WU FORMULAS FOR THE MOD 3 REDUCED POWER OPERATIONS

By

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Dedicated to Professor Tatsuji Kudo on his 60th birthday

Introduction

Let $\mathcal{P}_p^k$ be the mod p reduced power operation and let c_j be the universal Chern class. The purpose of this paper is to obtain the mod 3 Wu formulas, i.e., the complete description of $\mathcal{P}_3^k(c_j)$ as the polynomial in $c_1, c_2, \dots, c_q$, which was already done in [2]. There are two reasons for us not to avoid overlapping. First, the method of computation used here is quite different from that of [2]. Second, there are slight defects in the expression of Theorem 1 in [2] (although it does not contain any errors) which we will point out in §5.

This paper is arranged as follows. We treat in §1 and §2 somewhat combinatorial discussions for the partitions of an integer and the symmetric polynomials. In §3, we recall Shay's method [4], which is not to calculate $\mathcal{P}_p^k(c_j)$ ($c_j = \sum x_1 x_2 \cdots x_j$) directly but to calculate $\mathcal{P}_p^k(s_j)$ ($s_j = (-1)^{j-1} \sum x_i^j$) first. Theorem 3.2 is quite general in the sense that it is valid for any odd prime p , but the expression is monstrously complicated. In §4, for the case of $p=3$, we have succeeded to apply Theorem 3.2 and to obtain the explicit Wu formulas (Theorem 4.2), which agree with Theorem 1 in [2]. We exhibit in §5, the numerical table of all the coefficients in $\mathcal{P}_3^k(c_{n+k})$ for some k and n .

1. Partitions of an integer For a positive integer n , a tuple $(i_1, i_2, \dots, i_r)$ with $i_1 + i_2 + \cdots + i_r = n$ is called a partition of n . In order to fix the expression, we always arrange in non-increasing order. We denote $\Pi(n)$ the set of all partitions of n . For a partition $\omega = (i_1, i_2, \dots, i_r)$, denoting r_j the number of j 's in ω , we obtain another expression $R = (r_1, r_2, \dots)$ of the partition of n . Conversely, for given R we obtain ω for which we say that ω is associated to R . The number $n = r_1 + 2r_2 + \cdots$ is denoted by $\|R\|$ and $r = r_1 + r_2 + \cdots$ is denoted by $|R|$. It will

be convenient to illustrate a partition $\omega = (i_1, i_2, \dots, i_r)$ in Young diagram

Let $\tau(\omega)$ be the conjugate partition of ω . Then $\tau: \Pi(n) \rightarrow \Pi(n)$ is an involution; $\tau^2 = 1$.

We define an order in $\Pi(n)$ as follows. From Young diagram of ω , we obtain a Young diagram of $\eta = (j_1, j_2, \dots, j_s)$ by taking one block down so that $j_1 \geq j_2 \geq \dots \geq j_s > 0$ still holds. Then we define $\omega > \eta$ and $\Pi(n)$ is a partially ordered set with respect to the order generated by these relations.

Obviously $\Pi(n)$ has the maximum element (n) and the minimum element $(1, 1, \dots, 1)$, and $\omega > \eta$ if and only if $\tau(\omega) < \tau(\eta)$. We exhibit the Hasse's diagram of $\Pi(n)$ for some n . Note that τ reverses up and down in the Hasse's diagram.

Hasse's diagram of $\Pi(n)$

2. Symmetric polynomials Let $Z[x_1, x_2, \dots, x_q]$ be the graded polynomial algebra over the integer ring Z with generators $x_1, x_2, \dots, x_q$ of $\deg x_i = 1$. We denote by $A(n)$ the submodule of all the homogeneous symmetric polynomials of $\deg n$. Let c_i be the i -th fundamental symmetric polynomial in $x_1, \dots, x_q$ and let $\deg c_i = i$. Let $B(n)$ be the submodule of all the homogeneous polynomial of $\deg n$ in $Z[c_1, c_2, \dots, c_q]$. By the fundamental theorem of symmetric polynomials, there exists an isomorphism $\phi: A(n) \rightarrow B(n)$. Let $\sum x_1^{i_1} x_2^{i_2} \dots x_r^{i_r}$ be the minimal symmetric polynomial containing the monomial $x_1^{i_1} x_2^{i_2} \dots x_r^{i_r}$ and we write it briefly x^ω where $\omega = (i_1, i_2, \dots, i_r)$. Then the set $\{x^\omega; \omega \in \Pi(n)\}$ forms the basis of Z -module $A(n)$. Similarly we write $c_{j_1} c_{j_2} \dots c_{j_s}$ briefly c_η where $\eta = (j_1, j_2, \dots, j_s)$. Then the set $\{c_\eta; \eta \in \Pi(n)\}$ forms the basis of Z -module $B(n)$. Observing the following example

$$(\sum x_1)(\sum x_1^2 x_2^2 x_3 x_4 x_5) = \sum x_1^3 x_2^2 x_3 x_4 x_5 \\ + \binom{3}{1} \sum x_1^2 x_2^2 x_3^2 x_4 x_5 + \binom{4}{1} \sum x_1^2 x_2^2 x_3 x_4 x_5 x_6$$

and comparing the definition of order of $\Pi(n)$ (See §1), we easily obtain the following

PROPOSITION 2.1. (a) *In order to obtain inductively the description of x^ω as the polynomial in $c_1, c_2, \dots, c_q$, we only need the polynomials in $c_1, c_2, \dots, c_q$ for x^n with $\eta < \omega$ and $\eta \in \Pi(k)$ for $k < n$.*

(b) *In the polynomial of x^ω in $c_1, c_2, \dots, c_q$, only the monomials $\{c_\eta; \eta \geq \tau(\omega)\}$ possibly appear.*

COROLLARY 2.2. *The symmetric polynomial $\sum x_1^p x_2^p \cdots x_k^p x_{k+1} \cdots x_j$ is expressed by the linear combination of $\{c_\eta; \eta \geq (j, k, k, \dots, k)\}$.*

The following Proposition and Corollary were established and used in [2].

PROPOSITION 2.3. *Let $x^\omega = \sum x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}$ ($i_1 \geq \cdots \geq i_r$) be a symmetric polynomial of degree k . Then the coefficient in x^ω of $c_{j_1} c_{j_2} \cdots c_{j_s}$ ($j_1 \geq \cdots \geq j_s > 0$) with $s = i_1$ is equal to that of $c_{j_1-1} c_{j_2-1} \cdots c_{j_s-1}$ in $\sum x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}$.*

PROOF. Let $c'_1, c'_2, \dots, c'_{q+1}$ denote the elementary symmetric functions of $x_1, x_2, \dots, x_q, x_{q+1}$. Then $c'_i = c_i + c_{i-1} x_{q+1}$. Put $f(x_1, x_2, \dots, x_q, x_{q+1}) = \sum x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}$ and substitute $x_{q+1} = 1$. We easily see that the coefficient in $f(x_1, x_2, \dots, x_q, x_{q+1})$ of $c'_{j_1} c'_{j_2} \cdots c'_{j_s}$ is equal to that of $c_{j_1-1} c_{j_2-1} \cdots c_{j_s-1}$ in non-homogeneous symmetric function $f(x_1, x_2, \dots, x_q, 1)$.

COROLLARY 2.4. *The coefficient of $c_{j_1} c_{j_2} \cdots c_{j_p}$ ($j_1 \geq \cdots \geq j_p > 0$) in $\sum x_1^p x_2^p \cdots x_k^p x_{k+1} \cdots x_j$ is equal to that of $c_{j_1-1} c_{j_2-1} \cdots c_{j_p-1}$ in $\sum x_1^p \cdots x_{k-1}^p x_k \cdots x_{j-1}$.*

3. Integral case In this section, we repeat the methods of Shay [4] rapidly. Let p be a fixed odd prime. Let $\mathcal{A}^{ev}$ be the divided algebra over $\mathbb{Z}$ generated by the symbols $\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \dots$. Then $\mathcal{A}^{ev}$ turns out to be a Hopf algebra with comultiplication $\tilde{\mathcal{P}}_i \mapsto \sum_{j=0}^i \tilde{\mathcal{P}}_j \otimes \tilde{\mathcal{P}}_{i-j}$ with $\tilde{\mathcal{P}}_0 = 1$. For each x_i , we define the operation of $\tilde{\mathcal{P}}_i$ as follows:

$$\tilde{\mathcal{P}}_0(x_i) = x_i, \quad \tilde{\mathcal{P}}_1(x_i) = x_i^p, \quad \tilde{\mathcal{P}}_k(x_i) = 0 \quad \text{for } k > 1.$$

Then the operation of $\mathcal{A}^{ev}$ on $\mathbb{Z}[x_1, x_2, \dots, x_q]$ and on $\mathbb{Z}[c_1, c_2, \dots, c_q]$ is obtained. Denoting $s_i = (-1)^{i-1} \sum x_i^i$, we have

$$(3.1) \quad \tilde{\mathcal{P}}_k(s_i) = \binom{i}{k} s_{i+k(p-1)}.$$

In his paper [4], Shay made use of this fact skillfully. The elementary symmetric functions $c_1, c_2, \dots, c_q$ are related to $s_1, s_2, \dots, s_q$ by the Cauchy's formula of indeterminate z [3; Sect I, Chap I]:

$$(3.2) \quad z \frac{d}{dz} \log (1 + \sum_{i=1}^q c_i z^i) = \sum_{i=1}^q s_i z^i \pmod{z^{q+1}}.$$

In fact, setting $\sum_{i=0}^q c_i z^i = \prod_{j=1}^q (1 + x_j z)$, (3.2) is obtained from

$$z \frac{d}{dz} \log \left(\prod_{j=1}^q (1 + x_j z) \right) = \sum_{j=1}^q \frac{x_j z}{1 + x_j z} = \sum_{i=1}^{\infty} \left(\sum_{j=1}^q (-1)^{i-1} x_j^i \right) z^i.$$

From (3.2), making use of Talor expansion $\log(1-y) = -\sum_{r=1}^{\infty} \frac{1}{r} y^r$, s_j is described as the polynomial in $c_1, c_2, \dots, c_j$. Conversely, making use of Talor expansion $\exp y = \sum_{r=1}^{\infty} \frac{1}{r!} y^r$, c_j is described as the polynomial in $s_1, s_2, \dots, s_j$.

PROPOSITION 3.1.

$$(a) \quad s_j = \sum_{\|R\|=j} (-1)^{r-1} \frac{k}{r} \binom{r}{R} c^R, \quad r = |R|,$$

where $\binom{r}{R} = \binom{r}{r_1, r_2, \dots, r_q}$ and $c^R = c_1^{r_1} c_2^{r_2} \dots c_q^{r_q}$.

$$(b) \quad c_j = \sum_{\|L\|=j} \frac{1}{l!} \binom{l}{L} \left(\frac{s}{1} \right)^L, \quad l = |L|,$$

where $\binom{l}{L} = \binom{l}{l_1, l_2, \dots, l_j}$ and $\left(\frac{s}{1} \right)^L = \left(\frac{s_1}{1} \right)^{l_1} \left(\frac{s_2}{2} \right)^{l_2} \dots \left(\frac{s_j}{j} \right)^{l_j}$.

Let $\omega = (i_1, i_2, \dots, i_l)$ be the partition associated to L (See § 1), then (b) is rewritten

$$c_j = \sum_{\|L\|=j} \frac{1}{l!} \binom{l}{L} \frac{s_{i_1}}{i_1} \frac{s_{i_2}}{i_2} \dots \frac{s_{i_l}}{i_l}.$$

Operating $\tilde{\mathcal{P}}_k$ on c_j , we have

$$\tilde{\mathcal{P}}_k(c_j) = \sum_{\|L\|=j} \frac{1}{l!} \binom{l}{L} \sum_{\alpha} \frac{1}{i_1} \tilde{\mathcal{P}}_{k_1}(s_{i_1}) \dots \frac{1}{i_l} \tilde{\mathcal{P}}_{k_l}(s_{i_l})$$

where α runs through $(k_1, k_2, \dots, k_l)$ with $k_1 + \dots + k_l = k$ and $0 \leq k_t \leq i_t$ for $1 \leq t \leq l$. Applying (3.1), the term $\sum_{\alpha} \dots$ turns out to be

$$\sum_{\alpha} \prod_{t=1}^l \frac{1}{i_t} \binom{i_t}{k_t} s_{i_t+k_t(p-1)}.$$

Applying Proposition 3.1. (a), we have the following

THEOREM 3.2.

$$\tilde{\mathcal{P}}_k(c_j) = \sum_{\|L\|=j} \frac{1}{l!} \binom{l}{L} \sum_{\alpha} \prod_{t=1}^l \frac{1}{i_t} \binom{i_t}{k_t} \sum_{\beta(t)} (-1)^{r_t-1} \frac{i_t+k_t(p-1)}{r_t} \binom{r_t}{R_t} c^{R_t},$$

$$l=|L|, \quad r_t=|R_t|,$$

where α runs through $(k_1, k_2, \dots, k_j)$ with $k_1 + \dots + k_j = k$ and $0 \leq k_t \leq i_t$ for $1 \leq t \leq j$ and $\beta(t)$ runs through R_t with $\|R_t\| = i_t + k_t(p-1)$.

If one reduces this formula by mod p for any odd prime p , one should obtain the operation $\mathcal{P}_p^k$ on $c_j \dots$ at least theoretically. But actually it will be almost impossible because the expression is too complicated. When p is the smallest odd prime 3, we have succeeded to reduce by mod p . We show that in the next section.

4. Reduced power operations over $H^*(BU(q), \mathbb{Z}_3)$ In this section we treat the case $p=3$. We write $\mathcal{P}^k$ for $\mathcal{P}_3^k$. We have $\mathcal{P}^k(c_j) = \sum x_1^3 \dots x_k^3 x_{k-1} \dots x_j$ by Borel-Serre [1]. By Corollary 2.2, this is a linear combination of monomials $\{c_{j_1} c_{j_2} c_{j_3}; (j_1, j_2, j_3) \geq (j, k, k)\}$ and by Corollary 2.4, the coefficient of a monomial $c_{j_1} c_{j_2} c_{j_3}$ with $j_1 \geq j_2 \geq j_3 \geq 0$ is equal to that of the monomial $c_{j_1-j_3} c_{j_2-j_3}$ of $\mathcal{P}^{k-j_3}(c_{j-j_3})$. Therefore it is sufficient to obtain the coefficient of $c_i c_j$ in $\mathcal{P}^k(c_{n+k})$. Let D^1 be the ideal in $H^*(BU(q), \mathbb{Z}_3) = \mathbb{Z}_3[c_1, c_2, \dots, c_q]$, consisted of all elements of positive degree and let $D^i = D^1 D^{i-1}$ inductively. From Proposition 3.1 (b), we have

$$c_t = \frac{s_t}{t} + \frac{1}{2} \sum_{i=1}^{t-1} \frac{s_i}{i} \frac{s_{t-i}}{t-i}, \quad (\text{mod } D^3).$$

Hence

$$(4.1) \quad \mathcal{P}^k(c_t) = \frac{1}{t} \binom{t}{k} s_{t+2k} + \frac{1}{2} \sum_{i=1}^{t-1} \sum_{l=0}^k \frac{1}{i} \binom{i}{l} s_{i+2l} \cdot \frac{1}{t-i} \binom{t-i}{k-l} s_{t-i+2k-2}$$

5. Numerical tables We exhibit the numerical table of $\mathcal{P}^k(c_{n+k})$. Suppose one wish to get $\mathcal{P}^2(c_5)$, that is, $n=3$ and $k=2$. From the table of $n=3$, one will obtain

$k-l$	$\alpha_{i,j,l}^k$				
0	1				
1	0	2	1		
2	0	0	1	0	2

These are the coefficients of the corresponding monomials

$$\begin{array}{cccccc}
 c_5 c_2 c_2 & & & & & \\
 c_7 c_1 c_1 & c_6 c_2 c_1 & c_5 c_3 c_1 & & & \\
 c_9 & c_8 c_1 & c_7 c_2 & c_6 c_3 & c_5 c_4 &
 \end{array}$$

Hence we obtain

$$\mathcal{P}^2(c_5) = c_5 c_2^2 + 2c_6 c_2 c_1 + c_5 c_3 c_1 + c_7 c_2 + 2c_5 c_4.$$

Now we should point out two defects in Theorem 1 in [2]. First, since the binomial coefficient $\binom{n}{k}$ is understood to be 0 if $k > 0$ or $n > k$, $\sum_{i=0}^{\lfloor j/2 \rfloor}$ in (ii) should be replaced by $\sum_{i=\alpha}^{\lfloor j/2 \rfloor}$ where $\alpha = \max(0, j-k)$. Second, for the case of $\mathcal{P}^k(c_{n+k})$ with $k > n$, some of basis (monomials $c_i c_j c_l$) happens to appear twice. For example from Theorem 1 in [2], we obtain

$$\mathcal{P}^3(c_4) = \cdots + 2c_7 c_3 + c_6 c_4 + c_5^2 + c_4 c_6$$

which means of course

$$\mathcal{P}^3(c_4) = \cdots + 2c_7 c_3 + 2c_6 c_4 + c_5^2.$$

References

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- [4] P. B. SHAY; *mod p Wu formulas for the Steenrod algebra and the Dyer-Lashof algebra*, Proc. Amer. Math. Soc., **63** (1977) 339-347.

Taking note of modulo D^3 , we substitute Proposition 3.1 (a). Then (4.1) becomes

$$\begin{aligned} & \frac{t+2k}{t} \binom{t}{k} \left(c_{t+2k} - \frac{1}{2} \sum_{i=1}^{t+2k-1} c_i c_{t+2k-i} \right) \\ & + \frac{1}{2} \sum_{i=1}^{t-1} \sum_{l=0}^k \frac{i+2l}{i} \binom{i}{l} \cdot \frac{t-i+2k-2l}{t-i} \binom{t-i}{k-l} c_{i+2l} c_{t-i+2k-2l} \end{aligned}$$

Since $\frac{t+2k}{t} \binom{t}{k} = \frac{t+2k}{k} \binom{t-1}{k-1} \equiv \frac{t-k}{k} \binom{t-1}{k-1} = \binom{t-1}{k} \pmod{3}$, we have the following

PROPOSITION 4.1.

$$\begin{aligned} \mathcal{P}^k(c_t) &= \binom{t-1}{k} \left(c_{t+2k} - \frac{1}{2} \sum_{i=1}^{t+2k-1} c_i c_{t+2k-i} \right) \\ &+ \frac{1}{2} \sum_{i=1}^{t-1} \sum_{l=0}^k \binom{i-1}{l} \binom{t-i-1}{k-l} c_{i+2l} c_{t-i+2k-2l} \pmod{D^3}. \end{aligned}$$

Let $\alpha_{i,j,l}^k$ be the coefficient of $c_i c_j c_l$ in $\mathcal{P}^k(c_t) = \sum x_1^3 \cdots x_k^3 x_{k+1} \cdots x_t$. For each n with $1 \leq n \leq j+2k-1$, we write

$$\alpha(n) = \max(0, [(n-j+k+3)/3])$$

$$\omega(n) = \min(k, [(n-1)/3]) \quad ([m] \text{ is the Gauss notation}).$$

Our results are summarized in the following

THEOREM 4.2. (a) If $i \geq j \geq l > 0$, then $\alpha_{i,j,l}^k = \alpha_{i-l,j-l,0}^{k-l}$.

$$(b) \quad \alpha_{t+2k,0,0}^k = \binom{t-1}{k}.$$

If $t+2k-i > i > 0$, then

$$\alpha_{t+2k-i,i,0}^k = -\binom{t-1}{k} + \sum_{l=\alpha(i)}^{\omega(i)} \binom{i-2l-1}{l} \binom{t-i+2l-1}{k-l}.$$

If $t+2k-i = i$, then

$$\alpha_{t+2k-i,i,0}^k = -\frac{1}{2} \binom{t-1}{k} + \frac{1}{2} \sum_{l=\alpha(i)}^{\omega(i)} \binom{i-2l-1}{l} \binom{t-i+2l-1}{k-l}.$$

$\sum_{l=\alpha(i)}^{\omega(i)} \cdots$ is understood to be 0 if $\alpha(i) > \omega(i)$.

PROOF. Part (a) is a restatement of Corollary 2.4. Part (b) is obtained from Proposition 4.1.


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0      1      2      N = 2
1      2      2      1
2      0      1      0      0      2
3      1      0      2      2      0      2
4      2      2      1      1      2      1      1      1
5      0      1      1      0      0      1      0      0      1      0
6      1      0      2      2      0      2      2      0      2      2      0
7      2      2      1      1      2      1      1      2      1      1      2      1
8      0      1      0      0      1      1      0      0      1      0      0      2
9      1      0      2      2      0      2      2      0      2      2      0      2
10     2      2      1      1      2      1      1      2      1      1      2      1
11     0      1      0      0      1      0      0      1      0      0      1      0
12     1      0      2      2      0      2      2      0      2      2      0      2
13     2      2      1      1      2      1      1      2      1      1      2      1
14     0      1      0      0      1      0      0      1      0      0      1      0
15     1      0      2      2      0      2      2      0      2      2      0      2
16     2      2      1      1      2      1      1      2      1      1      2      1
17     0      1      0      0      1      0      0      1      0      0      1      0
18     1      0      2      2      0      2      2      0      2      2      0      2
19     2      2      1      1      2      1      1      2      1      1      2      1
20     0      1      0      0      1      0      0      1      0      0      1      0
21     1      0      2      2      0      2      2      0      2      2      0      2
22     2      2      1      1      2      1      1      2      1      1      2      1
23     0      1      0      0      1      0      0      1      0      0      1      0
24     1      0      2      2      0      2      2      0      2      2      0      2
25     2      2      1      1      2      1      1      2      1      1      2      1
26     0      1      0      0      1      0      0      1      0      0      1      0
27     1      0      2      2      0      2      2      0      2      2      0      2
28     2      2      1      1      2      1      1      2      1      1      2      1
29     0      1      0      0      1      0      0      1      0      0      1      0
30     1      0      2      2      0      2      2      0      2      2      0      2
31     2      2      1      1      2      1      1      2      1      1      2      1
32     0      1      0      0      1      0      0      1      0      0      1      0
33     1      0      2      2      0      2      2      0      2      2      0      2
34     2      2      1      1      2      1      1      2      1      1      2      1
35     0      1      0      0      1      0      0      1      0      0      1      0
36     1      0      2      2      0      2      2      0      2      2      0      2

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0      1      N = 3
1      0 2 1
2      0 0 1 0 2
3      1 0 0 2 2 1 1
4      0 2 1 0 1 2 0 0
5      0 0 1 0 2 2 0 1 0 0
6      1 0 0 2 2 1 2 1 2 2 0
7      0 2 1 0 1 2 0 0 0 0 2 1 0
8      0 0 1 0 2 2 0 1 0 0 0 1 0 2
9      1 0 0 2 2 1 2 1 2 2 0 0 2 2 1 1
10     0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0
11     0 0 1 0 2 2 0 1 0 0 0 1 0 2 2 0 1 0 0
12     1 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2 1 2 2 0
13     0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0 0 0 2 1 0
14     0 0 1 0 2 2 0 1 0 0 0 1 0 2 2 0 1 0 0 0 1 0 2
15     1 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2 1 2 2 0 0 2 2 1 1
16     0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0 0 0 2 1 0 0
17     0 0 1 0 2 2 0 1 0 0 0 1 0 2 2 0 1 0 0 0 1 0 2
18     1 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2 1 2 2 0 0 2 2 1 0
19     0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0 0 0 2 1 0 0
20     0 0 1 0 2 2 0 1 0 0 0 1 0 2 2 0 1 0 0 0 1 0 2
21     1 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2 1 2 2 0 0 2 2 1 1
22     0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0
23     0 0 1 0 2 2 0 1 0 0 0 1 0 2 2 0 1 0 0 0 1 0 2
24     1 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2 1 2 2 0 0 2 2 1 0
25     0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0
26     0 0 1 0 2 2 0 1 0 0 0 1 0 2 2 0 1 0 0 0 1 0 2
27     1 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2
28     0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0
29     0 0 1 0 2 2 0 1 0 0 0 1 0 2 2 0 1 0 0 0 1 0 2
30     1 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2 1 2 2 0 0 2 2 1 0
31     0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0
32     0 0 1 0 2 2 0 1 0 0 0 1 0 2 2 0 1 0 0 0 1 0 2
33     1 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2
34     0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0 0 0 2 1 0 1 2 0 0
35     0 0 1 0 2 2 0 1 0 0 0 1 0 2 2 0 1 0 0 0 1 0 2
36     1 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2 1 2 2 0 0 2 2 1 2

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0      1      N = 4
1      1 2 1
2      1 2 2 0 2
3      2 2 2 2 1 1 1
4      2 1 0 2 2 0 1 1 2
5      2 1 1 2 1 2 1 2 1 1
6      0 1 1 1 0 0 0 0 0 0 1 2
7      0 0 2 1 1 2 0 0 0 0 0 2 1
8      0 0 0 1 0 1 0 1 0 0 0 0 1 0 2
9      1 0 0 0 2 2 2 2 2 2 0 0 0 2 2 2
10     1 2 1 0 0 1 2 2 2 2 2 1 0 0 1 2 2 1
11     1 2 2 0 2 0 2 0 2 2 2 2 0 2 0 2 2 2
12     2 2 2 2 1 1 1 1 1 1 2 2 2 2 1 1 2 1
13     2 1 0 2 2 0 1 1 1 1 1 1 0 2 2 0 1 1 1
14     2 1 1 2 1 2 1 2 1 1 1 1 2 1 2 1 1 1 2 1
15     0 1 1 1 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0
16     0 0 2 1 1 2 0 0 0 0 0 2 1 1 2 0 0 0 0
17     0 0 0 1 0 1 0 1 0 0 0 0 1 0 1 0 1 0 0
18     1 0 0 0 2 2 2 2 2 2 0 0 0 2 2 2 2 2 0 0
19     1 2 1 0 0 1 2 2 2 2 2 1 0 0 1 2 2 2 2 1 0
20     1 2 2 0 2 0 2 0 2 2 2 2 0 2 2 2 2 2 2 0
21     2 2 2 2 1 1 1 1 1 1 2 2 2 2 1 1 1 1 1 2 2 1
22     2 1 0 2 2 0 1 1 1 1 1 1 0 2 2 0 1 1 1 1 0 2 2
23     2 1 1 2 1 2 1 2 1 1 1 1 2 1 2 1 1 1 1 2 1 2 1
24     0 1 1 1 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0
25     0 0 2 1 1 2 0 0 0 0 0 2 1 1 2 0 0 0 0 0 2 1 1
26     0 0 0 1 0 1 0 1 0 0 0 0 1 0 1 0 1 0 0 0 1 0
27     1 0 0 0 2 2 2 2 2 2 0 0 0 2 2 2 2 2 0 0 0 2
28     1 2 1 0 0 1 2 2 2 2 2 1 0 0 1 2 2 2 2 1 0 0
29     1 2 2 0 2 0 2 0 2 2 2 2 0 2 2 2 2 2 2 0 2 0
30     2 2 2 2 1 1 1 1 1 1 2 2 2 2 1 1 1 1 1 2 2 2
31     2 1 0 2 2 0 1 1 1 1 1 1 0 2 2 0 1 1 1 1 0 2 2
32     2 1 1 2 1 2 1 2 1 1 1 1 2 1 2 1 1 1 1 2 1 2 1
33     0 1 1 1 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1
34     0 0 2 1 1 2 0 0 0 0 0 2 1 1 2 0 0 0 0 0 2 1 1
35     0 0 0 1 0 1 0 1 0 0 0 0 1 0 1 0 1 0 0 0 1 0
36     1 0 0 0 2 2 2 2 2 2 0 0 0 2 2 2 2 2 0 0 0 2

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0      1      N = 5
1      2 2 1
2      0 1 0 0 2
3      2 0 2 2 0 1 1
4      1 1 2 1 2 1 2 1 2
5      0 2 0 0 0 0 0 0 0 0 1
6      0 0 1 1 0 0 0 0 0 0 0 1
7      0 0 0 2 1 2 0 0 0 0 0 0 2 2
8      0 0 0 0 1 0 0 0 1 0 0 0 0 1 0
9      1 0 0 0 0 2 2 0 2 2 0 0 0 0 2 2 0
10     2 2 1 0 0 0 1 2 1 1 2 1 0 0 0 1 2 1
11     0 1 0 0 2 0 0 2 0 0 1 0 0 2 0 0 2 0 0 2
12     2 0 2 2 0 1 1 0 1 1 0 2 2 0 1 1 0 1 1 0 2
13     1 1 2 1 2 1 2 1 2 2 2 1 2 1 2 1 2 2 1 1
14     0 2 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
15     0 0 1 1 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 1 1
16     0 0 0 2 1 2 0 0 0 0 0 0 0 0 0 0 2 1 2 0 0 0
17     0 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 1 0 0 0
18     1 0 0 0 0 2 2 0 2 2 0 0 0 0 0 2 2 0 2 2 0 0
19     2 2 1 0 0 0 1 2 1 1 2 1 0 0 0 0 1 2 1 1 2 1
20     0 1 0 0 2 0 0 2 0 0 1 0 0 2 0 0 2 0 0 1 0 0
21     2 0 2 2 0 1 1 0 1 1 0 2 2 0 1 1 0 1 1 0 2
22     1 1 2 1 2 1 2 1 2 2 2 1 2 1 2 1 2 1 2 1 2
23     0 2 0 0 0 0 0 0 0 0 0 2 0 0 0 0 0 0 0 0 2
24     0 0 1 1 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 1
25     0 0 0 2 1 2 0 0 0 0 0 0 2 1 2 0 0 0 0 0 2
26     0 0 0 0 1 0 0 1 0 0 0 0 0 1 0 0 1 0 0 0 0
27     1 0 0 0 0 2 2 0 2 2 0 0 0 0 2 2 0 2 2 0 0
28     2 2 1 0 0 0 1 2 1 1 2 1 0 0 0 1 2 1 1 2 1
29     0 1 0 0 2 0 0 2 0 0 1 0 0 2 0 0 2 0 0 1 0
30     2 0 2 2 0 1 1 0 1 1 0 2 2 0 1 1 0 1 1 0 2
31     1 1 2 1 2 1 2 1 2 2 2 1 2 1 2 1 2 1 2 1 2
32     0 2 0 0 0 0 0 0 0 0 0 2 0 0 0 0 0 0 0 0 2
33     0 0 1 1 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1
34     0 0 0 2 1 2 0 0 0 0 0 0 2 1 2 0 0 0 0 2 1
35     0 0 0 0 1 0 0 1 0 0 0 0 0 1 0 0 1 0 0 1 0
36     1 0 0 0 0 2 2 0 2 2 0 0 0 0 2 2 0 2 2 0 2

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0      1      N = 6
1      0 2 1
2      0 0 1 0 2
3      2 0 0 2 2 1 1
4      0 1 2 0 1 2 0 1 2
5      0 0 2 0 1 2 0 1 2 0 1
6      0 0 0 1 1 2 0 1 2 0 1 2 2
7      0 0 0 0 2 1 0 1 2 0 1 2 0 0
8      0 0 0 0 0 1 0 2 2 0 1 2 0 1 0 0
9      1 0 0 0 0 0 2 2 1 2 1 2 0 1 2 2 0
10     0 2 1 0 0 0 0 0 1 2 0 0 0 0 1 2 0 2 1 0
11     0 0 1 0 2 0 0 0 2 0 1 0 0 0 2 0 1 1 0 2
12     2 0 0 2 2 1 1 0 0 1 1 2 2 0 0 1 1 2 1 2 1
13     0 1 2 0 1 2 0 1 2 0 2 1 0 2 1 0 2 1 0 0 0 0
14     0 0 2 0 1 2 0 1 2 0 1 1 0 2 1 0 2 1 0 2 0 0
15     0 0 0 1 1 2 0 1 2 0 1 2 1 2 1 0 2 1 0 2 1 1
16     0 0 0 0 2 1 0 1 2 0 1 2 0 0 0 0 2 1 0 2 1 0
17     0 0 0 0 0 1 0 2 2 0 1 2 0 1 0 0 0 1 0 2 1 0
18     1 0 0 0 0 0 2 2 1 2 1 2 0 1 2 2 0 0 2 1 0 2
19     0 2 1 0 0 0 0 1 2 0 0 0 0 1 2 0 2 1 0 1 2 0
20     0 0 1 0 2 0 0 0 2 0 1 0 0 0 2 0 1 1 0 2 2 0
21     2 0 0 2 2 1 1 0 0 0 1 1 2 2 0 0 1 1 2 1 2 1
22     0 1 2 0 1 2 0 1 2 0 2 1 0 2 1 0 2 1 0 0 0 0
23     0 0 2 0 1 2 0 1 2 0 1 1 0 2 1 0 2 1 0 2 0 0
24     0 0 0 1 1 2 0 1 2 0 1 2 1 2 1 0 2 1 1 0 0 0
25     0 0 0 0 2 1 0 1 2 0 1 2 0 0 0 0 2 1 0 2 1 0
26     0 0 0 0 0 1 0 2 2 0 1 2 0 1 0 0 0 1 0 2 2 0
27     1 0 0 0 0 0 2 2 1 2 1 2 0 1 2 2 0 2 1 0 2 1
28     0 2 1 0 0 0 0 1 2 0 0 0 0 1 2 0 2 1 0 1 2 0
29     0 0 1 0 2 0 0 0 2 0 1 0 0 0 2 0 1 1 0 2 2 0
30     2 0 0 2 2 1 1 0 0 0 1 1 2 2 0 0 1 1 2 1 2 1
31     0 1 2 0 1 2 0 1 2 0 2 1 0 2 1 0 2 1 0 0 0 0
32     0 0 2 0 1 2 0 1 2 0 1 1 0 2 1 0 2 1 0 2 0 0
33     0 0 0 1 1 2 0 1 2 0 1 2 1 2 1 0 2 1 0 0 0 0
34     0 0 0 0 2 1 0 1 2 0 1 2 0 0 0 0 2 1 0 1 2 0
35     0 0 0 0 0 1 0 2 2 0 1 2 0 1 0 0 0 1 0 2 2 0
36     1 0 0 0 0 0 2 2 1 2 1 2 0 1 2 2 0 2 1 0 2 1

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0      1      N = 7
1      1 2 1
2      1 2 2 0 2
3      0 2 2 2 1 1 1
4      0 0 1 2 2 2 0 1 1 2
5      0 0 0 2 0 2 1 2 1 0 1
6      0 0 0 0 1 1 1 0 0 0 2 2 2
7      0 0 0 0 0 2 1 1 2 0 0 1 2 2 1
8      0 0 0 0 0 0 1 0 1 0 1 0 2 0 2 2
9      1 0 0 0 0 0 0 0 2 2 2 2 2 1 1 1 1 1
10     1 2 1 0 0 0 0 0 0 2 2 2 2 2 2 0 1 1 0 2
11     1 2 2 0 2 0 0 0 0 0 2 2 2 2 2 2 1 2 1 1 1
12     0 2 2 2 2 1 1 1 0 0 0 0 1 1 1 2 2 2 0 0 0 0 0
13     0 0 1 2 2 2 0 1 1 2 0 0 2 1 1 0 2 2 2 1 0 0 0 0 0
14     0 0 0 2 0 2 1 2 1 0 1 0 1 0 1 2 1 2 2 0 2 0 0 0 0
15     0 0 0 0 1 1 1 1 0 0 0 2 2 2 2 2 2 2 0 0 1 1 1 0 0
16     0 0 0 0 0 2 1 1 1 2 0 0 0 1 2 2 2 2 2 1 0 0 2 1 1
17     0 0 0 0 0 0 0 1 0 1 0 1 0 2 0 2 2 2 2 0 2 0 1 0
18     1 0 0 0 0 0 0 0 2 2 2 2 2 1 1 1 2 2 2 1 1 1 2 2 2 2 2 0 0
19     1 2 1 0 0 0 0 0 1 2 2 2 2 2 2 0 1 1 0 2 2 2 1 1 0 2 2 2 2 1 0 0 0
20     1 2 2 0 2 0 0 0 0 2 2 2 2 2 2 2 1 2 1 2 1 2 1 2 1 2 2 2 2 2 1 0 0 0
21     0 2 2 2 2 1 1 1 0 0 0 1 1 1 2 2 2 2 0 0 0 0 0 0 0 0 0 0 0 2 2 2 1 1 1 0 0
22     0 0 1 2 2 0 1 1 1 2 0 0 0 2 1 1 0 2 2 1 0 0 0 0 0 0 0 0 0 0 0 1 2 2 0 1 1 2
23     0 0 0 2 0 2 1 2 1 0 1 0 1 0 1 2 1 2 2 0 0 0 0 0 0 0 0 0 0 2 0 2 1 2 1
24     0 0 0 0 1 1 1 0 0 0 2 2 2 2 2 2 2 2 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1
25     0 0 0 0 0 2 1 1 2 0 0 1 1 2 2 2 2 2 1 0 0 2 1 1 2 0 0 0 0 0 0 2 1 1 2
26     0 0 0 0 0 0 1 0 1 0 1 0 2 0 2 2 2 2 0 2 0 1 0 1 0 1 0 0 0 0 0 0 0 1 0 1
27     1 0 0 0 0 0 0 2 2 2 2 2 2 1 1 1 2 2 2 1 1 1 2 2 2 2 2 0 0 0 0 0 0 2 2
28     1 2 1 0 0 0 0 0 1 2 2 2 2 2 2 0 1 1 0 2 2 0 1 1 0 2 2 2 2 1 0 0 0 0 0 1
29     1 2 2 0 2 0 0 0 0 2 0 2 2 2 2 2 1 2 1 2 1 2 1 2 1 2 1 2 2 2 2 0 2 0 0 0
30     0 2 2 2 1 1 1 0 0 0 1 1 1 2 2 2 0 0 0 0 0 0 0 0 0 0 0 0 0 2 2 2 1 1 1 0
31     0 0 1 2 2 0 1 1 2 0 0 2 1 1 0 2 2 1 0 0 0 0 0 0 0 0 0 0 0 1 2 2 0 1 1 2
32     0 0 0 2 0 2 1 2 1 0 1 0 1 0 1 2 1 2 0 2 0 0 0 0 0 0 0 0 0 0 2 2 1 1 2
33     0 0 0 0 1 1 1 0 0 0 2 2 2 2 2 2 2 0 0 1 1 1 0 0 0 0 0 0 0 0 1 1 1 0
34     0 0 0 0 0 2 1 1 2 0 0 1 2 2 2 2 2 1 0 0 2 1 1 2 0 0 0 0 0 0 2 1 1 2
35     0 0 0 0 0 0 1 0 1 0 1 0 2 0 2 2 2 2 0 2 0 1 0 1 0 1 0 0 0 0 0 1 0 1
36     1 0 0 0 0 0 0 0 2 2 2 2 2 2 1 1 1 2 2 2 1 1 1 2 2 2 2 2 0 0 0 0 0 2

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0      1      N = 8
1      2 2 1
2      0 1 0 0 2
3      0 0 2 2 0 1 1
4      0 0 0 1 2 1 2 1 2
5      0 0 0 0 2 0 0 0 0 0 1
6      0 0 0 0 0 1 1 0 0 0 0 2 2
7      0 0 0 0 0 0 2 1 2 0 0 0 1 2 1
8      0 0 0 0 0 0 0 0 1 0 0 1 0 0 2 0 0 2
9      1 0 0 0 0 0 0 0 2 2 0 2 2 0 1 1 0 2
10     2 2 1 0 0 0 0 0 0 1 2 1 1 2 1 2 1 2 1 1
11     0 1 0 0 2 0 0 0 0 0 2 0 0 1 0 0 0 0 0 0 0 0
12     0 0 2 2 0 1 1 0 0 0 0 1 1 0 2 2 0 0 0 0 0 0
13     0 0 0 1 2 1 2 1 2 0 0 0 2 1 2 1 2 1 0 0 0 0
14     0 0 0 0 2 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0
15     0 0 0 0 0 1 1 0 0 0 2 2 0 2 2 0 0 0 0 0 0 0
16     0 0 0 0 0 0 2 1 2 0 0 0 1 2 1 2 1 0 0 0 0 0
17     0 0 0 0 0 0 1 0 0 1 0 0 2 0 0 1 0 0 2 1 2 0 0 0
18     1 0 0 0 0 0 0 0 2 2 0 2 2 0 1 2 1 2 1 2 1 0 0 0
19     2 2 1 0 0 0 0 0 1 2 1 1 2 1 2 1 2 1 2 1 2 1 0 0
20     0 1 0 0 2 0 0 0 0 2 0 0 1 0 0 0 0 0 0 0 0 0
21     0 0 2 2 0 1 1 0 0 0 0 1 1 0 2 2 0 0 0 0 0 0
22     0 0 0 1 2 1 2 1 2 0 0 0 2 1 2 1 2 1 0 0 0 0
23     0 0 0 0 2 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0
24     0 0 0 0 0 1 1 0 0 0 2 2 0 2 2 0 0 0 0 0 0 0
25     0 0 0 0 0 2 1 2 0 0 0 1 2 1 1 2 1 0 0 0 2 1 2
26     0 0 0 0 0 0 1 0 0 1 0 0 2 0 0 1 0 0 2 0 0 1 0
27     1 0 0 0 0 0 0 2 2 0 2 2 0 1 1 0 2 2 0 2 2 0 0 0
28     2 2 1 0 0 0 0 0 1 2 1 1 2 1 2 1 2 1 2 1 2 1 0
29     0 1 0 0 2 0 0 0 0 2 0 0 1 0 0 0 0 0 0 0 0 0
30     0 1 0 2 2 0 1 1 0 0 0 1 1 0 2 2 0 0 0 0 0 0
31     0 0 0 1 2 1 2 1 2 0 0 0 2 1 2 1 2 1 0 0 0 0
32     0 0 0 0 2 0 0 0 0 0 1 0 0 1 0 0 0 0 0 2 0 0 0
33     0 0 0 0 0 1 1 0 0 0 2 2 0 2 2 0 0 0 0 1 1 0 0
34     0 0 0 0 0 0 2 1 2 0 0 1 2 1 1 2 1 0 0 0 2 1 2
35     0 0 0 0 0 0 1 0 0 1 0 0 2 0 0 1 0 0 2 0 0 1 0
36     1 0 0 0 0 0 0 0 2 2 0 2 2 0 1 1 0 2 2 0 1 1 0

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[illegible]