DERIVATIONS AND AFFINE STRUCTURES
OF SOME NILPOTENT LIE ALGEBRAS

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§ 1. Introduction and definitions

This is a continuation of our previous papers [11] and [12]. Throughout this paper, we say briefly a solvable Lie algebra $\mathfrak{g}$ admits an affine structure if a connected Lie group $G$ with $\mathfrak{g}$ as its Lie algebra admits a complete left-invariant locally flat affine structure, or equivalently, $\tilde{G}$, the universal covering group of $G$ operates on $\mathbb{R}^n$ ($n=\dim G$) as a simply transitive transformation group of affine motions. Henceforth $\mathfrak{g}$ always denotes a nilpotent Lie algebra over $\mathbb{R}$. $\mathbb{R}$ may be replaced by a field $K$ of characteristic 0 if we consider only $\mathfrak{g}$ itself.

It has been an open problem pointed out by L. Auslander [1] whether $\mathfrak{g}$ admits always an affine structure or not. This problem is interesting. For example, if the answer is yes, we can obtain many examples of compact complete affinely flat manifolds $M=\tilde{G}/\Gamma$ with $\pi_1(M)\cong \Gamma$, where $\Gamma$ is a uniform lattice of $\tilde{G}$.

On this problem, there are some results including solvable cases by L. Auslander, J. Milnor, J. Scheuneman, H. Fujiwara and us. In his paper [3], Fujiwara proved that every $\mathfrak{g}$ whose dimension is equal to or less than 6 can be graded by positive integers and consequently $\mathfrak{g}$ has a non-singular derivation. So, by the result due to Scheuneman [6], such $\mathfrak{g}$ always admits an affine structure. The existence of non-singular derivations is a strong sufficient condition but it is not a necessary one. In fact, J. Dixmier and W. G. Lister [2] have found already in 1957, independent of our problem, an 8-dimensional example, which we denote by $\mathfrak{g}_8$, without non-singular derivations and proposed to study a class of characteristically nilpotent Lie algebras. Here $\mathfrak{g}$ with the derivation algebra $\mathfrak{D}$ is said to be characteristically nilpotent if $\mathfrak{g}^{(k)}=\{0\}$ for some positive integer $k$, where $\mathfrak{g}^{(1)}$ is defined inductively by $\mathfrak{g}^{(1)}=\{\sum D_i x_i; x_i \in \mathfrak{g}, D_i \in \mathfrak{D}\}$ and $\mathfrak{g}^{(m)} = \mathfrak{D} \mathfrak{g}^{(m-1)} (m \geq 2)$. $\mathfrak{g}_8$ has no non-singular derivations but fortunately this is
3-step nilpotent, that is, $\mathfrak{g}^3 \mathfrak{g} = \{0\}$, where the sequence $\{\mathfrak{g}^k \mathfrak{g}\}$ denotes the descending central series for $\mathfrak{g}$. So $\mathfrak{g}_8$ admits an affine structure by the result due to Scheuneman [7] though it has no non-singular derivations.

Now we will explain the outline of this paper. To construct an algebra like $\mathfrak{g}_8$ for an arbitrary dimension seems to be somewhat difficult. We consider in § 3 a 7-dimensional and 6-step nilpotent Lie algebra $\mathfrak{g}_7 = \mathfrak{g}(a, b, c, d, e)$ and its derivation algebra. Then it is seen that $\mathfrak{g}_7$ has no non-singular derivations if $b = 0, c \neq 0, e \neq 0$, is characteristically nilpotent and admits an affine structure for arbitrary $a, b, c, d, e \in \mathbb{R}$. But this algebra is 6-step nilpotent. So the known results can not be applied to prove that it admits an affine structure. So, in § 2 we prepare a theorem concerning affine representations, which depends essentially on the fundamental result obtained by Scheuneman [8], and apply it to $\mathfrak{g}_7$. The theorem assures us that $\mathfrak{g}$ admits an affine structure if and only if there exists an affine representation $\rho : \mathfrak{g} \to \mathfrak{a}(n)$ ($n = \dim \mathfrak{g}$) as described in [12]. The algebras like $\mathfrak{g}_7$ always exist for each dimension $n$ ($n \geq 8$). In § 4, we investigate $\mathfrak{g}_n = \mathfrak{g}(a_1, a_2, \ldots, a_{n-6})$ ($a_i \in \mathbb{R}, n \geq 8$). We show that $\mathfrak{g}_n$ has no non-singular derivations if $a_1 \neq 0$, is characteristically nilpotent and admits an affine structure even if $a_1 = 0$.

So, together with the result due to Fujiwara, a distinctive character of $\mathfrak{g}$ appears at first in dimension 7 and subsequently.

From the result of § 2, it follows that if $\mathfrak{g}$ admits an affine structure, then it induces naturally an affine structure of $\mathfrak{g}/\mathbb{R}X_n$, where $X_n$ is some central element (See § 2). Geometrically, this means that the simply connected nilpotent Lie group $G(n)$ corresponding to $\mathfrak{g}$ is a fibre bundle over $G(n-1)$ corresponding to $\mathfrak{g}/\mathbb{R}X_n$ with a 1-dimensional fibre $\mathbb{R} = \{\exp tX_n; t \in \mathbb{R}\}$ and the affine structure $\rho$ (See § 2) under consideration is projectable onto $G(n-1)$. In the final section, we give an example such that the converse is not true.

**Definitions**

$\mathfrak{g}_8$ [2] is the 8-dimensional Lie algebra over $\mathbb{R}$ described in terms of a basis $\{X_1, X_2, \ldots, X_8\}$ by the following multiplication table:

1. $[X_1, X_2] = X_5$,  
2. $[X_1, X_3] = X_6$,  
3. $[X_1, X_4] = X_7$,  
4. $[X_1, X_5] = -X_8$,  
5. $[X_1, X_6] = X_5$,  
6. $[X_2, X_4] = X_6$,  
7. $[X_2, X_5] = -X_7$,  
8. $[X_3, X_4] = -X_5$,  
9. $[X_3, X_5] = -X_7$,  

$\mathfrak{g}_8$ has the following multiplication table:

- $[X_1, X_2] = X_5$
- $[X_1, X_3] = X_6$
- $[X_1, X_4] = X_7$
- $[X_1, X_5] = -X_8$
- $[X_1, X_6] = X_5$
- $[X_2, X_4] = X_6$
- $[X_2, X_5] = -X_7$
- $[X_3, X_4] = -X_5$
- $[X_3, X_5] = -X_7$
- $[X_4, X_6] = -X_8$
In addition $[X_i, X_j] = -[X_j, X_i]$ and $[X_i, X_i] = 0$ if it is not in the table above.

In the same way, $\mathfrak{g}_7 = g(a, b, c, d, e)$ $(a, b, c, d, e \in \mathbb{R})$ is described in terms of a basis $\{X_1, X_2, \ldots, X_7\}$ as follows:

1. $[X_1, K_k] = X_{k+1}$ $(2 \leq k \leq 6)$, $[X_1, X_7] = 0$,
2. $[X_2, X_3] = aX_4 + bX_5 + cX_6 + dX_7$,
3. $[X_2, X_4] = aX_5 + bX_6 + cX_7$,
4. $[X_2, X_5] = aX_6 + (b - e)X_7$,
5. $[X_2, X_6] = aX_7$,

And additional conditions are just the same as $\mathfrak{g}_8$.

$\mathfrak{g}_n = g(a_1, a_2, \ldots, a_{n-6})$ $(a_i \in \mathbb{R}, n \geq 8)$ is described in terms of a basis $\{X_1, X_2, \ldots, X_n\}$ as follows:

1. $[X_1, X_k] = X_{k+1}$ $(2 \leq k \leq n - 1)$,

$k$. $[X_2, X_{k+1}] = X_{k+3} + \sum_{j=1}^{n-5} a_j X_{j+k+4}$ $(2 \leq k \leq n - 5)$,

$n - 4$. $[X_2, X_{n-3}] = X_{n-1}$,

$n - 3$. $[X_2, X_{n-2}] = X_n$.

And additional conditions are just the same as $\mathfrak{g}_8$.

$\mathfrak{g}_n$ has been considered, independent of our problem, by G. Vrânceanu [9]. Once the aim is settled, then we only carry out elementary computations. So, in proofs, we shall confine ourselves only to write down necessary relations obtained from a number of equations.

§2. A theorem on affine structures

In this section, we shall prove a theorem concerning an affine representation which gives an affine structure of $\mathfrak{g}$. At present there are only a few general theorems on affine structure of $\mathfrak{g}$ even if $\mathfrak{g}$ is nilpotent. Our theorem depends
essentially on the most fundamental theorem due to J. Scheuneman [8], which we will call Theorem S in what follows and quote later in a slightly variant form.

Let \( \{e_1, e_2, \ldots, e_n\} \) be a basis of \( \mathbb{R}^n \). In what follows, a point \( x = \sum_{i=1}^{n} x_i e_i \in \mathbb{R}^n \) is identified as usual with a column vector \( (x_1, x_2, \ldots, x_n, 1) \in \mathbb{R}^{n+1} \) if \( \mathbb{R}^n \) is considered as a manifold on which affine motions operate. An affine motion is then an \( (n+1) \times (n+1) \) real matrix \( \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \) with \( A \in GL(n, \mathbb{R}) \) and \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \). The column vector \( v \) is called the translation part of \( \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \).

Assume the simply connected Lie group \( \mathcal{G} \) with \( \mathfrak{g} \) as its Lie algebra operates simply transitively on \( \mathbb{R}^n \), that is, there exists a representation \( F \) of \( \mathcal{G} \) into \( A(n) \), where \( A(n) = \{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} : A \in GL(n, \mathbb{R}), v \in \mathbb{R}^n \} \), such that \( F(\mathcal{G}) \) operates simply transitively on \( \mathbb{R}^n \). Let \( f = dF \) be the differential of \( F \). Then \( f : \mathfrak{g} \rightarrow \mathfrak{a}(n) \), where \( \mathfrak{a}(n) \) is the affine Lie algebra of \( A(n) \) has the following properties (cf. [8]):

(a) \( f(\mathfrak{g}) \) may be assumed to consist of upper triangular nilpotent matrices (within some affine conjugacy),

(b) the set of all translation parts of \( f(\mathfrak{g}) \) is the whole \( \mathbb{R}^n \).

Under the above situation, we have the following

**Theorem 1.** Assume \( \mathfrak{g} \) admits an affine structure and let \( f : \mathfrak{g} \rightarrow \mathfrak{a}(n) \) be as above. Then there exist a basis \( \{X_1, X_2, \ldots, X_n\} \) of \( \mathfrak{g} \) and a representation \( \rho : \mathfrak{g} \rightarrow \mathfrak{a}(n) \) which has the following properties:

\[
\rho(X) = \begin{bmatrix}
A(X) & \begin{bmatrix} x_1 \\
\vdots \\
x_n \\
0, \ldots, 0 \end{bmatrix} \\
0, \ldots, 0 & 0
\end{bmatrix} \quad (X = \sum_{i=1}^{n} x_i X_i \in \mathfrak{g}),
\]

where \( A : \mathfrak{g} \rightarrow \mathfrak{gl} (\mathfrak{g}) \) is a representation which satisfies

\[
A(X)Y - A(Y)X = [X, Y] \quad (X, Y \in \mathfrak{g})
\]

and

\[
A(X_k) = \begin{bmatrix}
0 & & & & 0 \\
& \ddots & & & 0 \\
& & 0 & & 0 \\
& & k & & 0 \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix} \quad (1 \leq k \leq n),
\]
(2) A basis \(\{X_1, X_2, \ldots, X_n\}\) is so chosen as \(b_k = \sum_{j=1}^{k} RX_j\) is an ideal of \(\mathfrak{g}\) for any \(k\) and \(X_k\) is a central element of the quotient algebra \(\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{h}_{k+1}\). Conversely the above \(\rho\) gives an affine structure of \(\mathfrak{g}\).

Before giving the proof, we state Theorem S.

**Theorem S** (Scheuneman [8]). Let \(f: \mathfrak{g} \to \mathfrak{a}(n)\) be a representation satisfying (a) and (b). Then there exist an origin preserving affine motion \(T = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{a}(n)\) and a central element \(Z\) of \(\mathfrak{g}\) such that \(T^{-1}f(g)T\) also satisfies (a) and (b), and \(T^{-1}f(Z)T\) takes the following form:

\[
T^{-1}f(Z)T = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.
\]

Using Theorem S, we have the following

**Proposition.** Let \(f\) be as above. Then there exist a basis \(\{Y_1, Y_2, \ldots, Y_n\}\) of \(\mathfrak{g}\) and \(T_n = \begin{pmatrix} A_n & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{a}(n)\) such that \(T_n^{-1}f(\mathfrak{g})T_n\) satisfies (a), (b) and

\[
T_n^{-1}f(Y_k)T_n = \begin{pmatrix} 0 & a_{ij}(Y_k) & \cdots & 0 \\ & 0 \\ & \vdots \\ & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (1 \leq k \leq n)
\]

(2) \(b'_k = \sum_{j=1}^{k} R Y_j\) is an ideal of \(\mathfrak{g}\) for any \(k\) and \(Y_{k+1}\) is a central element of \(\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{h}'_k\).

**Proof.** It is done by induction on \(n = \dim \mathfrak{g}\). If \(\dim \mathfrak{g} = 1\), it is sufficient to take \(Z\) (resp. \(T\)) in Theorem S as \(Y_1\) (resp. \(T_1\)). Assume the proposition is true for some \(n\). Let \(\mathfrak{g}\) be an \((n+1)\)-dimensional nilpotent Lie algebra and \(f: \mathfrak{g} \to \)
a(n+1) be a representation which satisfies (a), (b). Then by Theorem S, there exist a central element $Z$ and $T = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in A(n+1)$ such that $T^{-1}f(g)T$ and $T^{-1}f(Z)T$ satisfy the properties of Theorem S. We put $Z = Y_1$ and consider the matrix obtained by deleting the first row and the first column of $T^{-1}f(Y)T$ ($Y \in g$). Because of the shape (the property (a)) of the elements of $T^{-1}f(g)T$, this matrix depends only on a class $\bar{Y} (\mod R Y_1)$. Thus we get a well-defined mapping $\bar{f}: \bar{g} \rightarrow a(n)$, where $\bar{g}$ is a quotient algebra $g/R Y_1$ and $f(\bar{Y})$ is a matrix obtained as above from $T^{-1}f(Y)T$. Then it is easy to see that $\bar{f}$ is a representation which satisfies (a). The property (b) for $\bar{f}$ follows from that for $T^{-1}f(g)T$. Since $\dim \bar{g} = n$ and $\bar{f}$ satisfies the necessary properties, the induction hypothesis is applicable to $\bar{f}$ and $\bar{g}$. Therefore there exist a basis $\{W_1, W_2, ..., W_n\}$ ($W_i \in g$) of $\bar{g}$ and $S_n = \begin{pmatrix} B_n & 0 \\ 0 & 1 \end{pmatrix} \in A(n)$ such that they satisfy the properties of the proposition. Let $S$ be a matrix

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & B_n & 0 \\ 0 & 0 & 1 \end{bmatrix} \in A(n+1)$$

and put $T_{n+1} = T \cdot S$. We choose non-zero vectors $Y_2, ..., Y_{n+1}$, where $Y_{i+1} \in W_i$ ($1 \leq i \leq n$). Then it is not difficult to see that $\{Y_1 = Z, Y_2, ..., Y_{n+1}\}$ and $T_{n+1}$ satisfy the properties stated in the proposition. Thus the induction is completed.

Now we will prove Theorem 1. Let $\{Y_1, Y_2, ..., Y_n\}$ be a basis in the proposition and $\Phi$ be the following matrix:

$$\begin{bmatrix} 1 & v_1(Y_2) & v_1(Y_3) & \cdots & v_1(Y_n) \\ 1 & v_2(Y_3) & v_2(Y_n) \\ & & & \vdots \vdots \\ 0 & & & \cdots & v_{n-1}(Y_n) \\ & & & & 1 \end{bmatrix} \times 1.$$

Then $\Phi^{-1}T_n^{-1}f(Y_n)T_n\Phi$ takes the following form:
Let $\Psi$ be the following matrix:

\[
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]

Then $\Psi^{-1} \Phi^{-1} T_{n}^{-1} f(Y_{k}) T_{n} \Phi \Psi$ takes the following form:

\[
\begin{bmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & \vdots & \ddots & \vdots & \vdots \\
\lambda_{ij}(Y_{k}) & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]

where $\lambda_{ij}(Y_{k}) = b_{n-i+1,n-j+1}(Y_{k})$.

Now we put $X_{k} = Y_{n-k+1}$ $(1 \leq k \leq n)$ and define a representation $\rho: g \to \mathfrak{gl}(g)$ via

\[
\rho(X_{k})X = [(T_{n} \Phi \Psi)^{-1} \cdot f(Y_{n-k+1}) \cdot (T_{n} \Phi \Psi)]^{t}(x_{1}, x_{2}, \ldots, x_{n}, 1),
\]

where $X = \sum_{i=1}^{n} x_{i}X_{i} \in g$. Then it is easy to see that this $\rho$ is a desired one in Theorem 1. This proves the first half of Theorem 1.

Let $\rho$ be the representation of the form described in Theorem 1 and $G(n)$ be the analytic subgroup of $A(n)$ with the Lie algebra $\rho(g)$. To prove $G(n)$ operates simply transitively on $g \ (= \mathbb{R}^{n})$ it is sufficient to show that it operate transitively
on $g$ (cf. [12]). For this purpose, we consider the following equation for any $'a_1, a_2, \ldots, a_n, 1$

$$\exp x_n \rho(X_n) \cdot \exp x_{n-1} \rho(X_{n-1}) \cdot \cdots \exp x_1 \rho(X_1) \cdot 0$$

$= 'a_1, a_2, \ldots, a_n, 1,$

where $0 = '0, 0, \ldots, 0, 1$. Then it is seen that the left hand side of the above equation is equal to

$$(x_1, x_2 + P_1(x_1), \ldots, x_k + P_{k-1}(x_1, \ldots, x_{k-1}), \ldots, x_n + P_{n-1}(x_1, \ldots, x_{n-1}), 1),$$

where $P_{k-1}$ is a polynomial in $x_1, x_2, \ldots, x_{k-1}$. So we can determine all $x_i$ uniquely and successively by starting from $x_1 = a_1$. This implies the transitivity of $G(n)$. Q. E. D.

§ 3. Derivations and affine structures of $g_7$

1. Derivations of $g_7$

Let $g_7 = g(a, b, c, d, e)$ be as defined in the introduction. Then we have

**Lemma 1.** $g_7$ is a 6-step nilpotent Lie algebra for arbitrary $a, b, c, d, e \in \mathbb{R}$.

**Proof.** It is easy to check that the Jacobi identity holds. The 6-step nilpotency is obvious. Q. E. D.

Let $D$ be any derivation of $g_7$ and put $DX_i = \sum_{j=1}^{7} k_{ji} X_j$. Expressing both sides of the equation

$$D[X_i, X_j] = [DX_i, X_j] + [X_i, DX_j]$$

(*1)

in terms of $\{X_1, \ldots, X_7\}$ and comparing coefficients, we get a number of linear relations in terms of $k_{ij}$'s. These give a necessary and sufficient condition for $D = (k_{ij})$ to be a derivation. Here we write down them in order.

1. $[X_1, X_2] = X_3$;
   
   $k_{13} = k_{23} = 0$,
   
   $k_{33} = k_{11} + k_{22}$,
   
   $k_{43} = k_{32} - ak_{31}$,
   
   $k_{53} = k_{42} - bk_{31} - ak_{41}$,
\begin{align*}
k_{63} & = k_{52} - ck_{31} - bk_{41} - ak_{51}, \\
k_{73} & = k_{62} - dk_{31} - ck_{41} - (b - e)k_{51} - ak_{61}.
\end{align*}

(2) \[ [X_1, X_3] = X_4; \quad \text{and} \quad [X_1, X_4] = X_5; \]

(3) \[ [X_1, X_4] = X_5; \quad \text{and} \quad [X_1, X_5] = X_6; \]

(4) \[ [X_1, X_5] = X_6; \quad \text{and} \quad [X_1, X_6] = X_7; \]

(5) \[ [X_1, X_6] = X_7; \quad \text{and} \quad [X_2, X_3] = aX_4 + bX_5 + cX_6 + dX_7; \]

\[ \begin{align*}
ak_{44} & = k_{12} + ak_{22} + ak_{33}, \\
ak_{54} + bk_{55} & = bk_{22} + bk_{33} + ak_{43}, \\
ak_{64} + bk_{65} + ck_{66} & = ck_{22} + ck_{33} + bk_{43} + ak_{53}, \\
ak_{74} + bk_{75} + ck_{76} + dk_{77} & = ak_{42} + ak_{52} + ak_{62} + ak_{72}.
\end{align*} \]

(7) \[ [X_2, X_4] = aX_5 + bX_6 + cX_7; \]

(8) \[ [X_2, X_5] = aX_6 + (b - e)X_7; \]

\[ \begin{align*}
ak_{65} & = k_{12} + ak_{22} + ak_{55}, \\
ak_{76} + (b - e)k_{77} & = (b - e)k_{22} + (b - e)k_{55} + ak_{65}.
\end{align*} \]
(9) \[ [X_2, X_6] = aX_7; \]
\[ ak_{77} = k_{12} + ak_{22} + ak_{66}. \]
\[ ek_{77} = ek_{33} + ek_{44}. \]

Summing up, we see at first \( D \) must be of the following form:

\[
D = \begin{bmatrix}
  k_{11} & k_{12} & 0 \\
  k_{21} & k_{22} & 0 \\
  k_{33} & k_{44} & 0 \\
  * & * & k_{55} \\
  k_{66} & k_{77}
\end{bmatrix}
\]

and we have a recurrence formula

\[ k_{i+1i+1} = k_{11} + ak_{21} + k_{ii} \quad (3 \leq i \leq 6). \quad \text{(3.1)} \]

It follows from (3.1)

\[ k_{77} - k_{66} = k_{66} - k_{55} = k_{55} - k_{44} = k_{44} - k_{33} = k_{11} + ak_{21}, \]

\[ k_{77} = 4k_{44} - 3k_{33}. \quad \text{(3.2)} \]

Assume \( e \neq 0 \). Then it follows from (10)

\[ k_{77} = k_{44} + k_{33}. \quad \text{(3.3)} \]

From (3.2) and (3.3), we have

\[ k_{77} = \frac{7}{3} k_{33}. \quad \text{(3.4)} \]

Assume \( b = 0 \). Then the 3rd equation of (6) is

\[ ak_{64} + ck_{66} = ck_{22} + c k_{33} + a k_{53}. \quad \text{(3.5)} \]

On one hand, if \( a \neq 0 \), then the 5th equation of (2) is equivalent to

\[ ak_{64} = ack_{21} + ak_{53}. \quad \text{(3.6)} \]

It follows from (3.5) and (3.6)

\[ ak_{21} + ck_{66} = ck_{22} + c k_{33}. \]
Assume $c \neq 0$. Then we have
\[
 ak_{21} + k_{66} = k_{22} + k_{33}.
\]
Add $k_{11}$ to both sides of the above equation and use the 2nd equation of (1) and the 1st equation of (5). Then we have
\[
 k_{77} = 2k_{33}.
\] (3.7)
So we have $k_{33} = k_{77} = 0$ by (3.4) and (3.7). Then $k_{44} = 0$ by (3.4). Consequently we have
\[
 k_{ii} = 0 \quad (3 \leq i \leq 7), \quad k_{11} + ak_{21} = 0.
\] (3.8)
Then we have by the 1st equation of (6)
\[
 k_{21} + ak_{22} = 0.
\] (3.9)
From $k_{11} + k_{22} = k_{33} = 0$, we have
\[
 k_{22} = -k_{11}.
\] (3.10)
Therefore, from (3.8), (3.9) and (3.10), we have
\[
 D_1 \equiv \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = k_{11} \begin{bmatrix} 1 & a \\ -a^{-1} & -1 \end{bmatrix}.
\]
Let $A_1 = \begin{bmatrix} a & a^2 \\ 0 & a^{-1} \end{bmatrix}$. Then we have
\[
 A_1 D_1 A_1^{-1} = k_{11} \begin{bmatrix} 0 & 0 \\ -a^{-1} & 0 \end{bmatrix}.
\]
Let $A = A_1 \oplus I_5$, where $I_5$ is the $5 \times 5$ unit matrix. Then for every derivation $D$, $ADA^{-1}$ has the following form:
\[
 ADA^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ -k_{11}/a^3 & 0 \\ \ast & 0 & 0 \\ \ast & \ast & 0 \\ \ast & \ast & \ast & 0 \end{bmatrix}.
\]
If $a = 0$ in the above argument, we have
\[ D_1 = \begin{bmatrix} 0 & 0 \\ k_{21} & 0 \end{bmatrix}. \]

Therefore we have seen that for \( g(a, 0, c, d, e) \) \((c \neq 0, e \neq 0)\) every derivation \( D \) is singular and expressed by a lower triangular matrix with 0's on the main diagonal by a suitable choice of a basis (common to every \( D \)). Hence \( g^{[5]} = \{0\} \). Thus we have proved

**Theorem 2.** \( g_7 = g(a, 0, c, d, e) \) \((c \neq 0, e \neq 0)\) has no non-singular derivations and this algebra is characteristically nilpotent.

Next we consider \( g(a, 0, c, 0, e) \) \((a \neq 0, c \neq 0, e \neq 0)\). Then we can easily check that the following \( D \) satisfies the conditions (1) \~(10). Consequently \( D \) is a derivation.

\[
D = \begin{bmatrix}
-a & -a^2 & 0 & 0 & 0 & 0 & 0 \\
1 & a & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 & 0 & 0 \\
0 & c/2 & \alpha & 0 & 0 & 0 & 0 \\
0 & \beta & c/2 & \alpha & 0 & 0 & 0 \\
0 & \gamma & \beta & 3c/2 & \alpha & 0 & 0 \\
0 & \delta & \gamma & \beta & 5c/2 & \alpha & -e & 0
\end{bmatrix},
\]

where \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \) are arbitrary except \( \alpha = e \).

The rank of \( D \) is obviously 6. Thus we have proved

**Theorem 3.** \( g(a, 0, c, 0, e) \) \((a \neq 0, c \neq 0, e \neq 0)\) has a derivation \( D \) such that \( n_D = \dim (\ker D) = 1 \).

The above \( g \) is an example without non-singular derivations which has a derivation of the maximal rank \((= \dim g - 1)\).

**Remark 1.** \( g(0, a, b, c, d) \) was studied in [4] from a different viewpoint.

**Remark 2.** Parameters \( a, b, c, d, e \in \mathbb{R} \) are in general not essential, that is, different parameters may give isomorphic Lie algebras. We do not discuss this problem in this paper.
2. Affine structures of $\mathfrak{g}_7$.

As mentioned in the introduction, we have at present no simple condition to judge whether $\mathfrak{g}_7$ admits an affine structure or not. So following Theorem 1 in §2, we shall prove that for arbitrary $a, b, c, d, e \in \mathbb{R}$ $\mathfrak{g}_7$ admits an affine structure $\rho$.

At first we must construct a representation

$$\Lambda: \mathfrak{g}_7 \rightarrow \mathfrak{gl}(\mathfrak{g}_7)$$

which satisfies

$$\Lambda(X)Y - \Lambda(Y)X = [X, Y] \quad (X, Y \in \mathfrak{g}_7) \quad (*)$$

Take a basis $\{X_1, ..., X_7\}$ of $\mathfrak{g}_7$ as in the introduction and let $\Lambda(X_i) = (a_{ij}(X_i))$ with respect to this basis. Since $\Lambda$ must be a representation and satisfy $(*)$, we obtain a number of relations between matrix elements $a_{ij}(X_i)$'s and structural constants of $\mathfrak{g}_7$. By a quite elementary method, though some technique is needed, we can solve these relations. Here is a solution of the simplest form. The general solution contains some free parameters. For example, $a_{72}(X_2)$ can be chosen arbitrarily. Because of the relations $\Lambda(X_{k+1}) = [\Lambda(X_k), \Lambda(X_{k+1})]$ ($2 \leq k \leq 6$), it is sufficient to show only $\Lambda(X_1)$ and $\Lambda(X_2)$. But we shall write down all of them for the sake of completeness.

$$\Lambda(X_1) = \text{ad}X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda(X_2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & a & 0 & 0 & 0 & 0 \\ 0 & c & b & a & 0 & 0 & 0 \\ 0 & -d & c & b & -e & a & 0 \\ 0 & 0 & 0 & 0 & b & -2e & a & 0 \end{bmatrix}$$

$$\Lambda(X_3) = \begin{bmatrix} 0 \\ 0 & e & 0 & 0 & 0 & 0 \\ 0 & -d & c & e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Lambda(X_4) = \begin{bmatrix} 0 \\ 0 & -e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
\[
\begin{bmatrix}
0 \\
0 & -e & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \Lambda(X_5) = \Lambda(X_7) = 0.
\]

Then the representation \( \rho \) defined by this \( \Lambda \) gives an affine structure of \( g_7 \) by virtue of Theorem 1. Thus we have

**Theorem 4.** \( g_7 = g(a, b, c, d, e) \) admits an affine structure for arbitrary \( a, b, c, d, e \in \mathbb{R} \).

§ 4. Derivations and affine structures of \( g_n \) (\( n \geq 8 \))

There exists always \( g \) like \( g_7 \) which has no non-singular derivations for each dimension \( n \geq 8 \). It admits an affine structure and is characteristically nilpotent.

**Lemma 2.** \( g_n \) is an \( (n-1) \)-step nilpotent Lie algebra.

In what follows, results concerning \( n = \dim g \) must be proved precisely, for example, by the induction method. But it is not so difficult. So we omit proofs.

1. Derivations of \( g_n \)

Let \( D \) be a derivation of \( g_n \) and put \( DX_i = \sum_{j=1}^{n} k_{ij} X_j \). Applying \( D \) to both sides of \( [X_1, X_k] = X_{k+1} \) (\( 2 \leq k \leq n-1 \)) and comparing coefficients of \( X_i \) (\( 1 \leq i \leq k \), \( 2 \leq k \leq n-1 \)), we have the following relations:

\[
k_{ij} = 0 \quad (3 \leq j \leq n), \quad k_{ij} = 0 \quad (2 \leq i < j \leq n), \quad (4.1)
k_{ii} = k_{11} + k_{i-1 i-1} \quad (3 \leq k \leq n), \quad (4.2)
i.e., \quad k_{ii} - k_{i-1 i-1} = k_{11}.
\]

Next we apply \( D \) to both sides of the \((n-p)\)th equation \( (3 \leq p \leq n-2) \) and compare coefficients of \( X_{n-p+3} \). Then using (4.1), we have the following relations:

\[
k_{ii} = k_{22} + k_{i-2 i-2} \quad (5 \leq i \leq n), \quad (4.3)
i.e., \quad k_{ii} - k_{i-2 i-2} = k_{22}.
\]

Especially comparing coefficients of \( X_1 \) of the resultant relation from (2), we have \( k_{12} = 0 \). So together with (4.1), we have

\[
k_{ij} = 0 \quad (1 \leq i < j \leq n). \quad (4.4)
\]
From (4.2) and (4.3), it follows \( k_{45} - k_{44} = k_{44} - k_{33} \) and \( k_{55} = k_{33} + k_{22} \). So we have \( 2(k_{44} - k_{33}) = k_{22} \) and consequently

\[
k_{22} = 2k_{11}.
\] (4.5)

From (4.2), it follows \( k_{pp} = (p - 2)k_{11} \), i.e.,

\[
k_{pp} = pk_{11}.
\] (4.6)

Especially,

\[
k_{77} = 7k_{11}.
\] (4.7)

We apply \( D \) to both sides of (2) and compare coefficients of \( X_7 \). Then we have

\[
a_1k_{22} + a_1k_{33} + k_{55} = k_{75} + a_1k_{77}
\] (4.8)

In the same way, applying \( D \) to both sides of \([X_1, X_4] = X_5 \) (resp. \([X_1, X_3] = X_4 \)) and comparing coefficients of \( X_7 \) (resp. \( X_6 \)), we have \( k_{75} = k_{64} \) (resp. \( k_{64} = k_{53} \)). Thus we have \( k_{75} = k_{53} \). Then from (4.8) it follows

\[
a_1k_{22} + a_1k_{33} = a_1k_{77}.
\]

Now assume \( a_1 \neq 0 \). Then \( k_{77} = k_{22} + k_{33} \), i.e.,

\[
k_{77} = 2k_{11} + 3k_{11} = 5k_{11}.
\] (4.9)

From (4.7) and (4.9), it follows

\[
k_{11} = 0, \quad k_{77} = 0
\] (4.10)

Finally it follows from (4.6)

\[
k_{pp} = pk_{11} = 0 \quad (1 \leq p \leq n)
\] (4.11)

Summing up, we have seen that every derivation \( D \) is a lower triangular matrix with 0's on the main diagonal. Hence we have

**Theorem 5.** \( g_n \) has no non-singular derivations if \( a_1 \neq 0 \) and in this case it is characteristically nilpotent.

2. Affine structures of \( g_n \)

Next we consider affine structures of \( g_n \) for arbitrary \( a_1, a_2, \ldots, a_{n-6} \in \mathbb{R} \). The following representation \( \Lambda \) is one of what we seek, i.e., a representation \( \Lambda: g_n \to gl(g_n) \) such that \( \Lambda(X)Y - \Lambda(Y)X = [X, Y] \) for \( X, Y \in g_n \).
\[ A(X_1) = adX_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \]

\[ A(X_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & a_1 & 0 & 1 \\ 0 & a_2 & a_1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n-6} & a_{n-7} & a_{n-8} \\ 0 & 1 & 0 & 0 \end{bmatrix}, \]

where the \( k \)th column is \((0, \ldots, 0, 1, 0, a_1, a_2, \ldots, a_{n-k-3}) (k \geq 3) \) and \( A(X_j) = 0 \) \((3 \leq j \leq n)\). By virtue of Theorem 1, a representation \( \rho \) defined by this \( A \) gives an affine structure of \( g_0 \). Thus we have

**Theorem 6.** \( g_n \) admits an affine structure for arbitrary \( a_1, a_2, \ldots, a_{n-6} \in \mathbf{R} \).

\( g \) which is defined by \([X_1, X_k] = X_{k+1} (2 \leq k \leq n-1), [X_1, X_j] = X_j, [X_i, X_j] = 0 \) (otherwise) is obviously an \((n-1)\)-step nilpotent Lie algebra. \( g \) has a non-singular derivation because it is graded by positive integers. For example, let \( g^1 = RX_1 + RX_2 \) and \( g^k = RX_{k+1} (2 \leq k \leq n-1) \). Then \( g = \sum_{k=1}^{n-1} g^k \)
and \([g^i, g^j] \subseteq g^{i+j}\). Hence we have

**Corollary.** For each \(n \geq 7\), there exist at least two types of \(n\)-dimensional \((n-1)\)-step nilpotent Lie algebra such that one has a non-singular derivation and the other has no non-singular derivations and is characteristically nilpotent. Both algebras admit affine structures.

**Remark.** It is interesting to study a class of \(g\) which has no non-singular derivations and is not characteristically nilpotent.

§5. Some Remark

Suppose an \(n\)-dimensional \(g\) admits an affine structure \(f: g \to \alpha(n)\). Then we may assume within affine conjugacy that \(f(g)\) itself satisfies (a), (b) in §2 and there exists a central element \(Z\) such that \(f(Z)\) takes the form as described in Theorem S. Then we have seen in the proof of Proposition in §2, the representation \(f: \tilde{g} \to \alpha(n-1)\) induced naturally by \(f\) gives an affine structure, where \(\tilde{g} = g/\mathbb{R}Z\). Let \(\tilde{G}(n)\) be the simply connected nilpotent Lie group with the Lie algebra \(\tilde{g}\) and \(H\) be a 1-parameter subgroup \(\{\exp tZ; t \in \mathbb{R}\}\). Put \(\tilde{G}(n-1) = \tilde{G}(n)/H\). Then \(\tilde{G}(n-1)\) is also simply connected and admits an affine structure in question, since \(\tilde{g}\), the Lie algebra of \(\tilde{G}(n-1)\), admits an affine structure \(f\). \(\tilde{G}(n)\) is a trivial fibre bundle over \(\tilde{G}(n-1)\) with a 1-dimensional fibre \(H\) which is central, because \(\tilde{G}(n-1)\) is diffeomorphic to \(\mathbb{R}^{n-1}\). Thus a complete affinely flat structure which is left-invariant of \(\tilde{G}(n)\) is projectable onto that of \(\tilde{G}(n-1)\). On projectable connections and the local existence theorem of fibred spaces, refer to [13], for example. In case of \(\rho\) described in Theorem 1, \(\tilde{\rho}(Y)\) is defined as the matrix obtained by deleting the \(n\)th row and the \(n\)th column from \(\rho(Y)\) and the situation is same as above. Now we confine ourselves to affine structures of type \(\rho\) in Theorem 1. In [12], we gave to \(g_{5,6}\) an affine structure of the simplest form, that is,

\[
g_{5,6}: [X_1, X_k] = X_{k+1}, \quad (2 \leq k \leq 4), \quad [X_2, X_3] = X_5
\]

and \(\rho(X) = \begin{bmatrix} \Lambda(X) & X \\ 0 & 0 \end{bmatrix}\) \((X \in g_{5,6})\), where \(\Lambda(X_k) (k = 1, \ldots, 5)\) is given by
\[ \Lambda(X_1) = adX_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Lambda(X_2) = \begin{bmatrix} 0 \\ 0 & 0 & 1/2 & 0 & 0 \end{bmatrix}, \]

\[ \Lambda(X_3) = \begin{bmatrix} 0 \\ 0 & -1/2 & 0 & 0 & 0 \end{bmatrix}, \quad \Lambda(X_4) = \Lambda(X_5) = 0. \]

The 6-dimensional nilpotent Lie algebras \( \mathfrak{g} \) such that \( g_{5,6} \cong \mathfrak{g}/R \) \( X_6 \), where \( X_6 \) is a central element of \( \mathfrak{g} \), are just the following two [10]:

\[ g_{6,30}: \begin{align*}
[X_1, X_5] &= X_{k+1} & (2 \leq k \leq 4), \\
[X_2, X_3] &= X_5, \\
[X_2, X_5] &= X_6, \\
[X_3, X_4] &= -X_6,
\end{align*} \tag{5.1} \]

\[ g_{6,31}: \begin{align*}
[X_1, X_5] &= X_{k+1} & (2 \leq k \leq 5), \\
[X_2, X_3] &= X_5, \\
\end{align*} \tag{5.2} \]

The above affine structure \( \rho \) of \( g_{5,6} \) can be lifted to the following \( \tilde{\rho} \) of \( \tilde{\mathfrak{g}} = g_{6,31} \).

Put \( \tilde{\rho}(X) = \begin{bmatrix} \Lambda(X) \\ X \end{bmatrix} \) for \( X \in \tilde{\mathfrak{g}} \), where

\[ \begin{bmatrix} \Lambda(X) \\ X \end{bmatrix} = \begin{bmatrix} \Lambda(X_2) \\ 0 & 0 & 0 & 1/2 & 0 & 0 \end{bmatrix}, \quad \Lambda(X_3) = \begin{bmatrix} \Lambda(X_3) \\ 0 & 0 & 0 \end{bmatrix}, \]

\[ \begin{bmatrix} \Lambda(X_4) \\ 0 & -1/2 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Lambda(X_6) = \Lambda(X_6) = 0. \]

Now we investigate another affine structure of type \( \rho \) of \( \mathfrak{g} = g_{5,6} \):

\[ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 \\ 0 & 0 & 1/2 & 0 & 0 \end{bmatrix}, \quad \Lambda(X_3) = \begin{bmatrix} 0 \\ 0 & 0 & 1/3 & 0 & 0 \end{bmatrix} \]
\[
A(X_4) = \begin{bmatrix} 0 \\ 0 & 2/3 & 0 & 0 & 0 \end{bmatrix}, \quad A(X_5) = 0.
\]

Suppose \( \tilde{\rho} \) comes from some \( \tilde{\rho} \) of \( \mathfrak{g} = \mathfrak{g}_{6,30} \) or \( \mathfrak{g}_{6,31} \) by the procedure mentioned above, in other words, \( \rho \) is extendable to \( \tilde{\rho} \). Then \( \tilde{A}(X_k) \), where \( \tilde{A}(X) = \begin{bmatrix} \tilde{A}(X_k) | X \\ 0 & 0 \end{bmatrix} \) \((X \in \mathfrak{g})\), must be the following form:

\[
\tilde{A}(X_k) = \begin{bmatrix} A(X_k) \\ \lambda_{61}(X_k) \cdots \lambda_{65}(X_k) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (k = 1, 2, \ldots, 6)
\]

and satisfy the following equations:

\[
[\tilde{A}(X_i), \tilde{A}(X_j)] = \sum_{k=1}^{6} c_{ijk} \tilde{A}(X_k) \quad (i, j = 1, 2, \ldots, 6),
\]

\[
\tilde{A}(X_i) X_j - \tilde{A}(X_j) X_i = [X_i, X_j] \quad (i, j = 1, 2, \ldots, 6), \tag{**3}
\]

where \( \{c_{ijk}\} \) are the structure constants of (5.1) or (5.2) according to \( \mathfrak{g} = \mathfrak{g}_{6,30} \) or \( \mathfrak{g}_{6,31} \). After elementary computations, we see that there is no solution \( \{\lambda_{6j}(X_k); j, k = 1, 2, \ldots, 6\} \) which satisfies (**3) for \( \mathfrak{g} = \mathfrak{g}_{6,30} \) and the same is also true for \( \mathfrak{g} = \mathfrak{g}_{6,31} \).

Summing up, we have seen that an affine structure \( \rho \) of \( \mathfrak{g} \) is always projectable in a natural manner onto that of \( \mathfrak{g} = \mathfrak{g}/\mathfrak{r}X_6 \), but the converse procedure does not hold good in general. That is, there exist at least two affine structures \( \rho \) of \( \mathfrak{g}_{5,6} \). One of them is extendable to that of some \( \mathfrak{g} \) by the method which seems to be natural and the other is not.

Though affine structures under consideration are somewhat restricted, the above examples may suggest the difficulty of the application of the induction method on affine representations of \( \mathfrak{g} \) to the Auslander's conjecture.

References


