QUOTIENT AND INDUCED TOPOLOGIES OF THE S-TOPLOGY ON A BANACH SPACE

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Abstract

Following results are proved: The quotient of the S-topology is admissible (Theorem 1). The induced topology of the S-topology is not necessarily admissible (Theorem 2).

1. Introduction

In this paper we study the quotient and the induced topologies of the S-topology on a Banach space.

Let E be a Banach space on the real number field. Then a topology ρ on E is said to be admissible (see Prokhorov and Sazonov [8]) if it hold

- (1) ρ is a vector topology, and
- (2) for every random linear functional ϕ on E , the ρ -continuity of ϕ is equivalent to the $\sigma(E', E)$ -Radon extensibility of the corresponding cylinder set measure μ_ϕ on E' .

The weakest one of all admissible topologies is called the S-topology or the Sazonov topology. If there is an admissible topology then the S-topology also exists (see Okazaki [7], Theorem 4.1).

For other notations and definitions used here see Okazaki [7]. The existence of the S-topology was investigated by Mouchtari [6], Okazaki [7].

The main results are the followings.

- (1) The quotient of the S-topology is admissible, and
- (2) The induced topology of the S-topology is not necessarily admissible.

2. Quotient of the S-topology

Let E be a Banach space and M be a closed subspace of E . We assume

there is the S-topology τ on E .

THEOREM 1. The quotient topology τ/M of the S-topology τ is admissible on E/M .

PROOF. Let $\phi: E/M \rightarrow L^0(\Omega, \mathfrak{A}, \mathbf{P})$ be a random linear functional which is τ/M -continuous. Then the random linear functional $\phi \circ \pi$ on E is τ -continuous, here $\pi: E \rightarrow E/M$ is the quotient mapping. Thus for the unit ball B of E , $\phi \circ \pi(B)$ is bounded for the lattice structure of $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ (Badrikian and Chevet [2], Lemme 1). Consequently for the unit ball B/M of E/M , $\phi(B/M)$ is also bounded for the lattice structure of $L^0(\Omega, \mathfrak{A}, \mathbf{P})$. Hence the cylinder set measure μ_ϕ is $\sigma((E/M)', E/M)$ -Radon (see Badrikian and Chevet [2], Lemme 1).

Conversely assume μ_ϕ be $\sigma((E/M)', E/M)$ -Radon. Let $\pi': (E/M)' \rightarrow E'$ be the transpose of π . Then the image $\pi'(\mu_\phi)$ is $\sigma(E', E)$ -Radon. Remarking that the cylinder set measure $\pi'(\mu_\phi)$ corresponds to the random linear functional $\phi \circ \pi$ we can see $\phi \circ \pi$ is τ -continuous. By the definition of the quotient topology, ϕ is τ/M -continuous.

This proves the theorem.

COROLLARY 1. If there is the S-topology on E , then for every closed subspace M there is the S-topology on E/M .

We say a Banach space E is of dual $\mathfrak{S}_1$ if there is a decomposable positive measure space (Ω, μ) (see Hewitt and Stromberg [4], Definition (19.25)) such that the strong dual E' is embedded isomorphically into $L^1(\Omega, \mu)$.

LEMMA 1. For every positive measure space (Ω, μ) the S-topology exists on $L^\infty(\Omega, \mu)$.

PROOF. By Grothendieck [3], Pt. 1, Proposition 41, $L^\infty(\Omega, \mu)$ has the metric approximation property. Moreover the strong dual $(L^\infty(\Omega, \mu))'$ is isometric to $L^1(\nu)$ for a suitable Radon measure ν on a locally compact space (see Kakutani [5]), hence L^0 -embeddable. Thus by Okazaki [7], Theorem 4-4, there is the S-topology on $L^\infty(\Omega, \mu)$.

This proves the lemma.

PROPOSITION 1. Let E be a reflexive Banach space of dual $\mathfrak{S}_1$. Then there is the S-topology on E .

PROOF. Let (Ω, μ) be a decomposable measure space such that E' is em-

bedded isomorphically into $L^1(\Omega, \mu)$. Since E is reflexive and $(L^1(\Omega, \mu))' = L^\infty(\Omega, \mu)$ (see Hewitt and Stromberg [4], Theorem (20.20)), E is isomorphic to a quotient space of $L^\infty(\Omega, \mu)$. Thus by lemma 1 and corollary 1, the S-topology exists on E .

This proves the assertion.

3. Induced topology

We show the induced topology of the S-topology is not necessarily admissible. The following lemma is proved by the way same to lemma 1.

LEMMA 2. For every completely regular Hausdorff topological space K there is the S-topology on $C^b(K)$ the Banach space of all bounded continuous functions on K .

THEOREM 2. There is a Banach space E and a closed subspace M of E such that the S-topology exists on E and such that the induced topology of the S-topology on M is not admissible.

PROOF. Let p be $1 < p < 2$ and q be $1/p + 1/q = 1$. For the unit ball K of l_q , l_p is a closed subspace of $C^b(K)$. By lemma 2 there is the S-topology on $C^b(K)$, but there is not any admissible topology on l_p (see de Acosta [1], Example 4.2 and its corollary).

This proves the theorem.

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