FINITE COMPLEXES WHOSE SELF-HOMOTOPY EQUIVALENCES FORM CYCLIC GROUPS

Dedicated to Professor Tatsuji Kudo on his 60th birthday

By

Shichirô OKA

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The group $\mathcal{E}(X)$ of homotopy classes of homotopy equivalences of a based space X to itself, where the multiplication is the composition of maps, has been studied by several authors (e.g., [3], [4]). As pointed out by D. W. Kahn [3], many simple questions on $\mathcal{E}(X)$ are unfortunately not answered. One of them is the following

REALIZATION PROBLEM. *For a given group G , is there a space X with $\mathcal{E}(X) = G$?*

It is known that $\mathcal{E}(X)$ frequently contains an element of order 2, for example, when X is a suspension of a finite complex. Few examples of X with $Z_2 \subsetneq \mathcal{E}(X)$ are known, e.g., Kahn's examples [3] realizing the unit group and the infinite cyclic group, and the complex $S^4 \cup e^{11}$ with attaching map $h\Sigma^3h$, the composition of the Hopf map h with its 3-times suspension, which realizes the cyclic group Z_{15} of order 15(*).

The purpose of this note is to construct some examples of finite complexes X_n with $\mathcal{E}(X_n) = Z_n$, where $n \not\equiv 0 \pmod{8}$ or $n \equiv 16 \pmod{32}$. Our results are Theorems 1.8 (the case $n = p^r$ with p an odd prime), 3.3 (the case $n \equiv 1 \pmod{2}$), 3.6 (the case $n \equiv 2 \pmod{4}$), 4.1–4.2 (the case $n \equiv 4 \pmod{8}$), and Proposition 4.3 (the case $n \equiv 16 \pmod{32}$).

The essential part of the proofs is the construction of a p -local complex X_{p^r} with $\mathcal{E}(X_{p^r}) = Z_{p^r}$, p an odd prime, $r \geq 1$. Let $q = p^r$ and $M_q^m = S^{m-1} \cup_q e^m$. Then $[M_q^{4p-4}, M_q^{2p-1}] = Z/q \oplus Z/p \oplus Z/p$, where the first summand is the kernel of the suspension and $Z/p \oplus Z/p$ is isomorphic with the stable group $\{M_q^{4p-4}, M_q^{2p-1}\}$ (see (1.10)). The complex X_{p^r} is a mapping cone of a map $\phi: M_q^{4p-4} \rightarrow$

(*) $\mathcal{E}(S^4 \cup e^{11}) = Z_{15}$ is an immediate consequence of Theorem 3.15 in [4].

M_q^{2p-1} which corresponds with $(1, 1, 0) \in \mathbf{Z}/q \oplus \mathbf{Z}/p \oplus \mathbf{Z}/p$ (Definition 1.2). The construction of X_n for n odd is easy from the p -locality of X_{p^r} , that is, we can take X_n to be the wedge $\vee X_{q_i}$, where $n = \prod q_i$, $q_i = p_i^{t_i}$ with p_i odd primes and $p_i \neq p_j$ for $i \neq j$ (Definition 3.1). For other cases, the construction of X_n is similar (see §§ 3-4).

§1. Construction of X_{p^r}

Let $q = p^r$, $r \geq 1$, be a power of an odd prime p . Let $M_q^n = S^{n-1} \cup_q e^n$ be a Moore space of type $(\mathbf{Z}/q, n-1)$. M_q^{2p-1} has the first non-trivial homotopy group $\pi_{2p-2} = \mathbf{Z}/q$, and we consider the $(2p-2)$ -connective fiber space F over M_q^{2p-1} . Then we have a fiber space (up to homotopy type)

$$(1.1) \quad F \xrightarrow{j} M_q^{2p-1} \xrightarrow{k} K(\mathbf{Z}/q, 2p-2).$$

LEMMA 1.1. Let $K = M_p^{4p-4} \vee M_q^{4p-4} \vee M_p^{4p-3}$. Then there is a map $h: K \rightarrow F$ such that $h_*: \pi_i(K) \rightarrow \pi_i(F)$ is an isomorphism for $i \leq 4p-3$.

PROOF. From H. Cartan [2], we have

$$(1.2) \quad \begin{aligned} H^*(K(\mathbf{Z}/q, 2p-2); \mathbf{Z}/p) \\ = \{1, u, \beta_r u, \mathcal{P}^1 u, u^2, \beta \mathcal{P}^1 u, \beta_r u^2, \mathcal{P}^1 \beta_r u, \beta \mathcal{P}^1 \beta_r u, \dots\}, \end{aligned}$$

where $u \in H^{2p-2}$ is the fundamental class and β_r is the r -th order higher Bockstein operation ($\beta_1 = \beta$). Since $k_*: H^*(K(\mathbf{Z}/q, 2p-2); \mathbf{Z}/p) \rightarrow H^*(M_q^{2p-1}; \mathbf{Z}/p)$ is epic, the spectral sequence argument leads us to conclude that there are transgressive elements $x, x' \in H^{4p-5}(F; \mathbf{Z}/p)$ and $y \in H^{4p-4}(F; \mathbf{Z}/p)$ whose transgression images are $\tau(x) = \mathcal{P}^1 u$, $\tau(x') = u^2$, $\tau(y) = \mathcal{P}^1 \beta_r u$, and that

$$(1.3) \quad H^*(F; \mathbf{Z}/p) = \{1, x, x', \beta x, \beta_r x', y, \beta y\} \quad \text{for } \dim < 6p-8.$$

We apply Theorem 1.1 in [8] to (1.3) and obtain a p -local complex K and a map $h: K \rightarrow F$ such that $h_*: H_i(K; \mathbf{Z}/p) \rightarrow H_i(F; \mathbf{Z}/p)$ is isomorphic for $i \leq 4p-3$ and $H_i(K; \mathbf{Z}/p) = 0$ for $i > 4p-3$. To determine the homotopy type of K , we need the following results, (cf. [8, (1.5)], [1, §4], [5]).

$$(1.4.1) \quad \text{If } n \geq 4, [M_q^n, M_q^n] = \mathbf{Z}/q, [M_p^n, M_q^n] = \mathbf{Z}/p \text{ and } [M_q^n, M_p^n] = \mathbf{Z}/p.$$

$$(1.4.2) \quad \text{If } n \geq 5, \pi_n(M_a^n) = 0 \text{ and } [M_a^{n+1}, M_b^n] = 0 \text{ for } a, b \text{ odd.}$$

$$(1.4.3) \quad \text{If } n \geq 5, \pi_{n+1}(M_a^n) = 0 \text{ for } a \text{ odd.}$$

Here these groups are in the stable range by the assumption on n . By (1.4.1), (1.4.2) and Theorem 1.1 in [8], we can take $K = M_p^{4p-4} \vee M_q^{4p-4} \vee M_p^{4p-3}$ as desired. Since K and F are p -local, $h_*: \pi_i(K) \rightarrow \pi_i(F)$ is isomorphic for $i < 4p-3$ and epic for $i = 4p-3$. By (1.4.2) and (1.4.3), $\pi_{4p-3}(K) = 0$ and h_* is also isomorphic for $i = 4p-3$. Thus the lemma is proved.

Let $\rho: M_q^n \rightarrow M_p^n$ be the map realizing the projection $\mathbf{Z}/q \rightarrow \mathbf{Z}/p$, and define ϕ to be the composite

$$(1.5) \quad \phi: M_q^{4p-4} \xrightarrow{(\rho, \text{id})} M_p^{4p-4} \vee M_q^{4p-4} \xrightarrow{i} K \xrightarrow{h} F \xrightarrow{j} M_q^{2p-1},$$

where i is the inclusion.

DEFINITION 1.2. $X_q = M_q^{2p-1} \cup_{\phi} CM_q^{4p-4}$, the mapping cone of ϕ , where $q = p^r$.

LEMMA 1.3. $H^*(X_q; \mathbf{Z}/p) = \{1, x_1, \beta_r x_1, x_2, \beta_r x_2\}$, $\deg x_i = (2p-2)i$, and the generators x_i are chosen such that

$$(1.6) \quad (x_1)^2 = \mathcal{P}^1 x_1 = x_2, \quad \mathcal{P}^1 \beta_r x_1 = 0.$$

PROOF. Let Y, Z and U be the mapping cones of jhi, jh and j , respectively. Then there are maps

$$X_q \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{\bar{i}} U$$

which are identical on M_q^{2p-1} . By (1.1), there is an extension $\bar{k}: U \rightarrow K(\mathbf{Z}/q, 2p-2)$ of k , and by (1.3),

$$\bar{k}_*: H^i(K(\mathbf{Z}/q, 2p-2); \mathbf{Z}/p) \longrightarrow H^i(U; \mathbf{Z}/p)$$

is isomorphic for $i < 6p-7$. Since $h^*: H^*(F; \mathbf{Z}/p) \rightarrow H^*(K; \mathbf{Z}/p)$ is epic, $\bar{h}^*: H^*(U; \mathbf{Z}/p) \rightarrow H^*(Z; \mathbf{Z}/p)$ is also epic. Y is a subcomplex of Z and $\bar{i}$ is the inclusion. Hence we have, from (1.2),

$$H^*(Y; \mathbf{Z}/p) = \{1, v, \beta_r v, \mathcal{P}^1 v, v^2, \beta_r \mathcal{P}^1 v, \beta_r v^2\}, \quad \mathcal{P}^1 \beta_r v = 0,$$

where $v = \bar{i}^* \bar{h}^* \bar{k}^*(u)$.

Since the map $\rho: M_q^n \rightarrow M_p^n$ is of degree 1 on the bottom sphere S^{n-1} , we have $f^*(v^2) = f^*(\mathcal{P}^1 v)$. Putting $x_1 = f^*(v)$ and $x_2 = f^*(v^2)$, we obtain the lemma.

From now on, we shall compute the group $\mathcal{E}(X_q)$. We first consider $\mathcal{E}(M_q^{2p-1})$. By (1.4.1), $\mathcal{E}(M_q^{2p-1}) = \text{Aut}(\mathbf{Z}/q) = \{m \mid (m, p) = 1\}$. It is well known that $\text{Aut}(\mathbf{Z}/q)$, $q = p^r$, is the cyclic group of order $p^r - p^{r-1} = (q/p) \times (p-1)$, and hence we have

LEMMA 1.4. $\mathcal{E}(M_q^{2p-1}) = Z_{q/p} \times Z_{p-1}^{(*)}$,
where

$$(1.7) \quad Z_{q/p} = \{(1+ap)\text{id} \mid 0 \leq a < q/p\}$$

and $(1+ap)\text{id}$ is a generator of $Z_{q/p}$ iff $(a, p) = 1$. A self-homotopy equivalence lies in $Z_{q/p}$ if and only if it induces the identity homomorphism of the (co)-homology with $\mathbf{Z}/p$ -coefficient.

For any $h \in \mathcal{E}(X_q)$, the restriction $h|_{M_q^{2p-1}}$ is a homotopy equivalence and the map

$$R: \mathcal{E}(X_q) \longrightarrow \mathcal{E}(M_q^{2p-1}), \quad R(h) = h|_{M_q^{2p-1}},$$

is a homomorphism.

PROPOSITION 1.5. The image of R is the subgroup $Z_{q/p}$ in (1.7).

PROOF. Let $h \in \mathcal{E}(X_q)$ and consider $h^*: H^*(X_q; \mathbf{Z}/p) \rightarrow H^*(X_q; \mathbf{Z}/p)$. Put $h^*(x_1) = mx_1$. By (1.6) we have

$$m^2 x_1^2 = h^*(x_1^2) = \mathcal{P}^1 h^*(x_1) = m \mathcal{P}^1 x_1 = m x_1^2.$$

Then $m^2 \equiv m \pmod{p}$ and $m \equiv 1 \pmod{p}$. This means that $h^* = \text{id}$, by Lemma 1.3. Hence $R(h)^* = \text{id}: H^*(M_q^{2p-1}; \mathbf{Z}/p) \rightarrow H^*(M_q^{2p-1}; \mathbf{Z}/p)$, which implies $R(h) \in Z_{q/p}$ by Lemma 1.4.

Conversely, let $m = 1 + ap$ and $\bar{m}: K(\mathbf{Z}/q, 2p-2) \rightarrow K(\mathbf{Z}/q, 2p-2)$ realize the multiplication by m . Then $\bar{m}k = k(\bar{m}\text{id})$ and we have a commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{j} & M_q^{2p-1} & \xrightarrow{k} & K(\mathbf{Z}/q, 2p-2) \\ \downarrow \bar{m} & & \downarrow \text{id} & & \downarrow \bar{m} \\ F & \xrightarrow{j} & M_q^{2p-1} & \xrightarrow{k} & K(\mathbf{Z}/q, 2p-2) \end{array}$$

Since $m \equiv 1 \pmod{p}$, $\bar{m}^* = \text{id}: H^i(F; \mathbf{Z}/p) \rightarrow H^i(F; \mathbf{Z}/p)$, $i < 6p-8$, by (1.3). Using the cohomology with $\mathbf{Z}/q$ -coefficient, we have also a commutative diagram

$$\begin{array}{ccc} M_p^{4p-4} \vee M_q^{4p-4} & \xrightarrow{hi} & F \\ \downarrow \text{id} \vee m^2 \text{id} & & \downarrow \bar{m} \\ M_p^{4p-4} \vee M_q^{4p-4} & \xrightarrow{hi} & F \end{array}$$

(*) Z_n denotes the cyclic group of order n with operation written multiplicatively, while $\mathbf{Z}/n$ is the additive group isomorphic to Z_n .

Combining the above diagrams with (1.5), we obtain another commutative one

$$\begin{array}{ccccc} M_q^{4p-4} & \xrightarrow{\phi} & M_q^{2p-1} & \xrightarrow{i} & X_q \\ \downarrow m^2 \text{id} & & \downarrow \text{id} & & \downarrow h_m \\ M_q^{4p-4} & \xrightarrow{\phi} & M_q^{2p-1} & \xrightarrow{i} & X_q. \end{array}$$

The left square in the diagram induces an element $h_m \in [X_q, X_q]$ such that

$$(1.8) \quad R(h_m) = \text{id} \quad \text{and} \quad \pi h_m = (m^2 \text{id})\pi,$$

where we use the notation $[X, Y]$ for the set of based homotopy classes of maps from $(X, *)$ to $(Y, *)$ and denote the cofiber space for X_q by

$$(1.9) \quad M_q^{2p-1} \xrightarrow{i} X_q \xrightarrow{\pi} M_q^{4p-3}.$$

Since $m^2 \text{id}$ and id in the diagram are homotopy equivalences, we have $h_m \in \mathcal{C}(X_q)$ by the five lemma. Thus $\text{Im } R = \mathbb{Z}_{q/p}$ as desired.

LEMMA 1.6. (i) *The following sequence is exact.*

$$0 \longrightarrow [M_q^{4p-3}, X_q] \xrightarrow{\pi^*} [X_q, X_q] \xrightarrow{i^*} [M_q^{2p-1}, X_q].$$

(ii) $i_*: [M_q^{4p-3}, M_q^{2p-1}] \rightarrow [M_q^{4p-3}, X_q]$ is an isomorphism and $[M_q^{4p-3}, M_q^{2p-1}] = \mathbb{Z}/p$.

PROOF. From (1.4.2), $[M_q^{2p}, X_q] = 0$. Therefore (i) is the usual Puppe exact sequence. Since the pair (X_q, M_q^{2q-1}) is $(4p-5)$ -connected, we have an exact sequence

$$\begin{aligned} [M_q^{4p-2}, M_q^{4p-3}] &\longrightarrow [M_q^{4p-3}, M_q^{2p-1}] \xrightarrow{i_*} [M_q^{4p-3}, X_q] \\ &\xrightarrow{\pi_*} [M_q^{4p-3}, M_q^{4p-3}] \xrightarrow{\partial} [M_q^{4p-4}, M_q^{2p-1}]. \end{aligned}$$

By (1.4.2), i_* is monic. For $n=4p-4, 4p-3$, the homomorphisms

$$h_*: [M_q^n, K] \longrightarrow [M_q^n, F] \quad \text{and} \quad j_*: [M_q^n, F] \longrightarrow [M_q^n, M_q^{2p-1}]$$

are isomorphic by Lemma 1.1 and (1.1). $[M_q^n, K]$ is easily computed from Lemma 1.1, and we have

$$(1.10) \quad [M_q^{4p-3}, M_q^{2p-1}] = \mathbb{Z}/p, \quad [M_q^{4p-4}, M_q^{2p-1}] = \mathbb{Z}/p \oplus \mathbb{Z}/q \oplus \mathbb{Z}/p.$$

In particular, ϕ is of order q . Then ∂ is monic, and so i_* is epic. Thus (ii) is proved.

Let $l: X_q \rightarrow X_q \vee M_q^{4p-3}$ be the map smashing $(1/2) \times M_q^{4p-4}$ in the cone $CM_q^{4p-4} (\subset X_q)$ to the base point and define

$$\lambda: [M_q^{4p-3}, X_q] \cong \mathbf{Z}/p \longrightarrow \mathcal{E}(X_q)$$

by $\lambda(\alpha) = (\text{id} \vee \alpha)l$. By [4, Lemma 1.8 (i)] λ is well defined, that is, $\text{Im } \lambda \subset \mathcal{E}(X_q)$. By [4, Lemma 1.4] and Lemma 1.6 (ii), it is a homomorphism:

$$\lambda(\alpha + \beta) = \lambda(\alpha)\lambda(\beta), \quad \alpha, \beta \in [M_q^{4p-3}, X_q].$$

Then we have the following results, which will be proved in the next section.

PROPOSITION 1.7. *There is a short exact sequence*

$$0 \longrightarrow \mathbf{Z}/p \xrightarrow{\lambda} \mathcal{E}(X_q) \xrightarrow{R} \mathbf{Z}_{q/p} \longrightarrow 1.$$

THEOREM 1.8. *The group extension of the above is non-trivial, that is, $\mathcal{E}(X_q)$ is a cyclic group of order q , and it is generated by the element h_m in (1.8) with $m = 1 + ap$, $(a, p) = 1$.*

§2. Realization of $\mathbf{Z}_p$

We denote by $\{X, Y\}$ the group of stable homotopy classes of maps from X to Y , i.e., $\{X, Y\} = \text{dir lim}_n [\Sigma^n X, \Sigma^n Y]$, where Σ^n is the n -times suspension, and $\Sigma^\infty: [X, Y] \rightarrow \{X, Y\}$ the canonical map.

The groups $\{M_q^{n+k}, M_q^n\}$ are computed in [5]. We recall the following results.

$$(2.1.1) \quad \{M_q^{n-1}, M_q^n\} = \mathbf{Z}/q\{\delta\}, \quad \{M_q^n, M_q^n\} = \mathbf{Z}/q\{\text{id}\},$$

where δ is represented by the composite $M_q^{n-1} \rightarrow M_q^{n-1}/S^{n-2} = S^{n-1} \subset M_q^n$.

$$(2.1.2) \quad \{M_q^{n+2p-3}, M_q^n\} = \mathbf{Z}/p\{\delta\alpha\} \oplus \mathbf{Z}/p\{\alpha\delta\}, \quad \{M_q^{n+2p-2}, M_q^n\} = \mathbf{Z}/p\{\alpha\},$$

where the composite $S^{n+2p-3} \subset M_q^{n+2p-2} \xrightarrow{\alpha} M_q^n \rightarrow M_q^n/S^{n-1} = S^n$, which we call α_1 , is the first non-trivial element in the p -component of the stable homotopy of spheres.

LEMMA 2.1. $q \text{ id} = \pm i_* \pi^* \alpha$ in $\{X_q, X_q\}$.

PROOF. Since the only non-trivial $\mathcal{P}^1$ operation in $H^*(X_q; \mathbf{Z}/p)$ is $\mathcal{P}^1 x_1 = x_2$ and α is detected by $\mathcal{P}^1$, we have

$$\Sigma^\infty \phi = \delta \alpha.$$

H. Toda [9] had introduced the notion of “ $\mathbf{Z}/p$ -space” for p an odd prime and defined a derivation $\theta: \{X, Y\} \rightarrow \{\Sigma X, Y\}$ for $\mathbf{Z}/p$ -spaces X and Y , and we [6] extended his definitions to the case when p is any odd integer. In particular, it can be easily verified that Lemma 2.3 in [9], together with its proof, is valid for p an odd integer. Hence we have

$$q \text{ id} = i_* \pi^* \theta(\delta \alpha).$$

Here $\theta = D$ in [5] up to sign and we proved $D(\delta) = \text{id}$, $D(\alpha) = 0$ [5]. Thus $\theta(\delta \alpha) = \pm \alpha$.

PROPOSITION 2.2. $\{X_q, X_q\} = \mathbf{Z}/pq\{\text{id}\} \oplus \mathbf{Z}/(q/p)\{\pi^* \tilde{p}\}$, where $\tilde{p} \in \{M_q^{4p-3}, X_q\}$ satisfies $(q/p)\tilde{p} = 0$ and $\pi_* \tilde{p} = p \text{ id}$ in $\{M_q^{4p-3}, M_q^{4p-3}\}$.

PROOF. We apply $\{M_q^{4p-3}, -\}$ to (1.9) and obtain an exact sequence

$$0 \longrightarrow \{M_q^{4p-3}, M_q^{2p-1}\} \xrightarrow{i_*} \{M_q^{4p-3}, X_q\} \xrightarrow{\pi_*} \{M_q^{4p-3}, M_q^{4p-3}\} \xrightarrow{(\delta \alpha)_*} \{M_q^{4p-3}, M_q^{2p}\}.$$

Since $p\delta\alpha = 0$, there is an element $\tilde{p} \in \{M_q^{4p-3}, X_q\}$ with $\pi_* \tilde{p} = p \text{ id}$. The element $(q/p)\tilde{p}$ lies in the i_* -image of the Toda bracket $\langle \delta \alpha, p \text{ id}, (q/p) \text{ id} \rangle$. But we easily see that the bracket contains zero. Hence $(q/p)\tilde{p} = 0$ for some $\tilde{p}$. Then (2.1.2) and the exact sequence show

$$(2.2) \quad \{M_q^{4p-3}, X_q\} = \mathbf{Z}/(q/p)\{\tilde{p}\} \oplus \mathbf{Z}/p\{i_* \alpha\}.$$

Applying Lemma 2.1 and (2.2) to the stable version of the exact sequence in Lemma 1.6 (i), we obtain the result.

LEMMA 2.3. In $\{X_q, X_q\}$,

$$(a \text{ id} + b \pi^* \tilde{p})^n = a^n \text{ id} + \left\{ \sum_{i=1}^n \binom{n}{i} p^{i-1} a^{n-i} b^i \right\} \pi^* \tilde{p}, \quad a, b \in \mathbf{Z}.$$

PROOF. Induction on n .

COROLLARY 2.4. $(a \text{ id} + b \pi^* \tilde{p})^{q/p} = a^{q/p} \text{ id}$, $a, b \in \mathbf{Z}$.

PROOF. Let $v(m)$ be the maximal v such that $p^v | m$. Since $v\left(\frac{q/p}{i}\right) = v\left(\frac{p^{r-1}}{i}\right) = r-1-v(i)$, we have $p^{r-1} \left| \left(\frac{q/p}{i}\right) p^{i-1} \right.$ for $1 \leq i \leq p^{r-1}$. Therefore

$$\left(\frac{q}{p}\right)_i p^{i-1} \pi^* \tilde{p} = 0.$$

Now, we are ready to prove Proposition 1.7 and Theorem 1.8.

PROOF OF PROPOSITION 1.7. We consider the commutative diagram

$$(2.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & [M_q^{4p-3}, X_q] & \xrightarrow{\pi^*} & [X_q, X_q] & \xrightarrow{i^*} & [M_q^{2p-1}, X_q] \xrightarrow{\phi^*} \\ & & \downarrow \Sigma^\infty & & \downarrow \Sigma^\infty & & \cong \downarrow \Sigma^\infty \\ 0 & \longrightarrow & \{M_q^{4p-3}, X_q\} & \xrightarrow{\pi^*} & \{X_q, X_q\} & \xrightarrow{i^*} & \{M_q^{2p-1}, X_q\} \longrightarrow 0. \end{array}$$

By the same construction as in [8, (4.7)], the element α in (2.1.2) lies in $\Sigma^\infty[M_q^{2p+2}, M_q^4]$, and hence it is an image of a generator of $[M_q^{4p-3}, M_q^{2p-1}]$, which we also denote by α . Then $[M_q^{4p-3}, X_q] = \mathbb{Z}/p$ with generator $i_*\alpha$ by Lemma 1.6 (ii). Therefore the left Σ^∞ is monic.

It is well known that two co- H -structures on a double suspension $\Sigma^2 X = S^1 \wedge S^1 \wedge X$ defined from that of each S^1 are homotopic to each other. Then, using essentially the same homotopy, we see that

$$(2.4) \quad \Sigma l \text{ is homotopic to } (\text{id} \vee \Sigma \pi) d,$$

where d is the co- H -structure on ΣX_q . Then

$$(2.5) \quad \Sigma^\infty \lambda(\xi) = \text{id} + \pi^* \Sigma^\infty \xi \quad \text{for } \xi \in [M_q^{4p-3}, X_q].$$

In the right hand side of (2.5), π^* and Σ^∞ are monic, and hence λ is monic. The exactness at $\mathcal{E}(X_q)$ is Proposition 1.11 in [4].

PROOF OF THEOREM 1.8. Let $m = 1 + ap$, $(a, p) = 1$, and take $h_m \in \mathcal{E}(X_q)$ as in (1.8). We must determine the coefficient x in the relation $(h_m)^{q/p} = \lambda(xi_*\alpha)$.

By Proposition 2.2 and (1.8), an easy calculation yields

$$\Sigma^\infty h_m \equiv m \text{id} + am\pi^* \tilde{p} \pmod{q \text{id}}.$$

Then $\Sigma^\infty (h_m)^{q/p} = (1 + aq)\text{id}$ by Corollary 2.4. On the other hand, $\Sigma^\infty (h_m)^{q/p} = \Sigma^\infty \lambda(xi_*\alpha) = \text{id} + xi_*\pi^* \alpha = (1 \pm xq)\text{id}$ by (2.5) and Lemma 2.1. Therefore $x \equiv \pm a \pmod{p}$ and $(h_m)^{q/p} = \lambda(\pm ai_*\alpha) \neq \text{id}$. This means that h_m is of order q and $\mathcal{E}(X_q) = \mathbb{Z}_q$ with generator h_m .

§3. Realization of Z_n and Z_{2n} for odd n

Let n be an odd integer. n has a unique decomposition

$$n = q_1 q_2 \cdots q_s, \quad q_i = p_i^{r_i},$$

where p 's are distinct odd primes and $r_i \geq 1$.

DEFINITION 3.1. For the above n , we put

$$X_n = X_{q_1} \vee X_{q_2} \vee \cdots \vee X_{q_s},$$

where X_{q_i} is the complex defined in 1.2.

LEMMA 3.2. Let $k_i: X_{q_i} \rightarrow X_n$ be the inclusion to the i -th factor, and $\pi_i: X_n \rightarrow X_{q_i}$ the projection. Then $(k_i)_*: [X_{q_i}, X_{q_i}] \rightarrow [X_{q_i}, X_n]$ is bijective and $(\pi_i)_*$ is its inverse.

PROOF. Since $\pi_i k_i = \text{id}$, $(k_i)_*$ is injective. Let $\mathcal{C}$ be the class of finite abelian groups whose orders are relatively prime to p_i . Then $(k_i)_*: \pi_*(X_{q_i}) \rightarrow \pi_*(X_n)$ is a $\mathcal{C}$ -isomorphism, since $(k_i)_*: H_*(X_{q_i}; \mathbf{Z}/p_i) \rightarrow H_*(X_n; \mathbf{Z}/p_i)$ is isomorphic. Then, making use of the mod $\mathcal{C}$ five lemma and noting that $[M_{q_i}^m, X_{q_i}]$ is a $\mathbf{Z}/q_i$ -module, we see that

$$(k_i)_*: [M_{q_i}^m, X_{q_i}] \longrightarrow [M_{q_i}^m, X_n], \quad m \geq 4,$$

is an isomorphism.

Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} [M_{q_i}^{4p_i-3}, X_{q_i}] & \xrightarrow{\pi^*} & [X_{q_i}, X_{q_i}] & \xrightarrow{i^*} & [M_{q_i}^{2p_i-1}, X_{q_i}] & \xrightarrow{\phi^*} & [M_{q_i}^{4p_i-4}, X_{q_i}] \\ \cong \downarrow (k_i)_* & & \downarrow (k_i)_* & & \cong \downarrow (k_i)_* & & \cong \downarrow (k_i)_* \\ [M_{q_i}^{4p_i-3}, X_n] & \xrightarrow{\pi^*} & [X_{q_i}, X_n] & \xrightarrow{i^*} & [M_{q_i}^{2p_i-1}, X_n] & \xrightarrow{\phi^*} & [M_{q_i}^{4p_i-4}, X_n] \end{array}$$

We can apply the four lemma to the diagram even if $[X_{q_i}, X_{q_i}]$ and $[X_{q_i}, X_n]$ do not form groups and ϕ^* 's are not homomorphisms. Therefore $(k_i)_*: [X_{q_i}, X_{q_i}] \rightarrow [X_{q_i}, X_n]$ is surjective.

THEOREM 3.3. For any odd integer $n > 1$, $\mathcal{E}(X_n)$ is the cyclic group of order n .

PROOF. By Lemma 3.2, there are bijections

$$(3.1) \quad [X_n, X_n] \cong \prod_{i=1}^s [X_{q_i}, X_n] \cong \prod_{i=1}^s [X_{q_i}, X_{q_i}],$$

which preserve the composition of maps. Hence $\mathcal{E}(X_n)$ is isomorphic to the direct product $\prod \mathcal{E}(X_{q_i}) = \prod \mathbf{Z}_{q_i}$.

We next consider the case $n \equiv 2 \pmod{4}$. Define

$$X_2 = S^3 \cup_2 e^4.$$

Then $[X_2, X_2] = \{X_2, X_2\} = \mathbf{Z}/4\{\text{id}\}$ [1, §4], and we have

PROPOSITION 3.4. $\mathcal{E}(X_2) = \mathbf{Z}_2$.

DEFINITION 3.5. Let $n = 2n'$ with n' odd, and define X_n to be the wedge $X_2 \vee X_{n'}$.

Since X_2 is 2-local, the same discussion as in the proof of Lemma 3.2 yields the bijection $[X_n, X_n] \cong [X_2, X_2] \times [X_{n'}, X_{n'}]$. Therefore $\mathcal{E}(X_n) = \mathcal{E}(X_2) \times \mathcal{E}(X_{n'})$, which leads us the following

THEOREM 3.6. For $n \equiv 2 \pmod{4}$, $\mathcal{E}(X_n)$ is the cyclic group of order n .

§4. Some results on Z_n , $n \equiv 0 \pmod{4}$

We state some results on the realization of Z_n for $n \equiv 4 \pmod{8}$ and $n \equiv 16 \pmod{32}$. The construction is based on bijections analogous to (3.1).

THEOREM 4.1. Let $n = 4n'$ with n' odd. Put $X_4 = M_5^{13}$ and $X_n = X_4 \vee X_{n'}$, where $X_{n'}$ is the complex defined in 3.1. Then $\mathcal{E}(X_n) = \mathbf{Z}_n$.

PROOF. Let $q = 5^r$, $r \geq 1$. We have easily $[X_4, M_q^9] = 0$ and $[M_q^{17}, X_4] = 0$. Then $[X_4, X_q] = 0$ and $[X_q, X_4] = 0$. By the cellular approximation theorem, the maps

$$[X_4, X_4 \vee X_q] \longrightarrow [X_4, X_4 \times X_q],$$

$$[X_q, X_4 \vee X_q] \longrightarrow [X_q, X_4 \times X_q]$$

are bijective. Therefore $[X_4 \vee X_q, X_4 \vee X_q] \cong [X_4, X_4] \times [X_q, X_q]$. Then we have the result.

We give another example of a complex realizing Z_n , $n \equiv 4 \pmod{8}$.

THEOREM 4.2. $n = 4 \cdot 5^r n'$ with n' odd and $(n', 5) = 1$. Put $X_n = M_{5^{r+1}}^4 \vee X_{n'}$, where $X_{n'}$ is in Definition 3.1. Then $\mathcal{E}(X_n) = \mathbf{Z}_n$.

PROOF. $\mathcal{E}(X_n) \cong \mathcal{E}(M_{5^{r+1}}^4) \times \mathcal{E}(X_{n'}) \cong \text{Aut}(\mathbf{Z}/5^{r+1}) \times \mathbf{Z}_{n'} \cong \mathbf{Z}_{4 \cdot 5^r} \times \mathbf{Z}_{n'}$.

PROPOSITION 4.3. If $n \equiv 16 \pmod{32}$, the following complexes X_n realize the

cyclic group Z_n :

$$X_n = X_{16} \vee X_{n/16} \quad \text{with} \quad X_{16} = M_{17}^{49},$$

$$X_n = M_{17^{r+1}}^4 \vee X_{n/16 \cdot 17^r},$$

where r is the integer such that $17^r \mid n$ and $17^{r+1} \nmid n$.

PROOF. Similar.

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