman's book; Rational method in Lie algebras, Appendix A).

As for $q(\lambda)$, we have the useful Freudenthal's Formula (e.g., [8], [19] and [12], [17] for compact groups):

$$q(\lambda) = (\lambda + \delta, \lambda + \delta) - (\delta, \delta). \quad (11)$$

$\hat{\mathfrak{G}}(\mathfrak{H})$ is already specified by Theorem 5'. So we can theoretically compute $\zeta_{\mathcal{M}}(t)$.

In the rest of this paper, we specify $Sp(M, g)$ and give the minimal eigenvalue for each type of $\mathfrak{G}$. We use freely the notations and the results tabulated in Planches I~IX of Bourbaki [2] concerning with weights and roots of $\mathfrak{G}$.

At first we prepare the following

LEMMA. Let $\lambda \in \hat{\mathfrak{G}}$. If λ corresponds to the adjoint representation, then $q(\lambda) = 1$.

PROOF. Since $(\xi_{\lambda}, V_{\lambda}) = (\text{ad}, \mathfrak{G})$, the Casimir operator on V_{λ} is equal to $\sum_{i,j} a^{ij} \text{ad } X_i \cdot \text{ad } X_j$ and $\sum_{i,j} a^{ij} \text{ad } X_i \cdot \text{ad } X_j = q(\lambda) \cdot \mathbf{I}_n$ on $V_{\lambda} = \mathfrak{G}$ ($n = \dim \mathfrak{G}$). Computing the traces of both sides, it follows that $\sum_{i,j} a^{ij} T_r(\text{ad } X_i \cdot \text{ad } X_j) = \sum_{i,j} a^{ij} B(X_i, X_j) = \sum_{i,j} a^{ij} a_{ij} = n$ and $q(\lambda) \cdot T_r(\mathbf{I}_n) = q(\lambda) \cdot n$. So we have $q(\lambda) = 1$.

q. e. d.

We identify $\lambda = \sum_{i=1}^l p_i \alpha_i \in \hat{\mathfrak{G}}(\mathfrak{H})$ with $P = (p_1, \dots, p_l)$. Then $q(\lambda)$ is a polynomial in l variables $p_1, \dots, p_l$ without constant term. It is seen that there exists

only one point P at which $q(\lambda)$ attains its minimum. We denote this point by P_0 and also denote by λ_0 the element of $\hat{\mathfrak{G}}(\mathfrak{H})$ which corresponds to P_0 . It is known that the eigenvalues of the Casimir operator do not separate $\hat{\mathfrak{G}}$, i.e., different elements of $\hat{\mathfrak{G}}$ may have the same eigenvalue. This is true for $\hat{\mathfrak{G}}(\mathfrak{H})$ too.

Results are tabulated as follow.

Type A_l

$$(1) \quad \mathfrak{M}(\mathfrak{G}): \begin{cases} m_1 = 2p_1 - p_2 \geq 0, \\ m_2 = -p_1 + 2p_2 - p_3 \geq 0, \\ \vdots \\ m_{l-1} = -p_{l-2} + 2p_{l-1} - p_l \geq 0, \\ m_l = -p_{l-1} + 2p_l \geq 0. \end{cases}$$

$$(2) \quad q(\lambda) = \frac{1}{l+1} \left\{ \sum_{i=1}^l p_i^2 - \sum_{i=1}^{l-1} p_i p_{i+1} + \sum_{i=1}^l p_i \right\} \\ = \frac{1}{2(l+1)} \left\{ \sum_{i=1}^l m_i p_i + 2 \sum_{i=1}^l p_i \right\},$$

$$(3) \quad P_0 = (1, 1, \dots, 1) \longleftrightarrow \lambda_0 = \alpha_1 + \dots + \alpha_l \text{ (highest root),}$$

$$(4) \quad \xi_{\lambda_0} = \text{adjoint representation,}$$

$$(5) \quad q(\lambda_0) = 1.$$

Type B_l

$$(1) \quad \mathfrak{M}(\mathfrak{G}): \begin{cases} m_1 = 2p_1 - p_2 \geq 0, \\ m_2 = -p_1 + 2p_2 - p_3 \geq 0, \\ \vdots \\ m_{l-1} = p_{l-2} + 2p_{l-1} - p_l \geq 0, \\ m_l = -2p_{l-1} + 2p_l \geq 0 \text{ (i.e., } -p_{l-1} + p_l \geq 0). \end{cases}$$

$$(2) \quad q(\lambda) = \frac{1}{4l-2} \left\{ \sum_{i=1}^{l-1} 2p_i^2 + p_l^2 - 2 \sum_{i=1}^{l-1} p_i p_{i+1} + 2 \sum_{i=1}^{l-1} p_i + p_l \right\} \\ = \frac{1}{4l-2} \left\{ \sum_{i=1}^{l-1} m_i p_i + \frac{1}{2} m_l p_l + 2 \sum_{i=1}^{l-1} p_i + p_l \right\},$$

$$(3) \quad P_0 = (1, 1, \dots, 1) \longleftrightarrow \lambda_0 = \omega_1 = \alpha_1 + \dots + \alpha_l,$$

(4) ξ_{A_0} = representation with highest weight ω_1 ,

(5) $q(A_0) = \frac{l}{2l-1}$.

Type C_l

(1) $\mathfrak{M}(\mathbb{G}): \begin{cases} m_1 = 2p_1 - p_2 \geq 0, \\ m_2 = -p_1 + 2p_2 - p_3 \geq 0, \\ \vdots \\ m_{l-2} = -p_{l-3} + 2p_{l-2} - p_{l-1} \geq 0, \\ m_{l-1} = -p_{l-2} + 2p_{l-1} - 2p_l \geq 0, \\ m_l = -p_{l-1} + 2p_l \geq 0. \end{cases}$

(2) $q(A) = \frac{1}{2(l+1)} \left\{ \sum_{i=1}^{l-1} p_i^2 + 2p_l^2 - \sum_{i=1}^{l-2} p_i p_{i+1} - 2p_{l-1} p_l + \sum_{i=1}^{l-1} p_i + 2p_l \right\}$
 $= \frac{1}{4(l+1)} \left\{ \sum_{i=1}^{l-1} m_i p_i + 2m_l p_l + 2 \sum_{i=1}^{l-1} p_i + 4p_l \right\},$

(3) $P_0 = (1, 2, \dots, 2, 1) \longleftrightarrow A_0 = \omega_2,$

(4) ξ_{A_0} = representation with highest weight ω_2 ,

(5) $q(A_0) = \frac{4l-1}{4(l+1)}.$

Type D_l

(1) $\mathfrak{M}(\mathbb{G}): \begin{cases} m_1 = 2p_1 - p_2 \geq 0, \\ m_2 = -p_1 + 2p_2 - p_3 \geq 0, \\ \vdots \\ m_{l-3} = -p_{l-4} + 2p_{l-3} - p_{l-2} \geq 0, \\ m_{l-2} = -p_{l-3} + 2p_{l-2} - p_{l-1} - p_l \geq 0, \\ m_{l-1} = -p_{l-2} + 2p_{l-1} \geq 0, \\ m_l = -p_{l-2} + 2p_l \geq 0. \end{cases}$

(2) $q(A) = \frac{1}{2(l-1)} \left\{ \sum_{i=1}^l p_i^2 - \sum_{i=1}^{l-2} p_i p_{i+1} - p_{l-2} p_l + \sum_{i=1}^l p_i \right\}$
 $= \frac{1}{4(l-1)} \left\{ \sum_{i=1}^l m_i p_i + 2 \sum_{i=1}^l p_i \right\},$

- (3) $P_0 = (1, 2, \dots, 2, 1, 1) \longleftrightarrow \Lambda_0 = \text{highest root},$
- (4) $\xi_{\Lambda_0} = \text{adjoint representation},$
- (5) $q(\Lambda_0) = 1.$

Type E_6

- (1) $\mathfrak{M}(\mathfrak{G}): \begin{cases} m_1 = 2p_1 - p_3 \geq 0, m_2 = 2p_2 - p_4 \geq 0, \\ m_3 = -p_1 + 2p_3 - p_4 \geq 0, m_4 = -p_2 - p_3 + 2p_4 - p_5 \geq 0, \\ m_5 = -p_4 + 2p_5 - p_6 \geq 0, m_6 = -p_5 + 2p_6 \geq 0. \end{cases}$
- (2) $q(\Lambda) = \frac{1}{12} \left\{ \sum_{i=1}^6 p_i^2 - p_1 p_3 - p_2 p_4 - p_3 p_4 - p_4 p_5 - p_5 p_6 + \sum_{i=1}^6 p_i \right\}$
 $= \frac{1}{24} \left\{ \sum_{i=1}^6 m_i p_i + 2 \sum_{i=1}^6 p_i \right\},$
- (3) $P_0 = (1, 2, 2, 3, 2, 1) \longleftrightarrow \Lambda_0 = \omega_2 = \text{highest root},$
- (4) $\xi_{\Lambda_0} = \text{adjoint representation},$
- (5) $q(\Lambda_0) = 1.$

Type E_7

- (1) $\mathfrak{M}(\mathfrak{G}): \begin{cases} m_1 = 2p_1 - p_3 \geq 0, m_2 = 2p_2 - p_4 \geq 0, \\ m_3 = -p_1 + 2p_3 - p_4 \geq 0, \\ m_4 = -p_2 - p_3 + 2p_4 - p_5 \geq 0, \\ m_5 = -p_4 + 2p_5 - p_6 \geq 0, \\ m_6 = -p_5 + 2p_6 - p_7 \geq 0, m_7 = -p_6 + 2p_7 \geq 0. \end{cases}$
- (2) $q(\Lambda) = \frac{1}{18} \left\{ \sum_{i=1}^7 p_i^2 - p_1 p_3 - p_2 p_4 - p_3 p_4 - p_4 p_5 - p_5 p_6 - p_6 p_7 + \sum_{i=1}^7 p_i \right\}$
 $= \frac{1}{36} \left\{ \sum_{i=1}^7 m_i p_i + 2 \sum_{i=1}^7 p_i \right\},$
- (3) $P_0 = (2, 2, 3, 4, 3, 2, 1) \longleftrightarrow \Lambda_0 = \text{highest root},$
- (4) $\xi_{\Lambda_0} = \text{adjoint representation},$

$$(5) \quad q(\Lambda_0) = 1.$$

Type E_8

$$(1) \quad \mathfrak{M}(\mathfrak{G}): \begin{cases} m_1 = 2p_1 - p_3 \geq 0, \quad m_2 = 2p_2 - p_4 \geq 0, \\ m_3 = -p_1 + 2p_3 - p_4 \geq 0, \\ m_4 = -p_2 - p_3 + 2p_4 - p_5 \geq 0, \\ m_5 = -p_4 + 2p_5 - p_6 \geq 0, \quad m_6 = -p_5 + 2p_6 - p_7 \geq 0, \\ m_7 = -p_6 + 2p_7 - p_8 \geq 0, \quad m_8 = -p_7 + 2p_8 \geq 0. \end{cases}$$

$$(2) \quad q(\Lambda) = \frac{1}{30} \left\{ \sum_{i=1}^8 p_i^2 - p_1 p_3 - p_2 p_4 - \sum_{i=3}^7 p_i p_{i+1} + \sum_{i=1}^8 p_i \right\} \\ = \frac{1}{60} \left\{ \sum_{i=1}^8 m_i p_i + 2 \sum_{i=1}^8 p_i \right\},$$

$$(3) \quad P_0 = (2, 3, 4, 6, 5, 4, 3, 2) \longleftrightarrow \Lambda_0 = \text{highest root},$$

$$(4) \quad \xi_{\Lambda_0} = \text{adjoint representation},$$

$$(5) \quad q(\Lambda_0) = 1,$$

$$(6) \quad \hat{\mathfrak{G}}(\mathfrak{H}) = \hat{\mathfrak{G}}.$$

Type F_4

$$(1) \quad \mathfrak{M}(\mathfrak{G}): \begin{cases} m_1 = 2p_1 - p_2 \geq 0, \quad m_2 = -p_1 + 2p_2 - p_3 \geq 0, \\ m_3 = -2p_2 + 2p_3 - p_4 \geq 0, \quad m_4 = -p_3 + 2p_4 \geq 0. \end{cases}$$

$$(2) \quad q(\Lambda) = \frac{1}{18} \{ 2p_1^2 + 2p_2^2 + p_3^2 + p_4^2 - 2p_1 p_2 - 2p_2 p_3 - p_3 p_4 \\ + 2p_1 + 2p_2 + p_3 + p_4 \} \\ = \frac{1}{18} \left\{ m_1 p_1 + m_2 p_2 + \frac{1}{2} m_3 p_3 + \frac{1}{2} m_4 p_4 + 2p_1 + 2p_2 + p_3 + p_4 \right\},$$

$$(3) \quad P_0 = (1, 2, 3, 2) \longleftrightarrow \Lambda_0 = \omega_4,$$

$$(4) \quad \xi_{\Lambda_0} = \text{representation with highest weight } \omega_4,$$

$$(5) \quad q(\Lambda_0) = \frac{2}{3},$$

$$(6) \quad \hat{\mathfrak{G}}(\mathfrak{H}) = \hat{\mathfrak{G}}.$$

Type G_2

$$(1) \quad \mathfrak{M}(\mathfrak{G}): \begin{cases} m_1 = 2p_1 - 3p_2 \geq 0 \\ m_2 = -p_1 + 2p_2 \geq 0. \end{cases}$$

$$(2) \quad q(\Lambda_0) = \frac{1}{12} \{p_1^2 + 3p_2^2 - 3p_1p_2 + p_1 + 3p_2\} \\ = \frac{1}{24} \{m_1p_1 + 3m_2p_2 + 2p_1 + 6p_2\},$$

$$(3) \quad P_0 = (2, 1) \longleftrightarrow \Lambda_0 = \omega_1,$$

$$(4) \quad \xi_{\Lambda_0} = \text{representation with highest weight } \omega_1,$$

$$(5) \quad q(\Lambda_0) = \frac{1}{2},$$

$$(6) \quad \hat{\mathfrak{G}}(\mathfrak{H}) = \hat{\mathfrak{G}}.$$

Summing up, we have

THEOREM 7. (1) $\hat{\mathfrak{G}}(\mathfrak{H})$, (2) the eigenvalue $q(\Lambda)$ for $\Lambda \in \hat{\mathfrak{G}}(\mathfrak{H})$, (3) the unique $\Lambda_0 \in \hat{\mathfrak{G}}(\mathfrak{H})$ which gives the minimum of $q(\Lambda)$, (4) the representation ξ_{Λ_0} corresponding to Λ_0 and (5) the minimal eigenvalue for $\mathfrak{G}$ of each type are given as in the above table.

REMARK. The minimal eigenvalues of compact symmetric spaces were obtained in [11].

Taking up $\mathfrak{G} = A_2$ as an example, we give some propositions and remarks which are general.

$$\hat{\mathfrak{G}}(\mathfrak{H}) = \{(p_1, p_2): 2p_1 \geq p_2, 2p_2 \geq p_1 \ (p_i \geq 1, \text{ integer})\}.$$

$$q(\Lambda) = \frac{1}{3}(p_1^2 - p_1p_2 + p_2^2 + p_1 + p_2).$$

For $\Lambda = p_1\alpha_1 + p_2\alpha_2 \in \hat{\mathfrak{G}}(\mathfrak{H})$, we have

$$m(A) = \begin{cases} 2p_2 + 1 - p_1 & (p_1 > p_2), \\ p + 1 & (p = p_1 = p_2), \\ 2p_1 + 1 - p_2 & (p_1 < p_2). \end{cases}$$

So we have

PROPOSITION 5. $\{m(A): A \in \hat{\mathfrak{G}}(\mathfrak{H})\}$ is unbounded.

Since $q(A)$ is symmetric, we see that $q(A_1) = q(A_2)$ for $A_1 = (m, n)$, $A_2 = (n, m) \in \hat{\mathfrak{G}}(\mathfrak{H})$ ($m \neq n$), i.e.,

PROPOSITION 6. $q(A)$ does not separate points in $\hat{\mathfrak{G}}(\mathfrak{H})$.

Note that the above Proposition holds for other (non-symmetric) cases.

To determine $\#\{A \in \hat{\mathfrak{G}}(\mathfrak{H}); q(A) = \text{const.}\}$ is to determine the number of lattice points in $\hat{\mathfrak{G}}(\mathfrak{H})$ which lie on the surface in $\mathbf{R}^l$ determined by the equation $q(A) = \text{const.}$. This is a problem of lattice points in the elementary number theory and to determine the number is seemed to be almost impossible.

Suppose there exist $A_1, A_2 \in \hat{\mathfrak{G}}(\mathfrak{H})$ such that $\hat{\mathfrak{G}}(\mathfrak{H}) = \{A = m_1 A_1 + m_2 A_2; m_i \in \mathbf{Z}, m_i \geq 0 (i = 1, 2)\}$, where $A_1 = (p_1, p_2)$, $A_2 = (q_1, q_2)$. Then for $A = (1, 1)$, $(1, 1) = m_1 A_1 + m_2 A_2$ for some $m_1, m_2 \geq 0$. $m_1 A_1 + m_2 A_2 = (m_1 p_1 + m_2 q_1, m_1 p_2 + m_2 q_2) = (1, 1)$. From this, it follows $A_1 = (1, 1)$ or $A_2 = (1, 1)$. If $A_1 = (1, 1)$, $A_2 = (q_1, q_2)$, ($q_1 \geq q_2$), then $\hat{\mathfrak{G}}(\mathfrak{H}) \in A = (a, b)$ ($a \leq b$) can not be expressed as $A = m_1 A_1 + m_2 A_2$. We have in general

PROPOSITION 7. $\hat{\mathfrak{G}}(\mathfrak{H})$ has no free basis except for $\mathfrak{G} = \mathbf{E}_8, \mathbf{F}_4$ and $\mathbf{G}_2$ (cf. [13]).

As for generalized flag manifolds, to compute $Sp(M, g)$, it is necessary to know what representation of G contains the trivial representation of $H = C(T_1)$ and then to compute its multiplicity. For this purpose, it is sufficient to know explicitly how a given representation (ρ, V) of G decomposes when ρ is restricted to H . For an arbitrary subgroup H , this is a difficult problem (e.g., D. P. Želobenko's book: Compact Lie groups and their representations).

References

- [1] M. BERGER, P. GAUDUCHON and E. MAZET: Le spectre d'une variété riemannienne, Springer-Verlag, Berlin, 1971.

- [2] N. BOURBAKI: *Éléments de mathématique, groupes et algèbres de Lie*, Chapitres 4-6, Hermann, Paris, 1968.
- [3] R. S. CAHN and J. A. WOLF: *Zeta functions and their asymptotic expansions for compact symmetric spaces of rank one*, *Comment. Math. Hel.* **51** (1976), 1-21.
- [4] É. CARTAN: *Sur la détermination d'un système orthogonal complet dans un espace de Riemann symétrique clos*, *Rend. Circ. Mat. Palermo*, **53** (1927), 217-252.
- [5] H. FREUDENTHAL: *The existence of a vector of weight 0 in irreducible Lie groups without center*, *Proc. Amer. Math. Soc.* **7** (1956), 175-176.
- [6] S. HELGASON: *Differential geometry and symmetric spaces*, Academic Press, 1962.
- [7] A. IKEDA and Y. TANIGUCHI: *Spectra and eigenforms of the Laplacian on S^n and $P^n(C)$* , 1977 (preprint).
- [8] N. JACOBSON: *Lie algebras*, Interscience Publishers, New York and London, 1962.
- [9] S. KOBAYASHI and K. NOMIZU: *Foundations of differential geometry*, Vol. II, Wiley-Interscience, New York, 1969.
- [10] B. KOSTANT: *A formula for the multiplicity of a weight*, *Trans. Amer. Math. Soc.* **93** (1956), 53-73.
- [11] T. NAGANO: *On the minimum eigenvalues of the Laplacians in Riemannian manifolds*, *Sci. Papers Coll. Gen. Ed., Univ. Tokyo*, **11** (1961), 179-182.
- [12] M. SUGIURA: *Spherical functions and representations theory of compact Lie groups*, *Sci. Papers Coll. Gen. Ed., Univ. Tokyo*, **10** (1960), 187-193.
- [13] ———: *Representations of compact groups realized by spherical functions on symmetric spaces*, *Proc. Japan Acad.*, **38** (1962), 111-113.
- [14] M. TAKEUCHI: *Gendai no Kyukansu* (in Japanese), Iwanami Shoten, 1974.
- [15] H. URAKAWA: *The heat equation on compact Lie group*, *Osaka J. Math.*, **12** (1975), 285-297.
- [16] N. R. WALLACH: *Minimal immersions of symmetric spaces into spheres*, *Symmetric Spaces*, Marcel Dekker, New York, 1972.
- [17] ———: *Harmonic analysis on homogeneous spaces*, Marcel Dekker, New York, 1973.
- [18] H. C. WANG: *Closed manifolds with homogeneous complex structure*, *Amer. J. Math.*, **76** (1954), 1-32.
- [19] G. WARNER: *Harmonic analysis on semi-simple Lie groups I*, Springer-Verlag, Berlin, 1972.
- [20] A. WEIL: *L'intégration dans les groupes topologiques et ses applications*, Hermann, Paris, 1940.