HARMONIC ANALYSIS IN A BANACH SPACE

By

Yoshiaki OKAZAKI

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1. Introduction

In this paper we study the Bochner's theorem in an infinite-dimensional locally convex space and the Fourier transform of the Radon measures on the dual of a locally convex space. We investigate the following problems;

- (a) tightness of a family of Radon probability measures,

- (b) Radon extensibility of a cylinder set measure,
- (c) extension of the notion of the S-topology due to Sazonov [16] to general locally convex spaces, and
- (d) extension of the Bochner's theorem to infinite-dimensional locally convex spaces.

In section 3–section 6, we study above problems in a Banach space and in section 7, we extend them to locally convex spaces. Here we treat the case the Radon measures are given on the dual E' of a locally convex space E and “Radon” is a “ $\sigma(E', E)$ -Radon” measure. Thus problem (c) is formulated as follows. Is there a topology τ (the S-topology) on E such that for a cylinder set measure μ on E' , the τ -continuity of the Fourier transform $\mu^\wedge(x)$ of μ is equivalent to the $\sigma(E', E)$ -Radon extensibility of μ ? The answer of problem (c) is regarded as a topological solution of problem (d). We shall also seek after an analytical solution of problem (c).

In section 3, we are concerned with problems (a) and (b). The case E' is L^0 -embeddable is important relating problems (c) and (d). In section 4, we consider the problem (c). The material of section 4 is a refinement of the work of Mouchtari [12], especially Théorème 1 and Théorème 4 of [12]. Mouchtari proved if there is the S-topology then E' is L^0 -embeddable. So we shall consider the case E' is L^0 -embeddable in problem (c).

In section 4.1, applying the results in section 3 we show the Mouchtari's theorem, that is, if E is a Banach space with the metric approximation property and if E' is L^0 -embeddable, then the S-topology τ exists on E (theorem 4.4). For a separable Banach space this was proved by Mouchtari [12]. Our proof is essentially the same to Mouchtari's. The study of the S-topology was initiated by Sazonov [16], Gross [4] see also de Acosta [1], Dudley [3], Kuelbs [7], Kuelbs and Mandrekar [8], and Mouchtari [12].

In section 4.2, we prove that if E has the S-topology, E' separable (or E reflexive) and E' is of type p stable ($0 < p \leq 2$), then the S-topology is given by a family of L^p -seminorms (theorem 4.8). In particular, for $E = L^p$ ($2 \leq p < +\infty$) there is the S-topology on E which is locally convex.

In section 4.3, we characterize the S-topology on L^p ($2 \leq p < +\infty$) by the q -summing operators. This was implicitly given in Mouchtari [12], Théorème 4. In case of the Hilbert space the S-topology is given by the Hilbert-schmidt operators (Sazonov [16]). We shall prove the S-topology on L^p is given by the col-

lection of q -summing operators of L^p into $L^q([0, 1], dx)$, that is, the S-topology is given by the seminorms $|Rx|_{L^q[0,1]}$ where $R: L^p \rightarrow L^q[0, 1]$ be q -summing and q be arbitrary so that $1 < q < p'$, $1/p + 1/p' = 1$ (theorem 4.10 and 4.11).

In section 5, we give an analytical answer to problem (d). We state a necessary and sufficient condition such that a complex valued function ψ on E be the Fourier transform of the $\sigma(E', E)$ -Radon measure on E' (theorem 5.2).

In section 6, we prove a part of the Levy continuity theorem.

In section 7, we extend the preceding results to general locally convex spaces by the inductive (projective) limit procedure.

In the first half of section 7, we consider the inductive limit space. After stating solutions of problems (a) and (b) in this case, we prove the following result to problem (c). Let $E = \varinjlim E_n$ be the inductive limit of locally convex spaces $\{E_n\}$. If each E_n has the S-topology then E has the S-topology. The Bochner's theorem is also given in theorem 7.9.

In the latter half of section 7 we consider the projective limits. In section 7-2-1 and 7-2-2 we treat problems (a) and (b). In section 7-2-3 we prove the following theorem. Let $E = \varprojlim E_n$ be the projective limit of Banach spaces such that E'_n is L^0 -embeddable. Assume E has the metric approximation property and each E_n has the S-topology. Then there is the S-topology on E . In this case, in fact, the S-topology on E is the projective limit of the S-topologies of E_n .

2. Preliminaries

Let (X, Y) be a dual system of two vector spaces X and Y over the real numbers $\mathbf{R}$ with the bilinear form $\langle \cdot, \cdot \rangle$ and let $FD(X)$ be the set of all finite-dimensional subspaces of X . For every $F \in FD(X)$, the finite-dimensional space $Y/F^\perp$ has the natural Borel field $\mathfrak{B}(F)$, where

$$F^\perp = \{y \in Y; \langle x, y \rangle = 0 \text{ for every } x \in F\}.$$

We denote by $Z(Y, X)$ the algebra

$$\cup \{ \prod_{F \in FD(X)} \mathfrak{B}(F); F \in FD(X) \},$$

where $\prod_F$ is the natural projection of Y onto $Y/F^\perp$. The elements in $\prod_F \mathfrak{B}(F)$ are said to be cylinder sets based on F . A cylinder set measure μ on $Z(Y, X)$ is a non-negative finitely additive set function on $Z(Y, X)$ with $\mu(Y) = 1$ which is σ -additive on $\prod_F \mathfrak{B}(F)$ for every $F \in FD(X)$.

A random linear functional on X is a linear mapping of X into $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ the space of equivalence classes of all measurable functions on a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$.

Let

$$\phi: X \longrightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$$

be a random linear functional and (X, Y) be an arbitrary dual system. Then ϕ induces a cylinder set measure μ_ϕ on $Z(Y, X)$ such that

$$\begin{aligned} \mu_\phi\{y \in Y; \langle x_i, y \rangle \leq a_i, i=1, 2, \dots, k\} \\ = \mathbf{P}_\phi\{\omega \in \Omega_\phi; \phi(x_i)(\omega) \leq a_i, i=1, 2, \dots, k\}. \end{aligned}$$

Conversely if μ is a cylinder set measure on $Z(Y, X)$ then taking $\Omega_\phi = X^a$ (the algebraic dual), $\mathbf{P}_\phi =$ the σ -additive extension of μ , and $\phi(x) = \langle x, \cdot \rangle$ where $\langle \cdot, \cdot \rangle$ is the bilinear form of $X \times X^a$, a random linear functional ϕ satisfying above equalities exists (see Dudley [3]).

Let $\phi: X \rightarrow L^0(\Omega_\phi)$ be a random linear functional and μ_ϕ be the corresponding cylinder set measure on $Z(Y, X)$. We shall define the Fourier transform (the characteristic functional) $\hat{\mu}_\phi(x)$ by

$$\hat{\mu}_\phi(x) = \int_Y e^{i\langle x, y \rangle} d\mu_\phi(y), \quad x \in X.$$

If a topology is given on X , then we can consider the continuity of ϕ or of $\hat{\mu}_\phi$. It is known by Dudley [3], the continuity of ϕ implies the continuity of $\hat{\mu}_\phi$ and converse is also true if X is a topological vector space.

Let $\phi: X \rightarrow L^0(\Omega)$ be a random linear functional. We say the random linear functional ϕ is $\sigma(Y, X)$ -Radon if the cylinder set measure μ_ϕ can be extended to a $\sigma(Y, X)$ -Radon measure on Y . If X is a Banach space and $Y = X'$ (the topological dual), then μ_ϕ is $\sigma(X', X)$ -Radon if and only if μ_ϕ is cylindrically concentrated on the balls of X' .

We say that a topological vector space E has the metric approximation property if there is a net $\{T_\lambda\}_{\lambda \in A}$ of operators of finite rank of E such that $\{T_\lambda\}_{\lambda \in A}$ is equicontinuous and such that $T_\lambda x \rightarrow x$, uniformly on each compact subset. If E is a Banach space the equicontinuity is equivalent to $\sup_\lambda |T_\lambda| < +\infty$, where $|T_\lambda|$ is the operator norm of T .

Let E be a locally convex space. Then a topology τ is called the S-topology if (E, τ) is a topological vector space and τ is the weakest one satisfying for every

random linear functional $\phi: E \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$, the continuity of ϕ is equivalent to the $\sigma(E', E)$ -Radon extensibility of μ_ϕ . Here the dual E' is taken by the initial topology. If E is a barrelled space and $\phi: E \rightarrow L^0$ is $\sigma(E', E)$ -Radon then ϕ is continuous in the initial topology. Thus in this case the S-topology is (if it exists) weaker than the initial topology.

A linear metric space E is said to be L^0 -embeddable if there is an isomorphism of E into $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ for a suitable probability space $(\Omega, \mathfrak{A}, \mathbf{P})$. The L^0 -embeddability is equivalent to the accessibility of Kuelbs [7], see Okazaki [14].

Let $\{\mu_\alpha\}_{\alpha \in A}$ be a family of Radon probability measures on a topological space T . Then $\{\mu_\alpha\}_{\alpha \in A}$ is called tight if for each $\varepsilon > 0$ there is a compact set K such that

$$\inf_{\alpha \in A} \mu_\alpha(K) > 1 - \varepsilon.$$

3. Tightness of a family of Radon probability measures on E'

We prove a tightness condition of the family of $\sigma(E', E)$ -Radon probability measures on the dual E' of the Banach space E . We also prove a $\sigma(E', E)$ -Radon extensibility criterion for a cylinder set measure on $Z(E', E)$.

3.1. Tightness of measures

THEOREM 3-1. Let E be a Banach space and $\{\mu_\alpha\}_{\alpha \in A}$ be a family of $\sigma(E', E)$ -Radon probability measures on E' . Then the following conditions are equivalent.

- (1) $\{\mu_\alpha\}_{\alpha \in A}$ is tight on $E'_{\sigma(E', E)}$.
- (2) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'} \frac{|x'|_{E'}}{R + |x'|_{E'}} d\mu_\alpha(x') = 0.$$

- (3) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'} \left(1 - \exp\left[-\frac{|x'|}{R}\right]\right) d\mu_\alpha(x') = 0$$

- (4) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'} \left(1 - \exp\left[-\frac{|x'|^p}{R^p}\right]\right) d\mu_\alpha(x') = 0,$$

for every p , $0 < p < +\infty$.

(5) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\alpha} \int_{E'} 1 \wedge \left[\frac{|x'|}{R} \right] d\mu_{\alpha}(x') = 0,$$

where $a \wedge b = \min \{a, b\}$ and $|\cdot|$ is the dual norm in E' .

PROOF. We remark a $\sigma(E', E)$ -compact set K in E' is contained in a ball $\{x' \in E'; |x'| \leq R_0\}$ for $0 < R_0 < +\infty$, and the ball $\{|x'| \leq R_0\}$ is also $\sigma(E', E)$ -compact. Thus to see the tightness of $\{\mu_{\alpha}\}_{\alpha \in A}$, we show for every $\varepsilon > 0$, there is an R_0 , $0 < R_0 < +\infty$, such that

$$\sup_{\alpha \in A} \mu_{\alpha}\{x' \in E'; |x'| > R_0\} \leq \varepsilon.$$

First, let $\{\mu_{\alpha}\}_{\alpha \in A}$ be tight. Then for every $\varepsilon > 0$ there is an R_0 , $0 < R_0 < +\infty$ such that

$$\sup_{\alpha \in A} \mu_{\alpha}\{|x'| > R_0\} \leq \varepsilon.$$

We have

$$\begin{aligned} & \lim_{R \rightarrow \infty} \sup_{\alpha \in A} \int_{E'} \frac{|x'|}{R + |x'|} d\mu_{\alpha}(x') \\ & \leq \lim_{R \rightarrow \infty} \sup_{\alpha \in A} \int_{\{|x'| \leq R_0\}} \frac{R_0}{R + R_0} d\mu_{\alpha}(x') + \int_{\{|x'| > R_0\}} 1 d\mu_{\alpha}(x') \\ & = 0 + \varepsilon, \end{aligned}$$

which shows the validity of (2). Other cases are analogous.

To see the converse we use the following inequalities;

$$\begin{aligned} \mu(|x'| > R) & \leq 2 \int_{E'} \frac{|x'|}{R + |x'|} d\mu(x'), \\ \mu(|x'| > R) & \leq \frac{1}{1 - e^{-1}} \int_{E'} \left(1 - \exp \left[-\frac{|x'|^p}{R^p} \right] \right) d\mu(x'), \end{aligned}$$

and

$$\mu(|x'| > R) \leq \int_{E'} 1 \wedge \left[\frac{|x'|}{R} \right] d\mu(x').$$

By these inequalities, we have

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \mu_{\alpha}(|x'| > R) = 0,$$

under each one of the assumptions of (2)–(5).

This proves the theorem.

In the next theorem, we consider the case that E' (with the dual norm $|\cdot|$) can be embedded isomorphically into $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ for a suitable probability space $(\Omega, \mathfrak{A}, \mathbf{P})$. This theorem is important in showing the existence of the S-topology in the next section.

THEOREM 3-2. Let E be a Banach space and $\{\mu_\alpha\}_{\alpha \in A}$ be a family of $\sigma(E', E)$ -Radon probability measures on E' . Assume there is an isomorphism Ψ of E' into $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ for a suitable probability space $(\Omega, \mathfrak{A}, \mathbf{P})$. Then $\{\mu_\alpha\}_{\alpha \in A}$ is tight on $E'_{\sigma(E', E)}$ if and only if the following equality holds;

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'} \int_{\Omega} \frac{|\Psi(x')(\omega)|}{R + |\Psi(x')(\omega)|} d\mathbf{P}(\omega) d\mu_\alpha(x') = 0.$$

PROOF. Let $\{\mu_\alpha\}_{\alpha \in A}$ be tight. Then for every $\varepsilon > 0$ there is a positive R_0 such that

$$\sup_{\alpha \in A} \mu_\alpha(|x'| > R_0) \leq \frac{\varepsilon}{2}.$$

Since Ψ is an isomorphism, there is a $\delta > 0$ such that

$$|x'| \leq \delta \text{ implies } |\Psi(x')|_{L^0} \leq \frac{\varepsilon}{2}, \text{ where,}$$

$$|\Psi(x')|_{L^0} = \int_{\Omega} \frac{|\Psi(x')(\omega)|}{1 + |\Psi(x')(\omega)|} d\mathbf{P}(\omega).$$

Then for every $R > R_0/\delta$, we have

$$\begin{aligned} & \sup_{\alpha} \int_{E'} \int_{\Omega} \frac{|\Psi(x')(\omega)|}{R + |\Psi(x')(\omega)|} d\mathbf{P}(\omega) d\mu_\alpha(x') \\ &= \sup_{\alpha} \int_{E'} \int_{\Omega} \frac{\left| \Psi\left(\frac{x'}{R}\right)(\omega) \right|}{1 + \left| \Psi\left(\frac{x'}{R}\right)(\omega) \right|} d\mathbf{P}(\omega) d\mu_\alpha(x') \\ &\leq \sup_{\alpha} \int_{\{|x'| > R_0\}} 1 d\mu_\alpha(x') + \int_{\{|x'| \leq R_0\}} \frac{\varepsilon}{2} d\mu_\alpha(x') \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

since for $|x'| \leq R_0$, it holds

$$\left| \frac{x'}{R} \right| \leq \frac{R_0}{R} < \delta \quad \text{and} \quad \int \frac{\left| \Psi\left(\frac{x'}{R}\right)(\omega) \right|}{1 + \left| \Psi\left(\frac{x'}{R}\right)(\omega) \right|} d\mathbf{P}(\omega) = \left| \Psi\left(\frac{x'}{R}\right) \right|_{L^0} \leq \frac{\varepsilon}{2}.$$

Thus we have

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'} \int_{\Omega} \frac{|\Psi(x')(\omega)|}{R + |\Psi(x')(\omega)|} d\mathbf{P}(\omega) d\mu_{\alpha}(x') = 0.$$

Next we prove the converse.

For every $\varepsilon > 0$, there is a $\delta > 0$ such that $|\Psi(x')|_{L^0} < \delta$ implies $|x'| < \varepsilon/2$, that is, $|x'| \geq \varepsilon/2$ implies $|\Psi(x')|_{L^0} \geq \delta$. For every R , we have

$$\begin{aligned} \mu_{\alpha}(|x'| > R) &\leq \int_{E'} 1 \wedge \left[\frac{|x'|}{R} \right] d\mu_{\alpha}(x') \\ &\leq \int_{\{|x'| < \frac{\varepsilon}{2}\}} \frac{\varepsilon}{2} d\mu_{\alpha}(x') + \int_{\{|x'| \geq \frac{\varepsilon}{2}\}} 1 \wedge \left[\frac{|x'|}{R} \right] d\mu_{\alpha}(x') \\ &\leq \frac{\varepsilon}{2} + \int_{\{|x'| \geq R \frac{\varepsilon}{2}\}} \frac{1}{\delta} \left| \Psi\left(\frac{x'}{R}\right) \right|_{L^0} d\mu_{\alpha}(x'), \end{aligned}$$

since $|x'| \geq R\varepsilon/2$ implies $|\Psi(x'/R)|_{L^0} \geq \delta$, that is,

$$\frac{1}{\delta} \left| \Psi\left(\frac{x'}{R}\right) \right|_{L^0} \geq 1.$$

Thus we have

$$\begin{aligned} \mu_{\alpha}(|x'| > R) &\leq \frac{\varepsilon}{2} + \frac{1}{\delta} \int_{E'} \int_{\Omega} \frac{|\Psi(x'/R)(\omega)|}{1 + |\Psi(x'/R)(\omega)|} d\mathbf{P}(\omega) d\mu_{\alpha}(x') \\ &= \frac{\varepsilon}{2} + \frac{1}{\delta} \int_{E'} \int_{\Omega} \frac{|\Psi(x')(\omega)|}{R + |\Psi(x')(\omega)|} d\mathbf{P}(\omega) d\mu_{\alpha}(x'), \end{aligned}$$

for every $\alpha \in A$ and $R > 0$.

Hence by the assumption, we have

$$\lim_{R \rightarrow +\infty} \sup_{\alpha} \mu_{\alpha}(|x'| > R) \leq \varepsilon$$

for every $\varepsilon > 0$. This shows the tightness of $\{\mu_{\alpha}\}_{\alpha \in A}$.

This proves the assertion.

3.2. $\sigma(E', E)$ -Radon extensibility of a cylinder set measure

In the rest of this section we investigate the $\sigma(E', E)$ -Radon extensibility of the cylinder set measure on $Z(E', E)$, in case E has the metric approximation property.

THEOREM 3-3. Let E be a Banach space with the metric approximation property by a net $\{\pi_\lambda\}_{\lambda \in A}$ of operators of finite rank and μ be a cylinder set measure on $Z(E', E)$. Then μ is $\sigma(E', E)$ -Radon measure if and only if the following two conditions hold;

(1) The characteristic functional $\mu^\wedge(x)$ is continuous on E with the norm topology.

(2) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \int_{E'} \frac{|\pi'_\lambda x'|_{E'}}{R + |\pi'_\lambda x'|_{E'}} d\mu(x') = 0,$$

where π'_λ is the transpose of π_λ . We remark that the function $\frac{|\pi'_\lambda x'|}{R + |\pi'_\lambda x'|}$ is $Z(E', E)$ -measurable since π'_λ is also an operator of finite rank.

PROOF. Let μ be a $\sigma(E', E)$ -Radon measure. Then for every $\varepsilon > 0$, there is an R_0 such that $\mu(|x'| > R_0) < \varepsilon/4$. We take $\delta > 0$ so that $|1 - e^{it}| < \varepsilon/2$ for every $|t| < \delta$. Then for every x such that $|x| < \delta/R_0$, we have

$$\begin{aligned} |1 - \mu^\wedge(x)| &= \left| \int (1 - e^{i\langle x, x' \rangle}) d\mu(x') \right| \\ &\leq \int_{\{|x'| > R_0\}} 2 d\mu(x') + \int_{\{|x'| \leq R_0\}} |1 - e^{i\langle x, x' \rangle}| d\mu(x') \\ &\leq 2 \frac{\varepsilon}{4} + \int \frac{\varepsilon}{2} d\mu(x') = \varepsilon, \end{aligned}$$

which proves the continuity of $\mu^\wedge(x)$ on E . Since $\{\pi_\lambda\}_{\lambda \in A}$ is equicontinuous, if $|x'| \leq R_0$ then $|\pi'_\lambda x'| \leq |\pi'_\lambda| |x'| \leq \sup_{\lambda \in A} |\pi'_\lambda| \cdot R_0 < +\infty$. Hence we have

$$\begin{aligned} &\lim_{R \rightarrow \infty} \sup_{\lambda} \int_{E'} \frac{|\pi'_\lambda x'|}{R + |\pi'_\lambda x'|} d\mu(x') \\ &\leq \lim_{R \rightarrow \infty} \sup_{\lambda} \left[\int_{\{|x'| > R_0\}} 2 d\mu(x') + \int_{\{|x'| \leq R_0\}} \frac{C}{R + C} d\mu(x') \right] \\ &\leq \frac{\varepsilon}{2} + 0 = \frac{\varepsilon}{2}, \end{aligned}$$

which shows (2), where $C = \sup_{\lambda} |\pi'_{\lambda}| \cdot R_0$.

We prove the converse. Let $\mu_{\lambda} = \pi'_{\lambda}(\mu)$ be the image measure of μ by the finite rank operator π'_{λ} . Then the characteristic functional $\mu_{\lambda}^{\wedge}$ is given by $\mu_{\lambda}^{\wedge}(x) = \mu^{\wedge}(\pi_{\lambda}x)$. Remarking the equality

$$\int_{E'} \frac{|x'|}{R + |x'|} d\mu_{\lambda}(x') = \int \frac{|\pi'_{\lambda}x'|}{R + |\pi'_{\lambda}x'|} d\mu(x'),$$

we can see $\{\mu_{\lambda}\}_{\lambda \in A}$ is tight (theorem 3-1). We remark $\pi_{\lambda}x \rightarrow x$, uniformly on compact subsets, so by the continuity of $\mu^{\wedge}$, we have

$$\mu_{\lambda}^{\wedge}(x) = \mu^{\wedge}(\pi_{\lambda}x) \longrightarrow \mu^{\wedge}(x),$$

uniformly on compact subsets of E . Thus the tight family $\{\mu_{\lambda}\}_{\lambda \in A}$ converges to μ weakly and μ is also a Radon measure (Schwartz [17], Proposition (III; 5.2)).

This proves the theorem.

REMARK 3-4. The above condition (2) can be replaced by one of the following conditions;

$$(3) \quad \lim_{R \rightarrow +\infty} \sup_{\lambda} \int_E \left(1 - \exp \left[-\frac{|\pi'_{\lambda}x'|}{R} \right] \right) d\mu(x') = 0,$$

$$(4) \quad \lim_{R \rightarrow +\infty} \sup_{\lambda} \int_{E'} \left(1 - \exp \left[-\frac{|\pi'_{\lambda}x'|^p}{R^p} \right] \right) d\mu(x') = 0,$$

for every p , $0 < p < +\infty$, and

$$(5) \quad \lim_{R \rightarrow \infty} \sup_{\lambda} \int_{E'} 1 \wedge \left[\frac{|\pi'_{\lambda}x'|}{R} \right] d\mu(x') = 0.$$

In fact, by theorem 3-1, if one of these conditions holds, then the tightness of $\{\mu_{\lambda}\}_{\lambda \in A}$ follows.

Now we treat the case E' can be embedded into $L^0(\Omega, \mathfrak{A}, \mathbf{P})$.

THEOREM 3-5. Let E be a Banach space with the metric approximation property by a net $\{\pi_{\lambda}\}_{\lambda \in A}$ of finite rank operators and μ be a cylinder set measure on $Z(E', E)$. Assume there is an isomorphism Ψ of E' into $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ for a suitable probability space $(\Omega, \mathfrak{A}, \mathbf{P})$. Then μ is $\sigma(E', E)$ -Radon if and only if the following two conditions hold;

(1) The characteristic functional $\mu^{\wedge}(x)$ is continuous on E with the norm topology.

(2) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \int_{E'} \int_{\Omega} \frac{|\Psi(\pi'_\lambda x')(\omega)|}{R + |\Psi(\pi'_\lambda x')(\omega)|} d\mathbf{P}(\omega) d\mu(x') = 0.$$

Let μ_Ψ be the cylinder set measure on $Z(E, E')$ defined by the random linear functional $\Psi: E' \rightarrow L^0(\Omega, \mathfrak{A}, \mathbf{P})$. Then the condition (2) can be written

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \int_{E'} \int_E \frac{|\langle x, \pi'_\lambda x' \rangle|}{R + |\langle x, \pi'_\lambda x' \rangle|} d\mu_\Psi(x) d\mu(x') = 0.$$

PROOF. As shown in theorem 3-3, if μ is $\sigma(E', E)$ -Radon, then (1) holds. To see (2) we proceed by the way similar to the proof of theorem 3-2. For every $\varepsilon > 0$, take a constant R_0 such that $\mu(|x'| > R_0) \leq \varepsilon/2$. Since Ψ is an isomorphism, there is a $\delta > 0$ such that $|x'| \leq \delta$ implies $|\Psi(x')|_{L^0} \leq \varepsilon/2$. We put $C = \sup_{\lambda \in A} |\pi_\lambda| < +\infty$. Then we have

$$\mu(|\pi'_\lambda x'| > CR_0) \leq \mu(|x'| > R_0) \leq \frac{\varepsilon}{2}.$$

Now for every $R > CR_0/\delta$ and $\lambda \in A$ we have

$$\begin{aligned} & \int_{E'} \int_{\Omega} \frac{|\Psi(\pi'_\lambda x')(\omega)|}{R + |\Psi(\pi'_\lambda x')(\omega)|} d\mathbf{P}(\omega) d\mu(x') \\ &= \int_{E'} \left| \Psi\left(\frac{\pi'_\lambda x'}{R}\right)(\omega) \right|_{L^0(\Omega)} d\mu(x') \\ &\leq \int_{\{|\pi'_\lambda x'| > CR_0\}} 1 d\mu(x') + \int_{\{|\pi'_\lambda x'| \leq CR_0\}} \left| \Psi\left(\frac{\pi'_\lambda x'}{R}\right)(\omega) \right|_{L^0} d\mu(x') \\ &\leq \mu(|\pi'_\lambda x'| > CR_0) + \int_{\{|\pi'_\lambda x'| \leq CR_0\}} \frac{\varepsilon}{2} d\mu(x') \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

since $|\pi'_\lambda x'| \leq CR_0$ implies $\left| \frac{\pi'_\lambda x'}{R} \right|_{E'} \leq \delta$ for $R > CR_0/\delta$. This shows (2).

Next we prove the converse. Let $\mu_\lambda = \pi'_\lambda(\mu)$ be the image measure of μ by π'_λ . Remarking the equality

$$\begin{aligned} & \int_{E'} \int_{\Omega} \frac{|\Psi(x')(\omega)|}{R + |\Psi(x')(\omega)|} d\mathbf{P}(\omega) d\mu_\lambda(x') \\ &= \int_{E'} \int_{\Omega} \frac{|\Psi(\pi'_\lambda x')(\omega)|}{R + |\Psi(\pi'_\lambda x')(\omega)|} d\mathbf{P}(\omega) d\mu(x'), \end{aligned}$$

we can see $\{\mu_\lambda\}_{\lambda \in A}$ is tight (theorem 3-2). Since $\pi_\lambda x \rightarrow x$ in E , uniformly on every compact set, we have $\mu_\lambda^\wedge(x) = \mu^\wedge(\pi_\lambda x) \rightarrow \mu^\wedge(x)$, uniformly on compact subsets. Thus the tight family $\{\mu_\lambda\}_{\lambda \in A}$ converges to μ weakly and μ is also $\sigma(E', E)$ -Radon (Schwartz [17], Proposition (III; 5.2)).

This proves the theorem.

REMARK 3-6. In the above theorem we can replace $\{\pi_\lambda\}$ by $\{\pi_\lambda^2\}$ since $\{\pi_\lambda^2\}$ also gives the metric approximation property. Thus the condition (2) can be written as follows;

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \int_{E'} \int_E \frac{|\langle \pi_\lambda x, \pi'_\lambda x' \rangle|}{R + |\langle \pi_\lambda x, \pi'_\lambda x' \rangle|} d\mu_\Psi(x) d\mu(x') = 0.$$

4. S-topology

In this section first we study the existence of the S-topology, applying the results in section 3, secondly the local convexity of the S-topology, and lastly we give the S-topology by q-summing operators. The material of this section was investigated by Mouchtari [12] under a different formulation. We give here a refinement of the work of Mouchtari's.

4.1. Existence of the S-topology

THEOREM 4-1. Let E be a topological vector space with the separating dual E' . We assume there exists a topology τ_0 , not necessarily a vector space topology, satisfying for every random linear functional $\phi: E \rightarrow L^0$ the continuity of ϕ in τ_0 is equivalent to the $\sigma(E', E)$ -Radon extensibility of μ_ϕ . Then there exists the S-topology τ on E which is weaker than τ_0 , stronger than the weak topology $\sigma(E, E')$.

PROOF. We must prove the existence of the weakest vector space topology τ satisfying for every random linear functional ϕ on E , the continuity of ϕ in τ is equivalent to the $\sigma(E', E)$ -Radon extensibility of μ_ϕ .

Let τ be the weakest vector space topology generated by the functions of the form

$$\begin{aligned} \psi(x, \phi) &= \int_{E'} \frac{|\langle x, x' \rangle|}{1 + |\langle x, x' \rangle|} d\mu_\phi(x') \\ &= |\phi(x)|_{L^0(\Omega_\phi, \mathcal{A}_\phi, \mathbf{P}_\phi)}, \end{aligned}$$

where ϕ varies over all random linear functionals which is τ_0 -continuous, that is, ϕ is $\sigma(E', E)$ -Radon. We show this topology τ is the required S-topology.

First we prove the family $\{\psi(\cdot, \phi)\}$ defines in fact a topological vector space structure on E . Each $\psi(x, \phi)$ is a (F)-seminorm, that is, $\psi(x, \phi)$ satisfies;

- (F 1) $\psi(x, \phi) \geq 0$ for $x \in E$,
- (F 2) $\psi(\lambda x, \phi) \leq \psi(x, \phi)$ for $|\lambda| \leq 1, x \in E$,
- (F 3) $\psi(x + y, \phi) \leq \psi(x, \phi) + \psi(y, \phi)$ for $x, y \in E$,
- (F 4) $\psi(\lambda x_n, \phi) \rightarrow 0$, if $\psi(x_n, \phi) \rightarrow 0$ holds, and
- (F 5) $\psi(\lambda_n x, \phi) \rightarrow 0$, if $\lambda_n \rightarrow 0$ in $\mathbf{R}$.

Moreover for every pair ϕ_1, ϕ_2 of random linear functionals on E which are $\sigma(E', E)$ -Radon, the $\sigma(E', E)$ -Radon measure $\frac{1}{2}(\mu_{\phi_1} + \mu_{\phi_2})$ defines a random linear functional ϕ_3 such that $\mu_{\phi_3} = \frac{1}{2}(\mu_{\phi_1} + \mu_{\phi_2})$ which is $\sigma(E', E)$ -Radon, that is, τ_0 -continuous. Thus we have for every $\varepsilon_1, \varepsilon_2 > 0$

$$\begin{aligned} & \{x \in E; \psi(x, \phi_1) < \varepsilon_1\} \cap \{x \in E; \psi(x, \phi_2) < \varepsilon_2\} \\ & \supset \{x \in E; \psi(x, \phi_3) < \varepsilon\}, \end{aligned}$$

where $\varepsilon = \text{Min}\{\varepsilon_1, \varepsilon_2\}/2$. This shows the sets of the form $\{\psi(x, \phi) < \varepsilon\}$ gives a basis of neighborhood of 0 which defines a structure of a topological vector space on E . Remarking that every point $x' \in E'$ defines a random linear functional $\phi_{x'}$ so that $\mu_{\phi_{x'}} = \delta_{x'}$ (the point mass at x'), we see the topology τ is stronger than the weak topology $\sigma(E, E')$. Since each function $\psi(x, \phi)$ defining the topology τ is continuous in τ_0 , τ is weaker than τ_0 .

Next we prove the topology τ is the S-topology.

Let ϕ be a random linear functional which is $\sigma(E', E)$ -Radon. By the definition of τ , $\phi: (E, \tau) \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ is continuous. Conversely, let ϕ be a random linear functional that is continuous on (E, τ) . Since τ is weaker than τ_0 , ϕ is τ_0 -continuous. Thus by the assumption on τ_0 , μ_ϕ is extensible to a $\sigma(E', E)$ -Radon measure. By the definition of τ , τ is obviously the weakest topology satisfying for every random linear functional ϕ , the τ -continuity is equivalent to the $\sigma(E', E)$ -Radon extensibility of μ_ϕ . Thus τ is the S-topology.

This proves the theorem.

We have proved that if the S-topology exists, then it is given by the family

of (F)-seminorms of the form

$$x \rightarrow \int_{E'} \frac{|\langle x, x' \rangle|}{1 + |\langle x, x' \rangle|} d\mu_\phi(x'),$$

where ϕ varies over all $\sigma(E', E)$ -Radon random linear functionals.

REMARK 4-2. Even on a separable Hilbert space H , there are at least two distinct topologies satisfying for every random linear functional $\phi: H \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$, the continuity of ϕ for these topologies is equivalent to the $\sigma(H', H)$ -Radon (in this case also Radon in the norm of H) extensibility of μ_ϕ . One is the S-topology given by Sazonov [16] and another one is the topology defined by the measurable norms of Gross [4]. In this case, the S-topology in our sense is the S-topology of Sazonov [16].

We prove here the Mouchtari's theorem, that is, if E is a Banach space with the metric approximation property and if E' is L^0 -embeddable, then there is the S-topology on E . This result was first proved by Mouchtari [12], Théorème 1. Our proof is a modification of Mouchtari's.

The following lemma follows from Theorem 3-5.

LEMMA 4-3. Let E be a Banach space with the metric approximation property by a net $\{\pi_\lambda\}_{\lambda \in A}$ of operators of finite rank. We assume there is an into isomorphism $\Psi: E' \rightarrow L^0(\Omega, \mathfrak{A}, \mathbf{P})$ of strong dual E' for a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$. Then a norm-continuous random linear functional $\phi: E \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ is $\sigma(E', E)$ -Radon if and only if it holds

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \int_{E'} \int_E \frac{|\langle x, \pi'_\lambda x' \rangle|}{R + |\langle x, \pi'_\lambda x' \rangle|} d\mu_\Psi(x) d\mu_\phi(x') = 0,$$

where μ_ϕ (resp. μ_Ψ) is the cylinder set measure on $Z(E', E)$ (resp. $Z(E, E')$) induced by ϕ (resp. Ψ).

THEOREM 4-4 (Mouchtari [12]). Let E be a Banach space with the metric approximation property and E' is L^0 -embeddable. Then there is the S-topology on E .

PROOF. Let $\{\pi_\lambda\}_{\lambda \in A}$ be a net of operators of finite rank that gives the metric approximation property and let $\Psi: E' \rightarrow L^0(\Omega, \mathfrak{A}, \mathbf{P})$ be an isomorphic embedding.

Let τ be the vector space topology generated by the basis of neighborhoods of 0 of the form $\{x \in E; \psi(x, \phi) < \varepsilon\}$, where ϕ is a random linear functional on E

which is $\sigma(E', E)$ -Radon and $\psi(x, \phi) = |\phi(x)|_{L^0(\Omega_\phi)}$. By the way same to theorem 4-1, we can see τ defines a structure of a topological vector space and τ is stronger than the weak topology. Moreover as seen in section 2, τ is weaker than the norm topology of E .

We prove τ is the S-topology. Let $\phi: E \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ be a random linear functional which is $\sigma(E', E)$ -Radon. By the definition of τ , the set $\{x \in E; |\phi(x)|_{L^0(\Omega_\phi)} < \varepsilon\}$ is a neighborhood of 0 in τ . Hence ϕ is τ -continuous. Conversely let $\phi: E \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ be a random linear functional which is τ -continuous. We must prove that the cylinder set measure μ_ϕ is extensible to a $\sigma(E', E)$ -Radon measure. We show the condition of lemma 4-3 is satisfied. We remark ϕ is norm-continuous, since τ is weaker than the norm topology of E .

By the τ -continuity of ϕ and the definition of τ , for every $\varepsilon > 0$ there exist a $\delta > 0$ and a random linear functional $\phi_0: E \rightarrow L^0(\Omega_{\phi_0})$ which is $\sigma(E', E)$ -Radon such that $\{x \in E; \psi(x, \phi_0) < \delta\} \subset \{x \in E; |\phi(x)|_{L^0(\Omega_\phi)} < \varepsilon/2\}$. In fact the sets of the form $\{\psi(x, \phi_0) < \delta\}$ is a basis, not only a subbasis. Since ϕ_0 is $\sigma(E', E)$ -Radon, by lemma 4-3 there is a $K > 0$ such that

$$\sup_{\lambda \in A} \int_{E'} \int_E \frac{|\langle x, \pi'_\lambda x' \rangle|}{K + |\langle x, \pi'_\lambda x' \rangle|} d\mu_\psi(x) d\mu_{\phi_0}(x') < \frac{\varepsilon\delta}{2}.$$

Remark that the range of π_λ is finite-dimensional, hence the function $\frac{|\langle x, \pi'_\lambda x' \rangle|}{K + |\langle x, \pi'_\lambda x' \rangle|}$ is the tame function. So we have by the Fubini's theorem

$$\begin{aligned} & \int_{E'} \int_E \frac{|\langle x, \pi'_\lambda x' \rangle|}{K + |\langle x, \pi'_\lambda x' \rangle|} d\mu_\psi(x) d\mu_{\phi_0}(x') \\ &= \int_E \int_{F'} \frac{|\langle x, \pi'_\lambda x' \rangle|}{K + |\langle x, \pi'_\lambda x' \rangle|} d\mu_{\phi_0}(x') d\mu_\psi(x). \end{aligned}$$

We put $G_\lambda = \{x \in E; \int_{E'} \frac{|\langle \pi_\lambda x, x' \rangle|}{K + |\langle \pi_\lambda x, x' \rangle|} d\mu_{\phi_0}(x') < \delta\}$. Then we have $G_\lambda = \pi_\lambda^{-1} \left\{ z \in \pi_\lambda E; \psi\left(\frac{z}{K}, \phi_0\right) < \delta \right\}$, so G_λ belongs to $Z(E, E')$. For every $\lambda \in A$, we have

$$\begin{aligned} \frac{\varepsilon\delta}{2} &> \int_{F'} \int_E \frac{|\langle \pi_\lambda x, x' \rangle|}{K + |\langle \pi_\lambda x, x' \rangle|} d\mu_{\phi_0}(x') d\mu_\psi(x) \\ &\geq \int_{G_\lambda} \delta d\mu_\psi(x) = \delta \mu_\psi(G_\lambda), \end{aligned}$$

that is $\mu_\psi(G_\lambda^\varepsilon) < \varepsilon/2$ for every $\lambda \in \Lambda$. For $x \in G_\lambda$, it holds $\pi_\lambda x \in K \cdot \{y \in E; \psi(y, \phi_0) < \delta\} \subset K \cdot \{y \in E; |\phi(x)|_{L^0(\Omega_\phi)} < \varepsilon/2\}$, that is,

$$\int_{E'} \frac{|\langle \pi_\lambda x, x' \rangle|}{K + |\langle \pi_\lambda x, x' \rangle|} d\mu_\phi(x') < \frac{\varepsilon}{2}.$$

Consequently we have

$$\begin{aligned} & \int_E \int_{E'} \frac{|\langle \pi_\lambda x, x' \rangle|}{K + |\langle \pi_\lambda x, x' \rangle|} d\mu_\phi(x') d\mu_\psi(x) \\ & \leq \int_{G_\lambda} \frac{\varepsilon}{2} d\mu_\psi(x) + \int_{G_\lambda^c} 1 d\mu_\psi(x) \\ & \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

for every $\lambda \in \Lambda$. Thus the condition of lemma 4-3 is valid for a random linear functional ϕ , so μ_ϕ is extensible to a $\sigma(E', E)$ -Radon measure.

By the definition of τ , τ is the weakest (see theorem 4-1) and hence τ is the S-topology.

This proves the theorem.

REMARK 4-5. From the above theorem 4-4, there is the S-topology on $E = l^p$ ($2 \leq p < +\infty$). But it is known if $E = l^p$ for $1 < p < 2$, then the S-topology on E does not exist (see de Acosta [1], Corollary, § 4).

4.2. Representation of the S-topology

We have shown if E is a Banach space with the metric approximation property and if E' is L^0 -embeddable, then the S-topology τ on E exists and τ is given by a system of (F)-seminorms. In the rest of this section, we investigate the local convexity of the S-topology. If E is reflexive or E' separable, then it is shown the S-topology is given by a system of p -seminorms ($0 < p < 1$). It is also shown if E is reflexive or E' separable and E' is of type p -stable for $1 \leq p \leq 2$, then the S-topology is locally convex. The proof is analogous to Mouchtari [12], Théorème 4.

Let E be a vector space and $\psi: E \rightarrow \mathbf{R}$ a function. Then ψ is said to be a p -norm ($0 < p \leq 1$) if ψ satisfies

$$(P\ 1) \quad \psi(x) \geq 0 \quad \text{for } x \in E,$$

$$(P\ 2) \quad \psi(\lambda x) = |\lambda|^p \psi(x) \quad \text{for } \lambda \in \mathbf{R}, x \in E, \text{ and}$$

$$(P\ 3) \quad \psi(x+y) \leq \psi(x) + \psi(y) \quad \text{for } x, y \in E.$$

If $\{\psi_\alpha\}_{\alpha \in A}$ is a family of p -seminorms on a vector space E such that

(1) for every $\alpha, \beta \in A$, there exists $\gamma \in A$ such that $\psi_\alpha(x) \leq \psi_\gamma(x)$ and $\psi_\beta(x) \leq \psi_\gamma(x)$ for every $x \in E$, and

$$(2) \quad \bigcap_{\alpha \in A} \{x \in E; \psi_\alpha(x) = 0\} = \{0\},$$

then $\{\psi_\alpha\}_{\alpha \in A}$ determines a structure of a Hausdorff topological vector space on E .

LEMMA 4-6. Let F be a Banach space of type p -stable ($0 < p \leq 2$). Then for every positive Radon measure μ on F of p -th order, the cylinder set measure ν defined by

$$\nu^\wedge(x') = \exp \left[- \int | \langle x, x' \rangle |^p d\mu(x) \right],$$

for every $x' \in F'$, is a Radon measure on F . Moreover for every $q, 0 \leq q < p$, the spaces $L^q(\mathfrak{B}(F), d\nu)$ and $L^q(\mathfrak{B}(F), d\mu)$ induce the equivalent topologies on F' . If $p=2$, we can also take $q=2$.

PROOF. The assertion follows from the Wiener integral by the stable random measure, see Mouchtari [12], Okazaki [13].

A Radon measure ν on a Banach space F is said to be strictly p -stable ($0 < p \leq 2$) if the characteristic functional $\nu^\wedge(x')$ is of the form

$$\nu^\wedge(x') = \exp \left[- \int_F | \langle x, x' \rangle |^p d\mu(x) \right],$$

for a suitable Radon measure μ of second order on F . If F is of type p -stable, then there are sufficiently many strictly p -stable Radon measures on F (lemma 4-6).

Every Banach space is of type p -stable for every p so that $0 < p < 1$ (see Pisier [15]). Hence we have the following theorem.

THEOREM 4-7. Let E be a Banach space with the metric approximation property and E' is L^0 -embeddable. If E is reflexive or E' separable, then the S -topology is given by a system of p -seminorms for every p fixed, $0 < p < 1$. Precisely the S -topology is given by the p -seminorms

$$x \longrightarrow \int_{E'} | \langle x, x' \rangle |^p d\nu(x'),$$

where ν varies over all strictly r -stable Radon probability measures on E' and r be fixed such as $0 < p < r < 1$.

PROOF. Let p, r be $0 < p < r < 1$ fixed. We show the sets of the form

$$(*) \quad \left\{ x \in E; \int_{E'} |< x, x' >|^p dv(x') < \varepsilon \right\},$$

where v is a strictly r -stable Radon probability, become a basis of neighborhoods of 0 and gives the S -topology. We remark that every $\sigma(E', E)$ -Radon measure is also a Radon measure for the norm topology of E' , since E is reflexive or E' separable.

Let $V(v_i, \varepsilon_i) = \left\{ x \in E; \int_{E'} |< x, x' >|^p dv_i(x') < \varepsilon_i \right\}$ be two sets of the form $(*)$, $i=1, 2$. Since v_i is strictly r -stable there is a Radon probability measure μ_i on E' of second order such that

$$\int_{E'} e^{i< x, x' >} dv_i(x') = \exp \left[- \int_{E'} |< x, x' >|^r d\mu_i(x') \right],$$

and such that $L^r(d\mu_i)$ and $L^p(dv_i)$ induce the same topology on E for $i=1, 2$. We put $\mu = \frac{1}{2}(\mu_1 + \mu_2)$. By lemma 4-6, there is a strictly r -stable Radon measure v on E' such that

$$\int_{E'} e^{i< x, x' >} dv(x') = \exp \left[- \int_{E'} |< x, x' >|^r d\mu(x') \right],$$

and such that $L^r(\mathfrak{B}(E'), d\mu)$ and $L^p(\mathfrak{B}(E'), dv)$ induce the same topology on E . For $\varepsilon = \min \{\varepsilon_1, \varepsilon_2\}$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\int_{E'} |< x, x' >|^r d\mu_i(x') < \delta$ implies $\int |< x, x' >|^p dv_i(x') < \varepsilon$ for $i=1, 2$. And there exists a $\eta = \eta(\delta) > 0$ such that $\int_{E'} |< x, x' >|^p dv(x') < \eta$ implies $\int_{E'} |< x, x' >|^r d\mu(x') < \delta/2$. Thus $\int_{E'} |< x, x' >|^p dv(x') < \eta$ implies $\int_{E'} |< x, x' >|^p dv_i(x') < \varepsilon$ for $i=1, 2$. We have proved that

$$V(v, \eta) \subset V(v_1, \varepsilon_1) \cap V(v_2, \varepsilon_2),$$

which shows the sets of the form $(*)$ make a basis of neighborhoods of 0. Hence the system of p -seminorms

$$x \longrightarrow \int_{E'} |< x, x' >|^p dv(x')$$

where v are strictly r -stable Radon probabilities, defines a structure of a topological vector space. This topology τ_0 is a Hausdorff topology. In fact, for every $x_0 \neq 0$

in E , take $x'_0 \in E'$ such that $\langle x_0, x'_0 \rangle \neq 0$. If we take a strictly r -stable measure ν determined by $\delta_{x'_0}$, that is, $\hat{\nu}(x') = \exp \left[- \int |\langle x, x' \rangle|^r d\delta_{x'_0} \right]$, then $\int |\langle x, x' \rangle|^p d\nu(x') = \left(\int |\langle x, x' \rangle|^r d\delta_{x'_0} \right)^{p/r} \times \int |t|^p d\gamma_r(t)$, where $\gamma_r(t)$ is a symmetric r -stable measure on the real line $\mathbf{R}$. In particular we have

$$\int |\langle x_0, x' \rangle|^p d\nu(x') = |\langle x_0, x'_0 \rangle|^p \int |t|^p d\gamma_r(t) \neq 0.$$

We show the above topology τ_0 is the S-topology. Let $\phi: E \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ be a random linear functional such that μ_ϕ is a Radon measure on E' . For every $\varepsilon > 0$, there is an $R > 0$ such that $L = \mu_\phi\{x' \in E'; |x'| \leq R\} > 1 - \varepsilon/2$. If we put $\mu(A) = \mu_\phi(A \cap \{|x'| \leq R\})$, the restriction of μ_ϕ to $\{|x'| \leq R\}$, then μ is a Radon measure of second order. By lemma 4-6 the cylinder set measure ν defined by the characteristic functional $\hat{\nu}(x) = \exp \left[- \int_{E'} |\langle x, x' \rangle|^r d\mu(x') \right]$ is a Radon measure on E' , $L^r(\mathfrak{B}(E'), d\mu)$ and $L^p(\mathfrak{B}(E'), d\nu)$ induce the same topology on E . So there is a $\delta > 0$ such that $\int_{E'} |\langle x, x' \rangle|^p d\nu(x') < \delta$ implies $\int_{E'} |\langle x, x' \rangle|^r d\mu(x') < \varepsilon/2$. By the Tehebichev's inequality we have for x so that $\int |\langle x, x' \rangle|^p d\nu(x') < \delta$,

$$\begin{aligned} & \mu\{x' \in E'; |\langle x, x' \rangle| > 1\} \\ & \leq \int_{E'} |\langle x, x' \rangle|^r d\mu(x') < \frac{\varepsilon}{2}. \end{aligned}$$

Thus we have for x so that $\int |\langle x, x' \rangle|^p d\nu(x') < \delta$,

$$\begin{aligned} & \mathbf{P}_\phi\{\omega \in \Omega_\phi; |\phi(x)(\omega)| > 1\} \\ & = \mu_\phi\{x' \in E'; |\langle x, x' \rangle| > 1\} \\ & \leq \mu_\phi\{x' \in E'; |\langle x, x' \rangle| > 1, |x'| \leq R\} + \frac{\varepsilon}{2} \\ & = \mu\{x' \in E'; |\langle x, x' \rangle| > 1\} + \frac{\varepsilon}{2} \\ & \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which show the random linear functional ϕ is continuous on (E, τ_0) into $L^0(\Omega_\phi,$

$\mathfrak{A}_\phi, \mathbf{P}_\phi$). Conversely, let $\phi: E \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ be a τ_0 -continuous random linear functional. Since for every strictly r -stable Radon measure ν the spaces $L^0(\mathfrak{B}(E'), d\nu)$ and $L^p(\mathfrak{B}(E'), d\nu)$ induce the same topology on E , the topology τ_0 is also given by a system of (F)-seminorms

$$x \longrightarrow \int_{E'} \frac{|\langle x, x' \rangle|}{1 + |\langle x, x' \rangle|} d\nu(x')$$

where ν is a strictly r -stable Radon probability measure. Hence τ_0 is weaker than the S-topology (see the proof of theorem 4-1). Thus ϕ is a Radon measure on E' .

This completes the proof.

By the manner same to the proof of theorem 4-7, we can show the following theorem

THEOREM 4-8 (Mouchtari [12], Théorème 4). Let E be a Banach space with the metric approximation property and E' is L^0 -embeddable. We assume E is reflexive or E' separable and E' is of type p -stable for $1 < p \leq 2$. Then the S-topology is given by the family of seminorms of the form

$$x \longrightarrow \left(\int_{E'} |\langle x, x' \rangle|^q d\nu(x') \right)^{1/q},$$

where ν varies over all strictly p -stable Radon probability measures and q be $1 < q < p \leq 2$ fixed. If $p=2$, then we take $q=2$.

In the above theorem, the S-topology is also given by the family of seminorms

$$x \longrightarrow \left(\int_{E'} |\langle x, x' \rangle|^p d\mu(x') \right)^{1/p},$$

where μ is a Radon measure on E of second order, since $L^q(d\nu)$ and $L^p(d\mu)$ induce the same topology on E , where

$$\hat{\nu}(x) = \exp \left[- \int |\langle x, x' \rangle|^p d\mu \right].$$

THEOREM 4-9. Let (S, Σ, μ) be a positive measure space and p be $2 \leq p < +\infty$. Then the S-topology on $L^p(S, \Sigma, \mu)$ exists and locally convex.

PROOF. By Grothendieck [5], § 5, n°3, Proposition 4.1, $L^p(S, \Sigma, \mu)$ has the metric approximation property. Moreover the dual $L^{p'}(S, \Sigma, \mu)$ can be em-

bedded isomorphically into L^0 , where $1/p + 1/p' = 1$, $1 < p' \leq 2$ (see Maurey [11], § 2, Remarque 2). It is also well known that $L^{p'}(S, \Sigma, \mu)$ is of type q -stable for every $q < p'$ (see Pisier [15]). Thus taking q so that $1 < q < p'$, we see there is the S-topology which is locally convex by theorem 4-8.

4.3. Operator characterization of the S-topology on $L^p(2 \leq p < +\infty)$

Let E be a Banach space with the metric approximation property and E' is L^0 -embeddable. Let E be reflexive or E' be separable. Then, if E' is of type p -stable, the S-topology τ on E exists and given by the seminorms

$$x \longrightarrow \left(\int_{E'} |\langle x, x' \rangle|^p d\mu(x') \right)^{1/p}$$

for every Radon measure μ of second order. We remark for every Radon measure μ on E' of p -th order, the cylinder set measure ν on E' defined by $\nu^{\wedge}(x) = \exp \left[- \int_{E'} |\langle x, x' \rangle|^p d\mu(x') \right]$ is also a Radon measure on E' (lemma 4-6).

The S-topology on L^2 is given by Hilbert-schmidt operators (Sazonov [16]). We treat here the case $E = L^p$, $2 < p < +\infty$.

THEOREM 4-10. Let $E = L^p(S, \Sigma, \mu)$, where (S, Σ, μ) is a positive measure space and p be $2 < p < +\infty$. We fix q such as $1 < q < p'$, where $1/p + 1/p' = 1$ and hence $1 < p' < 2$. Then the S-topology is given by the family of seminorms

$$(**) \quad x \longrightarrow |Rx|_{L^q},$$

where $R: E \rightarrow L^q(\Omega_R, \mathfrak{A}_R, \mathbf{P}_R)$ varies over all q -summing operators of E into $L^q(\Omega_R, \mathfrak{A}_R, \mathbf{P}_R)$ for a probability space $(\Omega_R, \mathfrak{A}_R, \mathbf{P}_R)$.

PROOF. Let μ be a Radon probability measure on E' of q -th order, then by lemma 4-6, the cylinder set measure ν such that $\nu^{\wedge}(x) = \exp \left[- \int |\langle x, x' \rangle|^q d\mu(x') \right]$ is Radon on E' . Thus by Maurey [11], Theorem 2, the natural mapping $R: E \rightarrow L^q(\mathfrak{B}(E'), d\mu)$ is q -summing. Consequently, the S-topology is weaker than the topology determined by the seminorms of the form $(**)$ (see theorem 4-8).

Conversely let $R: E \rightarrow L^p(\Omega_R, \mathbf{P}_R)$ be q -summing. Then the transpose $R^*: L^{q'}(\Omega_R) \rightarrow E'$ is also q -summing by Maurey [11], Théorème 2, and the image measure of γ_q under R^* is Radon measure on E' , where

$$\gamma_q^{\wedge}(f) = \exp \left[- \int_{\Omega_R} |f(\omega)|^q d\mathbf{P}_R(\omega) \right], \quad f \in L^q(\Omega_R).$$

Thus the cylinder set measure λ_q on E' defined by

$$\hat{\lambda}_q(x) = \exp \left[- \int |(Rx)(\omega)|^q d\mathbf{P}_R(\omega) \right]$$

is a Radon measure on E' . In particular the seminorm $|Rx|_{L^q(\Omega_R)}$ is continuous with respect to the S-topology.

This proves the theorem.

Let $\{\mu_\alpha\}_{\alpha \in A}$ be the collection of all Radon probabilities of q -th order on $(L^p(S, \Sigma, \mu))'$, $1 < q < p'$, $1/p + 1/p' = 1$, $2 < p < +\infty$. Let R_α be the natural mapping

$$R_\alpha: L^p(S, \Sigma, \mu) \longrightarrow L^q(\mathfrak{B}(L^{p'}(S)), \mu_\alpha).$$

Then the S-topology is given by the seminorms $|R_\alpha x|_{L^q(\mu_\alpha)}$.

We remark that all spaces $L^q(\mu_\alpha)$ is isometric to a subspace of $L^q([0, 1], dx)$, where dx is the Lebesgue measure. Thus we have the following theorem.

THEOREM 4-11. Let p be $2 < p < +\infty$ and $1 < q < p'$, $1/p + 1/p' = 1$. Then the S-topology on $L^p(S, \Sigma, \mu)$ is given by the system of all seminorms

$$x \longrightarrow |Rx|_{L^q([0, 1], dx)},$$

where $R: L^p(S, \Sigma, \mu) \rightarrow L^q([0, 1], dx)$ is a q -summing operator.

5. Bochner's theorem

We prove a necessary and sufficient condition for a complex valued function ψ on E is the Fourier transform of a $\sigma(E', E)$ -Radon measure on E' (the Bochner's theorem).

THEOREM 5-1. Let E be a Banach space with the metric approximation property and E' be L^0 -embeddable. Then a complex valued function ψ on E is the characteristic functional of a $\sigma(E', E)$ -Radon probability measure on E' if and only if

- (1) ψ is positive-definite with $\psi(0) = 1$,
- (2) ψ is continuous with respect to the Sazonov topology.

PROOF. The assertion follows from theorem 4-4.

We prove another form of the Bochner's theorem which is similar to Kuelbs [7], Theorem 3-1. Let $\Psi: E' \rightarrow L^0(\Omega, \mathfrak{A}, \mathbf{P})$ be an into isomorphism. We remark that the sets of the form

$$\left\{f \in L^0; \operatorname{Re} \left[\int_{\Omega} (1 - e^{i f(\omega)}) d\mathbf{P}(\omega) \right] < \varepsilon \right\},$$

where $\operatorname{Re} z$ is the real part of the complex number z , make a basis of neighborhoods of 0 of $L^0(\Omega, \mathfrak{A}, \mathbf{P})$, as easily shown by the truncation inequality (Loeb [10], p. 196). Let μ_{Ψ} be a cylinder set measure on E induced by the random linear functional Ψ . Since Ψ is an isomorphism, the sets of the form

$$\left\{x' \in E'; \operatorname{Re} \left[\int_E (1 - e^{i \langle x, x' \rangle}) d\mu_{\Psi}(x) \right] < \varepsilon \right\}$$

make a basis of neighborhoods of 0 of E' .

THEOREM 5-2. Let E be a Banach space with the metric approximation property by a net $\{\pi_{\lambda}\}_{\lambda \in A}$ of operators of finite rank. Assume there is an into isomorphism $\Psi: E' \rightarrow L^0(\Omega, \mathfrak{A}, \mathbf{P})$ for a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$. Then a complex valued function ψ on E is the characteristic functional of a $\sigma(E', E)$ -Radon probability measure on E' if and only if the following three conditions hold.

- (1) ψ is positive-definite on E with $\psi(0)=1$,
- (2) ψ is continuous in norm of E , and
- (3) it holds that

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \operatorname{Re} \left[\int_E \left(1 - \psi \left(\frac{\pi_{\lambda} x}{R} \right) \right) d\mu_{\Psi}(x) \right] = 0,$$

where μ_{Ψ} is the cylinder set measure on E induced by Ψ .

PROOF. Assume $\psi(x) = \hat{\mu}(x)$ for a $\sigma(E', E)$ -Radon measure μ . Then (2) holds, see theorem 3-3(a). To see (3), we proceed as the proof of theorem 3-2. We can rewrite (3) as follows;

$$\begin{aligned} 0 &= \lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \operatorname{Re} \left[\int_E \left(1 - \hat{\mu} \left(\frac{\pi_{\lambda} x}{R} \right) \right) d\mu_{\Psi}(x) \right] \\ &= \lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \operatorname{Re} \left[\int_E \int_{E'} (1 - e^{i \langle \frac{\pi_{\lambda} x}{R}, x' \rangle}) d\mu(x') d\mu_{\Psi}(x) \right] \\ &= \lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \operatorname{Re} \left[\int_{E'} \int_E (1 - e^{i \langle x, \frac{\pi_{\lambda} x'}{R} \rangle}) d\mu_{\Psi}(x) d\mu(x') \right], \end{aligned}$$

where we have used the Fubini's theorem. Since Ψ is isomorphic, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|x'| \leq \delta$ implies $\operatorname{Re} \left[\int_E (1 - e^{i \langle x, x' \rangle}) d\mu_{\Psi}(x) \right] < \varepsilon/2$. We put $C = \sup_{\lambda \in A} |\pi_{\lambda}| < +\infty$. Let R_0 be such that $\mu(|x'| > R_0) < \varepsilon/4$, then we have

$\mu(|\pi'_\lambda x'| > CR_0) \leq \mu(|x'| > R_0) \leq \varepsilon/4$. Thus for every $R > CR_0/\delta$ and every $\lambda \in A$, we have

$$\begin{aligned} & \operatorname{Re} \left[\int_{E'} \int_E (1 - e^{i \langle x, \frac{\pi'_\lambda x'}{R} \rangle}) d\mu_\Psi(x) d\mu(x') \right] \\ & \leq \int_{\{|\pi'_\lambda x'| > CR_0\}} 2d\mu(x') \\ & \quad + \int_{\{|\pi'_\lambda x'| \leq CR_0\}} \operatorname{Re} \left[\int_E (1 - e^{i \langle x, \frac{\pi'_\lambda x'}{R} \rangle}) d\mu_\Psi(x) \right] d\mu(x') \\ & \leq \frac{\varepsilon}{2} + \int_{E'} \frac{\varepsilon}{2} d\mu(x') = \varepsilon, \end{aligned}$$

which shows (3).

Let ψ be a complex function on E satisfying (1), (2) and (3). Let μ be the cylinder set measure such that $\mu^\wedge(x) = \psi(x)$ and let $\mu_\lambda = \pi'_\lambda(\mu)$. Since $\mu^\wedge_\lambda(x) = \mu^\wedge(\pi_\lambda x) = \psi(\pi_\lambda x) \rightarrow \psi(x) = \mu^\wedge(x)$, uniformly on compact subsets, to see μ is $\sigma(E', E)$ -Radon, it is sufficient to show the tightness of the family $\{\mu_\lambda\}_{\lambda \in A}$ (Schwartz [17], Proposition (III; 5.2)). Since Ψ is isomorphic, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\operatorname{Re} \left[\int_E (1 - e^{i \langle x, x' \rangle}) d\mu_\Psi(x) \right] < \delta$ implies $|x'| < \varepsilon/2$. Hence $\left| \frac{x'}{R} \right| \geq \varepsilon/2$ implies $\operatorname{Re} \left[\int_E (1 - e^{i \langle x, \frac{x'}{R} \rangle}) d\mu_\Psi(x) \right] \geq \delta$, so we have

$$\begin{aligned} \mu_\lambda(|x'| > R) & \leq \int_{E'} 1 \wedge \left[\frac{|x'|}{R} \right] d\mu_\lambda(x') \\ & \leq \int_{\{|\frac{x'}{R}| < \frac{\varepsilon}{2}\}} \frac{\varepsilon}{2} d\mu_\lambda(x') + \int_{\{|\frac{x'}{R}| \geq \frac{\varepsilon}{2}\}} 1 \wedge \left[\frac{|x'|}{R} \right] d\mu_\lambda(x') \\ & \leq \frac{\varepsilon}{2} + \int_{\{|\frac{x'}{R}| \geq \frac{\varepsilon}{2}\}} \frac{1}{\delta} \operatorname{Re} \left[\int_E (1 - e^{i \langle x, \frac{x'}{R} \rangle}) d\mu_\Psi(x) \right] d\mu_\lambda(x') \\ & \leq \frac{\varepsilon}{2} + \frac{1}{\delta} \operatorname{Re} \left[\int_E \int_{E'} (1 - e^{i \langle x, \frac{\pi'_\lambda x'}{R} \rangle}) d\mu(x') d\mu_\Psi(x) \right] \\ & = \frac{\varepsilon}{2} + \frac{1}{\delta} \operatorname{Re} \left[\int_E \left(1 - \psi \left(\frac{\pi_\lambda x}{R} \right) \right) d\mu_\Psi(x) \right], \end{aligned}$$

for every $\lambda \in A$. Thus we have

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \mu_\lambda(|x'| > R) = 0,$$

which shows the tightness of $\{\mu_\lambda\}$.

REMARK 5-3. If E' is L^0 -embeddable, then we can take a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ such that there is an isomorphism $\Psi: E' \rightarrow L^0(\Omega)$ and such that the cylinder set measure μ_Ψ on E induced by Ψ is symmetric, that is, $\int e^{i\langle x, x' \rangle} d\mu_\Psi(x)$ is a real function of $x \in E$, see Okazaki [13]. Considering this measure μ_Ψ , the condition (3) can be rewritten as

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \int_E \left(1 - \psi\left(\frac{\pi_\lambda x}{R}\right)\right) d\mu_\Psi(x) = 0.$$

6. The Levy continuity theorem

An infinite-dimensional version of the Levy continuity theorem was studied by Gross [4], Kuelbs [7], Kuelbs and Mandrekar [8]. Here we prove a part of the Levy continuity theorem for a sequence of Radon probability measures on E' where E is a Banach space such that E is reflexive or E' separable and such that E' is L^0 -embeddable.

First we define the generalized Fourier transform $\mu^\sim(x)$ following Kuelbs [7], § 5, LeCam [9] where μ is a Radon probability measure on E' and $x \in (E')^a$ (the algebraic dual of E').

Let E be a Banach space which is reflexive or E' separable. Assume there is an isomorphism $\Psi: E' \rightarrow L^0(\Omega, \mathfrak{A}, \mathbf{P})$ for a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$. Let μ_Ψ be a σ -additive cylindrical measure on $(E')^a$ induced by Ψ . We can take μ_Ψ symmetric (see Remark 5-3). Let μ be a Radon measure on E' and $\{\mu_\alpha\}$ be a net of probability measures such that $\mu_\alpha \rightarrow \mu$ weakly and each μ_α has a finite support. Then $\mu_\alpha^\sim(x)$ is defined for $x \in (E')^a$ and $\{\mu_\alpha^\sim\}$ is a Cauchy net in $L^2((E')^a, d\mu_\Psi)$. We put $\mu^\sim(x) = \text{l.i.m. } \mu_\alpha^\sim(x)$, then we have

$$\int_{(E')^a} \mu^\sim(x) d\mu_\Psi(x) = \int_{E'} \left[\int_\Omega e^{i\Psi(x')(\omega)} d\mathbf{P}(\omega) \right] d\mu(x').$$

For details, see Kuelbs [7] and LeCam [9]. We remark $\int_\Omega e^{i\Psi(x')(\omega)} d\mathbf{P}(\omega)$ is a real function on E' and the sets of the form

$$\left\{ x' \in E'; 1 - \int_\Omega e^{i\Psi(x')(\omega)} d\mathbf{P}(\omega) < \varepsilon \right\}$$

determines the norm topology of E' .

THEOREM 6-1. Let E be a Banach space such that E is reflexive or E' is

separable. Assume there is an into isomorphism $\Psi: E' \rightarrow L^0(\Omega, \mathfrak{A}, \mathbf{P})$ for a probability space inducing the symmetric cylinder set measure μ_Ψ on $(E')^a$. Then a sequence of Radon probability measures $\{\mu_k\}$ on E' converges weakly to a Radon probability measure μ on E' if the following conditions hold;

(1) $\mu_k^\wedge(x) \rightarrow \mu^\wedge(x)$ for every $x \in E$, and

(2) $\{\mu_k^\wedge\}$ converges in probability to $\mu^\wedge$ in $L^0((E')^a, d\mu_\Psi)$, where we mean the weak convergence $\mu_k \rightarrow \mu$ is for the weak* topology $\sigma(E', E)$.

PROOF. By a finite-dimensional Levy continuity theorem $\mu_k \rightarrow \mu$ cylindrically (Schwartz [17]). Thus to see the assertion, it is sufficient to show the tightness of $\{\mu_k\}$. For every $\varepsilon > 0$, there is a $\delta > 0$ such that $|x'| \geq \varepsilon$ implies $1 - \int_\Omega e^{i\Psi(x')(\omega)} d\mathbf{P}(\omega) \geq \delta$. Hence for every $R > 0$, we have

$$\begin{aligned} \mu_k\{|x'| > R\} &= \mu_k\left\{\left|\frac{\varepsilon}{R}x'\right| > \varepsilon\right\} \\ &\leq \mu_k\left\{x'; 1 - \int_\Omega e^{i\Psi\left(\frac{\varepsilon}{R}x'\right)(\omega)} d\mathbf{P}(\omega) \geq \delta\right\} \\ &\leq \frac{1}{\delta} \int_{E'} \int_\Omega (1 - e^{i\Psi\left(\frac{\varepsilon}{R}x'\right)(\omega)}) d\mathbf{P}(\omega) d\mu_k \\ &= \frac{1}{\delta} \int_{(E')^a} \left(1 - \mu_k^\wedge\left(\frac{\varepsilon}{R}x\right)\right) d\mu_\Psi(x), \end{aligned}$$

which shows

$$\begin{aligned} \overline{\lim}_k \mu_k\{|x'| > R\} &\leq \frac{1}{\delta} \int_{(E')^a} \left(1 - \tilde{\mu}\left(\frac{\varepsilon}{R}x\right)\right) d\mu_\Psi(x) \\ &= \frac{1}{\delta} \int_{E'} \int_\Omega (1 - e^{i\Psi\left(\frac{\varepsilon}{R}x'\right)(\omega)}) d\mathbf{P}(\omega) d\mu(x'). \end{aligned}$$

Since μ is Radon, there is an $R_0 > 0$ so that $\mu(|x'| > R_0) < \delta \cdot \varepsilon/4$. Thus for every $R > R_0/\varepsilon$, we have

$$\begin{aligned} \overline{\lim}_k \mu_k\{|x'| > R\} &\leq \frac{1}{\delta} \int_{\{|x'| \leq R_0\}} \int_\Omega (1 - e^{i\Psi\left(\frac{\varepsilon}{R}x'\right)(\omega)}) d\mathbf{P}(\omega) d\mu(x') \\ &\quad + \frac{1}{\delta} 2\mu\{|x'| > R_0\} \\ &\leq \frac{1}{\delta} \int_{\{|x'| \leq R_0\}} \int_\Omega (1 - e^{i\Psi\left(\frac{\varepsilon}{R}x'\right)(\omega)}) d\mathbf{P}(\omega) d\mu(x') + \frac{\varepsilon}{2}. \end{aligned}$$

Now we can take R_1 so that $|x'| \leq 1/R_1$ implies $\int_{\Omega} (1 - e^{i\Psi(x')(\omega)}) d\mathbf{P}(\omega) \leq \delta\epsilon/2$. Hence for $R > R_1/\epsilon$, we have

$$\frac{1}{\delta} \int_{\{|x'| \leq R_0\}} \int_{\Omega} (1 - e^{i\Psi(\frac{\epsilon}{R}x')(\omega)}) d\mathbf{P}(\omega) < \frac{\epsilon}{2}.$$

We have proved that $\lim_k \mu_k\{|x'| > R\} < \epsilon$ for sufficiently large R . Hence there is a k_0 such that $\sup_{k \geq k_0} \mu_k\{|x'| > R\} < \epsilon$.

The proof is complete.

REMARK 6-2. If $\mu_k \rightarrow \mu$ weakly for the norm topology of E' , then (1) and (2) hold by the way same to Kuelbs [7], Theorem 5.1.

7. Extension to general locally convex spaces

The tightness of a family of $\sigma(E', E)$ -Radon probability measures, the Radon extension of a cylinder set measure and the existence theorem of the S-topology is extended to a locally convex space.

7.1. LB space

First we consider the inductive limit.

7-1-1. Tightness of measures

The next theorem corresponds to theorem 3-1.

THEOREM 7-1. Let $E = \varinjlim (E_n, |\cdot|_n)$ be the inductive limit of Banach spaces $\{(E_n, |\cdot|_n)\}_{n=1}^{\infty}$. Let $\{\mu_{\alpha}\}_{\alpha \in A}$ be a family of $\sigma(E', E)$ -Radon probability measures on E' . Then the following conditions are equivalent.

- (1) $\{\mu_{\alpha}\}_{\alpha \in A}$ is tight on $E'_{\sigma(E', E)}$.
- (2) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'_n} \frac{|x'|_{E'_n}}{R + |x'|_{E'_n}} d\mu_{\alpha}^n(x') = 0,$$

for every $n=1, 2, \dots$, where μ_{α}^n is the image measure $\psi_n(\mu_{\alpha})$, where $\psi_n: E' \rightarrow E'_n$ is the canonical projection.

- (3) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'_n} \left(1 - \exp\left[-\frac{|x'|_{E'_n}}{R}\right]\right) d\mu_{\alpha}^n(x') = 0,$$

for every n , where $\mu_n^\alpha = \psi_n(\mu_\alpha)$.

(4) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'} \left(1 - \exp \left[- \frac{|x'|_{E'_n}^p}{R^p} \right] \right) d\mu_\alpha^n(x') = 0,$$

for every n and every p , $0 < p < +\infty$, where $\mu_\alpha^n = \psi_n(\mu_\alpha)$.

(5) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'_n} 1 \wedge \left[\frac{|x'|_{E'_n}}{R} \right] d\mu_\alpha^n(x') = 0,$$

for every n , where $\mu_\alpha^n = \psi_n(\mu_\alpha)$.

PROOF. We have $E' = \varprojlim_n E'_n$ the projective limit of $\{E'_n\}_{n=1}^\infty$. Let $\psi_n: E' \rightarrow E'_n$ be the canonical projection which is the transpose of the natural injection $\iota_n: E_n \rightarrow E$ and μ^n be the image measure $\psi_n(\mu)$, where μ is a $\sigma(E', E)$ -Radon measure.

Let $\{\mu_\alpha\}_{\alpha \in A}$ be tight in $E'_{\sigma(E', E)}$. Then the family $\{\mu_\alpha^n\}_{\alpha \in A}$ is tight in $(E'_n, \sigma(E'_n, E_n))$ for every n since ψ_n is continuous. Thus by theorem 3-1, the conditions (2), (3), (4) and (5) hold.

Conversely, assume one of the conditions (2), (3), (4) and (5) holds. Then by theorem 3-1, the family $\{\mu_\alpha^n\}_{\alpha \in A}$ is tight in $(E'_n, \sigma(E'_n, E_n))$ for every n . Thus by the way similar to Bourbaki [2], Théorème 2, § 4, n° 3, the family $\{\mu_\alpha\}_{\alpha \in A}$ is tight in $E'_{\sigma(E', E)}$.

This proves the theorem.

The next theorem can be shown analogously by theorem 3-2.

THEOREM 7-2. Let $E = \varinjlim_n (E_n, |\cdot|_n)$ be the inductive limit of Banach spaces $\{(E_n, |\cdot|_n)\}_{n=1}^\infty$ and $\{\mu_\alpha\}_{\alpha \in A}$ be a family of $\sigma(E', E)$ -Radon probability measures on $E'_{\sigma(E', E)}$. Assume each E'_n can be embedded into $L^0(\Omega_n, \mathfrak{A}_n, \mathbf{P}_n)$ for a probability space $(\Omega_n, \mathfrak{A}_n, \mathbf{P}_n)$. Then $\{\mu_\alpha\}_{\alpha \in A}$ is tight in $E'_{\sigma(E', E)}$ if and only if it holds

$$\lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'_n} \int_{\Omega_n} \frac{|\Psi_n(x')(\omega)|}{R + |\Psi_n(x')(\omega)|} d\mathbf{P}_n(\omega) d\mu_\alpha^n(x') = 0$$

for every n , where $\Psi_n: E'_n \rightarrow L^0(\Omega_n)$ is the isomorphism and $\mu_\alpha^n = \psi_n(\mu_\alpha)$.

7-1-2. $\sigma(E', E)$ -Radon extensibility of a cylinder set measure

As for the $\sigma(E', E)$ -Radon extensibility of the cylinder set measure on E' ,

we have the following theorem.

THEOREM 7-3. Let $E = \varinjlim_n (E_n, |\cdot|_n)$ be the inductive limit of Banach spaces $\{(E_n, |\cdot|_n)\}_{n=1}^\infty$ and μ be a cylinder set measure on $Z(E', E)$. Assume each E_n has the metric approximation property by a net $\{\pi_\lambda^n\}_{\lambda \in A(n)}$ of operators of finite rank. Then μ is $\sigma(E', E)$ -Radon if and only if

- (1) The characteristic functional $\mu^\wedge(x)$ is continuous on E .
- (2) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A(n)} \int_{E'_n} \frac{|\pi_\lambda^n x'|_{E'_n}}{R + |\pi_\lambda^n x'|_{E'_n}} d\mu^n(x') = 0$$

for every n , where $\mu^n = \psi_n(\mu)$.

PROOF. We remark that the characteristic functional $\mu^\wedge(x)$ is continuous in E if and only if $\mu^\wedge(x)$ is continuous in E_n for every n , that is, $\widehat{\mu^n}(x) = \mu^\wedge(\iota_n x)$ is continuous in E_n for every n , where $\iota_n: E_n \rightarrow E$ is the canonical injection.

Let μ be $\sigma(E', E)$ -Radon. Then μ^n is $\sigma(E'_n, E_n)$ -Radon for every n , hence (1) and (2) hold by theorem 3-3.

Conversely if the conditions (1) and (2) are satisfied, then μ^n is $\sigma(E'_n, E_n)$ -Radon on E_n for every n by theorem 3-3. Thus by the way same to Bourbaki [2], Théorème 2, § 4, n°3, μ is a $\sigma(E', E)$ -Radon measure.

The above condition (2) can be replaced by one of the conditions similar to (3), (4) and (5) of theorem 7-1, see remark 3-4.

THEOREM 7-4. Let $E = \varinjlim_n (E_n, |\cdot|_n)$ be the inductive limit of Banach spaces and μ be a cylinder set measure on $Z(E', E)$. We assume each E_n has the metric approximation property by a net $\{\pi_\lambda^n\}_{\lambda \in A(n)}$ of operators of finite rank and for every n there is an isomorphism Ψ_n of E'_n into $L^0(\Omega_n, \mathfrak{A}_n, \mathbf{P}_n)$ for a probability space $(\Omega_n, \mathfrak{A}_n, \mathbf{P}_n)$.

Then μ is $\sigma(E', E)$ -Radon if and only if it holds

- (1) The characteristic functional $\mu^\wedge(x)$ is continuous in E .
- (2) It holds

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A(n)} \int_{E'_n} \int_{\Omega_n} \frac{|\Psi_n(\pi_\lambda^n x')(\omega)|}{R + |\Psi_n(\pi_\lambda^n x')(\omega)|} d\mathbf{P}_n(\omega) d\mu^n(x') = 0$$

for every n where $\mu_n = \psi_n(\mu)$.

Let μ_{Ψ_n} be the cylinder set measure on $Z(E'_n, E'_n)$ defined by the random linear functional Ψ_n . Then the condition (2) can be written

$$\lim_{R \rightarrow \infty} \sup_{\lambda \in A(n)} \int_{E'_n} \int_{E_n} \frac{|\langle \pi_\lambda^n x, x' \rangle|}{R + |\langle \pi_\lambda^n x, x' \rangle|} d\mu_{\Psi_n}(x) d\mu^n(x') = 0.$$

PROOF. Using theorem 3-5, a proof similar to the one of theorem 7-3 can be applied.

7-1-3. S-topology

We prove here if each locally convex space E_n has the S-topology, then inductive limit $E = \varinjlim E_n$ also has the S-topology.

THEOREM 7-5. Let $E = \varinjlim_n E_n$ be the inductive limit of locally convex spaces $\{E_n\}_{n=1}^\infty$. If each E_n has the S-topology, then E has the S-topology.

PROOF. The topological dual E' is $E' = \varprojlim_n E'_n$ the projective limit of $\{E'_n\}$. Let τ_n be the S-topology on E_n for every n . We remark that the topology τ_n can be given by a family of (F)-norms which are derived by all the $\sigma(E'_n, E_n)$ -Radon random linear functionals, see theorem 4-1. Hence τ_n is stronger than τ_{n+1} , that is, the injection $\iota_{n,n+1}: (E_n, \tau_n) \rightarrow (E_{n+1}, \tau_{n+1})$ is continuous for every n , since $\iota'_{n,n+1}: E'_{n+1} \rightarrow E'_n$ is weak *-continuous. Thus we may consider the inductive limit topology τ_0 of $\{\tau_n\}_{n=1}^\infty$ on E . We show this topology τ_0 satisfies that for every random linear functional $\phi: E \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$, the continuity of ϕ in τ_0 is equivalent to the $\sigma(E', E)$ -Radon extensibility of μ_ϕ . Then the assertion follows from theorem 4-1.

Let $\phi: (E, \tau_0) \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ be continuous linear mapping. Then ϕ is continuous in (E_n, τ_n) for every n . Let μ_ϕ^n be the $\sigma(E'_n, E_n)$ -Radon measure on E'_n derived by ϕ_n which is the restriction of ϕ to E_n . We shall show $\{\mu_\phi^n\}_{n=1}^\infty$ forms a projective system of $\sigma(E'_n, E_n)$ -Radon measures, that is, $\iota'_{n,n+1}(\mu_\phi^{n+1}) = \mu_\phi^n$ for every n . For every cylinder set

$$A = \{x' \in E'_n; \langle x_i, x' \rangle \leq a_i, a_i \in \mathbf{R}, x_i \in E_n \quad i = 1, 2, \dots, k\}$$

we have

$$\begin{aligned} & \iota'_{n,n+1}(\mu_\phi^{n+1})(A) \\ &= \mu_\phi^{n+1}\{x' \in E'_{n+1}; \langle x', \iota_{n,n+1}(x_i) \rangle \leq a_i\} \\ &= \mathbf{P}_\phi\{\omega \in \Omega_\phi; \phi(\iota_{n+1} \circ \iota_{n,n+1}(x_i))(\omega) \leq a_i, \quad i = 1, 2, \dots, k\} \\ &= \mathbf{P}_\phi\{\omega \in \Omega_\phi; \phi(\iota_n(x_i))(\omega) \leq a_i, \quad i = 1, 2, \dots, k\} \end{aligned}$$

$$\begin{aligned}
&= \mu_\phi^n \{x' \in E'_n; \langle x', x_i \rangle \leq a_i, \quad i=1, 2, \dots, k\} \\
&= \mu_\phi^n(A),
\end{aligned}$$

which proves the assertion. Hence there is the projective limit $\bar{\mu}_\phi$ of $\{\mu_\phi^n\}$, that is, $\bar{\mu}_\phi$ is a $\sigma(E', E)$ -Radon measure on E' such that $\psi_n(\bar{\mu}_\phi) = \mu_\phi^n$, where $\psi_n: E' \rightarrow E'_n$ is the natural projection (see Bourbaki [2], Théorème 2, § 4, n°3). Obviously $\bar{\mu}_\phi$ is the $\sigma(E', E)$ -Radon extension of μ_ϕ .

Conversely let μ_ϕ be $\sigma(E', E)$ -Radon. We show the random linear functional ϕ is τ_0 -continuous. Since the image $\psi_n(\mu_\phi) = \mu_\phi^n$ is also $\sigma(E'_n, E_n)$ -Radon, ϕ is continuous on (E_n, τ_n) into $L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ for every n . Thus by the definition of the inductive topology, ϕ is τ_0 -continuous.

This completes the proof.

COROLLARY 7-6. Let $E = \varinjlim (E_n, |\cdot|_n)$ be the inductive limit of Banach spaces $\{(E_n, |\cdot|_n)\}_{n=1}^\infty$. We assume each E_n has the metric approximation property and E'_n is L^0 -embeddable for each n . Then there is the Sazonov topology on E .

PROOF. The assertion follows from theorem 4-4 and theorem 7-5.

Let F be a Fréchet space and μ be a Radon measure on E . Then we say μ is of p -th order, $0 < p < +\infty$, if for every continuous seminorm $|\cdot|$ in F , we have $\int_F |x|^p d\mu(x) < +\infty$.

LEMMA 7-7. Let F be a Fréchet space with a basis $\{|\cdot|_n\}_{n=1}^\infty$ of continuous seminorms such that the each associated normed space $F/\{x \in F; |x|_n = 0\}$ is of type p -stable, $0 < p \leq 2$. Then for every positive Radon measure μ on F of p -th order, the cylinder set measure ν defined by

$$\nu^\wedge(x') = \exp \left[- \int_F |\langle x, x' \rangle|^p d\mu(x) \right], \quad x' \in F',$$

is again a Radon measure on F . Moreover for every q , $0 \leq q < p$, the two topologies on F' induced by $L^q(\mathfrak{B}(F), d\nu)$ and $L_p(\mathfrak{B}(F), d\mu)$ are equivalent. If $p=2$, we may take $q=2$.

A Radon measure ν on a Fréchet space F is said to be strictly p -stable ($0 < p \leq 2$) if the characteristic functional $\nu^\wedge(x')$ is given by

$$\nu^{\wedge}(x') = \exp \left[- \int_F | \langle x, x' \rangle |^p d\mu(x) \right], \quad x' \in F',$$

for a suitable Radon measure μ of second order.

THEOREM 7-8. Let $E = \varinjlim_n (E_n, | \cdot |_n)$ be the inductive limit of Banach spaces $\{(E_n, | \cdot |_n)\}_{n=1}^{\infty}$ such that each E_n has the metric approximation property and E'_n is L^0 -embeddable. If each E_n is reflexive or E'_n separable, then the S-topology is given by a system of p -seminorms for every p but fixed such as $0 < p < 1$. Moreover, if each E'_n is of type r -stable ($1 \leq r \leq 2$), then the S-topology is given by the family of seminorms

$$x \longrightarrow \left(\int_{E'} | \langle x, x' \rangle |^r d\mu(x') \right)^{1/r},$$

where μ is a Radon measure of second order on the Fréchet E' .

PROOF. We remark that every $\sigma(E', E)$ -Radon measure is also Radon for the strong dual (Fréchet) topology in E' . Thus by using lemma 7-7, the theorem can be proved by the way similar to the proof of theorem 4-7. See also a remark after the theorem 4-8.

The proof is complete.

7-1-4. Bochner's theorem

Now we extend the Bochner's theorem. Let $E = \varinjlim_n (E_n, | \cdot |_n)$ be the inductive limit of Banach spaces and φ be a complex valued positive-definite function on E with $\varphi(0)=1$. Then if E satisfies the conditions in corollary 7-6, φ is the characteristic functional of a Radon measure on $(E', \sigma(E', E))$ if and only if φ is continuous in the S-topology.

THEOREM 7-9. Let $E = \varinjlim_n (E_n, | \cdot |_n)$ be the inductive limit of Banach spaces such that each E_n has the metric approximation property by a net $\{\pi_{\lambda}^n\}_{\lambda \in \Lambda(n)}$ of operators of finite rank and such that there is an isomorphism Ψ_n of E'_n into $L^0(\Omega_n, \mathfrak{A}_n, \mathbf{P}_n)$ for every n . Then a complex valued function φ on E is the characteristic functional of a $\sigma(E', E)$ -Radon probability measure on E' if and only if the following three conditions hold.

- (1) φ is positive-definite on E with $\varphi(0)=1$,
- (2) φ is continuous in E , and
- (3) it holds that

$$\lim_{R \rightarrow +\infty} \sup_{\lambda \in A(n)} \operatorname{Re} \left[\int_{E_n} \left(1 - \varphi \left(\frac{\pi_\lambda^n x}{R} \right) d\mu_{\Psi_n}(x) \right) \right] = 0$$

for every n , where μ_{Ψ_n} is the cylinder set measure on E_n induced by Ψ_n .

PROOF. By the definition of the inductive topology φ is continuous in E if and only if φ is continuous in E_n for every n . Assume $\varphi(x) = \mu^\wedge(x)$ for a $\sigma(E', E)$ -Radon measure μ . Then for every $x \in E_n$ we have $\varphi(x) = \hat{\mu}^n(x)$, where $\mu^n = \psi_n(\mu)$ and $\psi_n: E' \rightarrow E'_n$ is the natural projection. Thus applying theorem 5-2 to $\varphi|_{E_n}$, condition (b) holds.

Conversely assume (1), (2) and (3). Let μ be the cylinder set measure determined by φ (the conditions (1) and (2) define the cylinder set measure μ). Then by (3) and theorem 5-2, $\mu^n = \psi_n(\mu)$ is a $\sigma(E'_n, E_n)$ -Radon measure on E'_n for every n . Thus μ is also $\sigma(E', E)$ -Radon by Bourbaki [2], Théorème 2, §4, n°3.

This proves the theorem.

7.2. Fréchet space

In the rest of this section, we treat the projective limit.

7-2-1. Tightness of measures

Let $E = \varprojlim_n E_n$ be the projective limit of locally convex spaces $\{E_n\}_{n=1}^\infty$, $\rho_n: E \rightarrow E_n$ be the natural projection and $\rho_{n,n+1}$ be the mapping $E_{n+1} \rightarrow E_n$ so that $\rho_{n,n+1} \circ \rho_{n+1} = \rho_n$. In the sequel we assume always $\rho_n(E_n)$ is dense in E_n for every n . Then we have $E' = \varprojlim_n E'_n$ as a set (see Köthe [6], p. 290).

Let E be a Fréchet space with a basis $\{|_n\}$ of continuous seminorms such that $|_n \leq |_{n+1}$ for every n . Let E_n be the completion of the normed space $E/\{|_n=0\}$, then we have $E = \varprojlim_n E_n$. We denote by $|_{E'_n}$ or $|'_n$ the dual norm in E'_n . We consider the norm $|_{E'_n}$ as a function on E' by setting

$$|x'|_{E'_n} = \begin{cases} |y'|_{E'_n} & \text{if } x' = \rho'_n(y'), y' \in E'_n \\ +\infty & \text{if there is not } y' \in E'_n \text{ such as } x' = \rho'_n(y'). \end{cases}$$

We remark the transpose ρ'_n is one-to-one since $\rho_n(E)$ is dense. We also set for positive R

$$\frac{|x'|_{E'_n}}{R + |x'|_{E'_n}} = 1,$$

if there is no y' in E'_n so that $x' = \rho'_n(y')$.

THEOREM 7-10. Let E be a Fréchet space and $\{\mu_\alpha\}_{\alpha \in A}$ be a family of $\sigma(E', E)$ -Radon probability measures on E' . Then the following conditions are equivalent.

- (1) $\{\mu_\alpha\}$ is tight in $E'_{\sigma(E', E)}$.
- (2) It holds

$$\lim_n \lim_{R \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'} \frac{|x'|'_n}{R + |x'|'_n} d\mu_\alpha(x') = 0.$$

- (3) It holds

$$\lim_{n \rightarrow +\infty} \sup_{\alpha \in A} \int_{E'} \frac{|x'|'_n}{n + |x'|'_n} d\mu_\alpha(x') = 0.$$

PROOF. Assume $\{\mu_\alpha\}_{\alpha \in A}$ be tight. Then for every $\varepsilon > 0$ there is a $\sigma(E', E)$ -compact set K such that $\inf_{\alpha \in A} \mu_\alpha(K) \geq 1 - \varepsilon/2$. Since E is a Fréchet space, the polar $K^\circ \subset E$ is a neighborhood of 0 in E . Thus there is an n_0 such that $\{|x|_{n_0} \leq \delta\} \subset K^\circ$ for some $\delta > 0$. Hence the bipolar $K^{\circ\circ} \supset K$ is a $\sigma(E', E)$ -compact set which is contained in $\rho'_{n_0}(E'_{n_0})$. So there is an R_0 such that $K \subset \{x' \in E'; |x'|'_{n_0} \leq R_0\}$. For every $n \geq n_0$ and $R > R_0$, we have

$$\begin{aligned} & \sup_{\alpha \in A} \int_{E'} \frac{|x'|'_n}{R + |x'|'_n} d\mu_\alpha(x') \\ & \leq \sup_{\alpha \in A} \int_{\{|x'|'_n \leq R_0\}} \frac{R_0}{R + R_0} d\mu_\alpha(x') \\ & \quad + \sup_{\alpha \in A} \int_{\{|x'|'_n > R_0\}} 1 d\mu_\alpha(x') \\ & \leq \frac{R_0}{R + R_0} + \frac{\varepsilon}{2}, \end{aligned}$$

since $|x'|'_n \leq |x'|'_{n_0}$ for $n \geq n_0$ and $\{|x'|'_n > R_0\} \subset \{|x'|'_{n_0} > R_0\}$. This shows (2).

Obviously (2) implies (3), since $|x'|'_n \geq |x'|'_{n+1}$.

Let $U_n = \{x \in E; |x|_n \leq \frac{n}{1}\}$, then we have $\bigcup_{n=1}^{\infty} U_n^\circ = E'$ and U_n° is $\sigma(E', E)$ -compact. We shall show for every $\varepsilon > 0$ there is an n_0 such that $\inf_{\alpha \in A} \mu_\alpha(U_{n_0}^\circ) \geq 1 - \varepsilon$. We have

$$\begin{aligned} \mu_\alpha((U_n^\circ)^c) &= \mu_\alpha\{x'; x' \notin U_n^\circ\} \\ &= \mu_\alpha\{x' \in E'; +\infty \geq \sup_{x \in U_n} |\langle x, x' \rangle| > 1\} \end{aligned}$$

$$\begin{aligned}
&= \mu_\alpha \{x' \in E'; \sup_{|x|_n \leq 1} |<x, x'>| > n\} \\
&= \mu_\alpha \left\{x' \in E'; \left| \frac{x'}{n} \right|_{E'_n} \right\} > 1 \\
&\leq 2 \int_{E'} \frac{|x'|'_n}{n + |x'|'_n} d\mu_\alpha(x').
\end{aligned}$$

By (3), we can see $\{\mu_\alpha\}$ is tight.

This completes the proof.

THEOREM 7-11. Let E be a Fréchet space with a basis $\{|_n\}$ of continuous seminorms such that $|_n \leq |_{n+1}$ for every n . We assume there is an isomorphism Ψ_n of E'_n into $L^0(\Omega_n, \mathfrak{A}_n, \mathbf{P}_n)$. Then a family $\{\mu_\alpha\}_{\alpha \in A}$ of $\sigma(E', E)$ -Radon probability measures is tight in $\sigma(E', E)$ if and only if it holds

$$\lim_{n \rightarrow \infty} \lim_{R \rightarrow +\infty} \sup_{\beta \in A} \int_{E'} \int_{\Omega_n} \frac{|\Psi_n(x')(\omega)|}{R + |\Psi_n(x')(\omega)|} d\mathbf{P}_n(\omega) d\mu_\alpha(x') = 0,$$

where if $x' \notin \rho'_n(E'_n)$, we put $\Psi_n(x') \equiv +\infty$ and if $x' \in \rho'_n(E'_n)$, we put $\Psi_n(x') = \Psi_n(\rho_n'^{-1}(x'))$.

PROOF. Let $\{\mu_\alpha\}_{\alpha \in A}$ be tight. Then by the way similar to the proof of theorem 7-10, for every $\varepsilon > 0$, we can find $R_0 > 0$ and n_0 such that

$$\sup_{\alpha \in A} \mu_\alpha \{x' \in E'; |x'|'_{n_0} > R_0\} < \frac{\varepsilon}{2}.$$

Thus remarking $|x'|'_n \leq |x'|'_{n_0}$ for every $n \geq n_0$, we have for $n \geq n_0$, $R \geq R_0$,

$$\sup_{\alpha \in A} \mu_\alpha \{x' \in E'; |x'|'_n > R\} < \frac{\varepsilon}{2}.$$

We put $N_0 = \max \{n_0, [R_0] + 1\}$, where $[R]$ is the largest integer not exceeding R . Then we have for every $n \geq N_0$

$$\sup_{\alpha \in A} \mu_\alpha \{x' \in E'; |x'|'_n > n\} < \frac{\varepsilon}{2}.$$

Thus the theorem is reduced to E'_n . Since Ψ_n is an isomorphism, there is a δ_n , $0 < \delta_n < 1$, such that $|x'|'_n \leq \delta_n$ implies $|\Psi_n(x')|_{L^0(\Omega_n)} < \varepsilon/2$. For every $R_n > n/\delta_n$, we have

$$\begin{aligned}
& \sup_{\alpha \in A} \int_{E'} \int_{\Omega_n} \frac{|\Psi_n(x')(\omega)|}{R_n + |\Psi_n(x')(\omega)|} d\mathbf{P}_n(\omega) d\mu_\alpha(x') \\
& \leq \sup_{\alpha} \int_{\{|x'|'_n > n\}} 1 d\mu_\alpha(x') + \sup_{\alpha} \int_{\{|x'|'_n \leq n\}} \frac{\varepsilon}{2} d\mu_\alpha(x') \\
& \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

hence we have for $n \geq N_0$

$$\lim_{n \rightarrow +\infty} \sup_{\alpha} \int_{E'} \int_{\Omega_n} \frac{|\Psi_n(x')(\omega)|}{R + |\Psi_n(x')(\omega)|} d\mathbf{P}_n(\omega) d\mu_\alpha(x') < \varepsilon.$$

Thus we have

$$\overline{\lim}_{n \rightarrow +\infty} \lim_{R \rightarrow +\infty} \sup_{\alpha} \int_{E'} \int_{\Omega_n} \frac{|\Psi_n(x')(\omega)|}{R + |\Psi_n(x')(\omega)|} d\mathbf{P}_n(\omega) d\mu_\alpha(x') = 0.$$

We show the converse. For every $\varepsilon > 0$, there is an N such that for every $n \geq N$

$$\lim_{R \rightarrow +\infty} \sup_{\alpha} \int_{E'} \int_{\Omega_n} \frac{|\Psi_n(x')(\omega)|}{R + |\Psi_n(x')(\omega)|} d\mathbf{P}_n(\omega) d\mu_\alpha(x') < \frac{\varepsilon}{2}.$$

There is a δ such that $|\Psi_N(x')|_{L^0(\Omega_N)} < \delta$ implies $|x'|'_N < \varepsilon/2$. We have

$$\begin{aligned}
& \mu_\alpha(|x'|'_N > R) \\
& \leq \int_{E'} 1 \wedge \left[\frac{|x'|'_N}{R} \right] d\mu_\alpha(x') \\
& \leq \int_{\{|x'|'_N < \frac{\varepsilon}{2}\}} \frac{\varepsilon}{2} d\mu_\alpha(x) + \int_{\{|x'|'_N > \frac{\varepsilon}{2}\}} 1 \wedge \left[\frac{|x'|'_N}{R} \right] d\mu_\alpha(x') \\
& \leq \frac{\varepsilon}{2} + \int_{\{|x'|'_N > R \cdot \frac{\varepsilon}{2}\}} \frac{1}{\delta} \left| \Psi_N\left(\frac{x'}{R}\right) \right|_{L^0(\Omega_N)} d\mu_\alpha(x').
\end{aligned}$$

Thus we have

$$\begin{aligned}
& \mu_\alpha\{|x'|'_N > R\} \\
& \leq \frac{\varepsilon}{2} + \frac{1}{\delta} \int_{E'} \int_{\Omega_N} \frac{|\Psi_N(x')(\omega)|}{R + |\Psi_N(x')(\omega)|} d\mathbf{P}_N(\omega) d\mu_\alpha(x'),
\end{aligned}$$

which show for sufficiently large R

$$\sup_{\alpha} \mu_\alpha(|x'|'_N > R) < \varepsilon.$$

The proof is complete.

7-2-2. $\sigma(E', E)$ -Radon extensibility of a cylinder set measure

Here we shall prove the $\sigma(E', E)$ -Radon extensibility for a cylinder set measure μ on E .

THEOREM 7-12. Let E be a Fréchet space satisfying the conditions of theorem 7-11. We assume also E has the metric approximation property by a net $\{\pi_\lambda\}_{\lambda \in A}$ of operators of finite rank. Then a cylinder set measure μ is $\sigma(E', E)$ -Radon if and only if it holds

- (1) the characteristic functional $\mu^\wedge(x)$ is continuous in E , and
- (2) it holds

$$(***) \quad \lim_{n \rightarrow +\infty} \lim_{R \rightarrow +\infty} \sup_{\lambda \in A} \int_{E'} \int_{\Omega_n} \frac{|\Psi_n(x')(\omega)|}{R + |\Psi_n(x')(\omega)|} d\mathbf{P}_n(\omega) d\mu_\lambda(x') = 0,$$

where $\mu_\lambda = \pi'_\lambda(\mu)$.

PROOF. Let μ be $\sigma(E', E)$ -Radon. Then for every $\varepsilon > 0$, there is a $\sigma(E', E)$ -compact set K such that $\mu(K) \geq 1 - \varepsilon$. Since $\{\pi_\lambda\}$ is equicontinuous, there is a neighborhood U of 0 in E such that $\pi_\lambda(U) \subset K^\circ$ for every $\lambda \in A$, where K° is the polar of K which is a neighborhood of 0 in E . In particular we have $(\pi'_\lambda)^{-1}(U^\circ) \supset K^{\circ\circ}$ (the bipolar), hence $\mu((\pi'_\lambda)^{-1}(U^\circ)) \geq 1 - \varepsilon$, which shows the tightness of $\{\mu_\lambda\}_{\lambda \in A}$. By theorem 7-11, the condition (***) holds.

Conversely assume (***) is valid. Then by theorem 7-11, $\{\mu_\lambda\}_{\lambda \in A}$ is tight. Remarking $\pi_\lambda x \rightarrow x$, uniformly on compact sets, we have $\mu_\lambda^\wedge(x) = \mu^\wedge(\pi_\lambda x) \rightarrow \mu^\wedge(x)$, uniformly on compact sets. Thus by Schwartz [17], Proposition (III; 5.2), μ is a $\sigma(E', E)$ -Radon measure and $\mu_\lambda \rightarrow \mu$, weakly in $E'_{\sigma(E', E)}$.

This completes the proof.

Let μ_{Ψ_n} be the cylinder set measure on $Z(E_n, E'_n)$ defined by the isomorphism $\Psi_n: E'_n \rightarrow L^0(\Omega_n, \mathfrak{A}_n, \mathbf{P}_n)$. Then the condition (***) is

$$\lim_{n \rightarrow +\infty} \lim_{R \rightarrow +\infty} \sup_{\lambda} \int_{E'} \int_{E_n} \frac{|\langle x, \pi'_\lambda x' \rangle|}{R + |\langle x, \pi'_\lambda x' \rangle|} d\mu_{\Psi_n}(x) d\mu(x') = 0,$$

where if $x' \notin \rho'_n(E'_n)$ we put $\frac{|\langle x, x' \rangle|}{R + |\langle x, x' \rangle|} = 1$ for x in E_n , and if $x' \in \rho'_n(E'_n)$ we put $\langle x, x' \rangle = \langle x, (\rho'_n)^{-1}(x') \rangle$.

7-2-3. S-topology

We prove the existence of the S-topology on a Fréchet space. A result

corresponding to this case was announced in Mouchtari [12], § 4, which is slightly different from ours.

THEOREM 7-13. Let $E = \varprojlim E_n$ be the projective limit of Banach spaces such that there is an isomorphism $\Psi_n: E'_n \rightarrow L^0(\Omega_n, \mathfrak{A}_n, \mathbf{P}_n)$. Assume E has the metric approximation property. Then if each E_n has the S-topology τ_n , then there is the S-topology τ on E .

PROOF. Let τ_0 be the vector space topology generated by the (F)-seminorms

$$x \longrightarrow \int_{E'} \frac{|\langle x, x' \rangle|}{1 + |\langle x, x' \rangle|} d\mu_\phi(x') = \psi(x, \phi),$$

where ϕ varies over all $\sigma(E', E)$ -Radon random linear functionals. The topology τ_0 is stronger than the weak topology and weaker than the Fréchet topology of E , see theorem 4-1. We prove τ_0 is the S-topology.

Let ϕ be a random linear functional which is $\sigma(E', E)$ -Radon. Then obviously ϕ is continuous in τ_0 .

Conversely let $\phi: (E, \tau_0) \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ be a continuous random linear functional. Since the (F)-seminorms given above is a basis, not necessarily sub-basis, for every $\varepsilon > 0$, there is a $\delta > 0$ and a $\sigma(E', E)$ -Radon random linear functional $\phi_0: E \rightarrow L^0(\Omega_0)$ such that

$$\{x \in E; \psi(x, \phi_0) < \delta\} \subset \{x \in E; |\phi(x)|_{L^0(\Omega_\phi)} < \varepsilon\}.$$

Since ϕ_0 is $\sigma(E', E)$ -Radon, by theorem 7-12 there are N and R_0 such that

$$\sup_{\alpha \in A} \int_{E'} \int_{E_n} \frac{|\langle x, \pi'_\alpha x' \rangle|}{R + |\langle x, \pi'_\alpha x' \rangle|} d\mu_{\Psi_n}(x) d\mu_{\phi_0}(x') < \frac{\varepsilon \delta}{2}$$

for every $n \geq N$, $R \geq R_0$, where $\{\pi_\alpha\}_{\alpha \in A}$ is the collection of operators of finite rank which gives the metric approximation property. By the Fubini's theorem we have

$$\begin{aligned} & \int_{E'} \int_{E_n} \frac{|\langle x, \pi'_\alpha x' \rangle|}{R + |\langle x, \pi'_\alpha x' \rangle|} d\mu_{\Psi_n}(x) d\mu_{\phi_0}(x') \\ &= \int_{(\pi'_\alpha)^{-1}\rho'_n(E'_n)} \int_{E_n} \frac{|\langle x, \pi'_\alpha x' \rangle|}{R + |\langle x, \pi'_\alpha x' \rangle|} d\mu_{\Psi_n}(x) d\mu_{\phi_0}(x') \\ & \quad + \int_{(\pi'_\alpha)^{-1}\rho'_n(E'_n)^c} 1 d\mu_{\phi_0}(x') \end{aligned}$$

$$\begin{aligned}
&= \int_{\rho'_n(E'_n)} \int_{E_n} \frac{|\langle x, x' \rangle|}{R + |\langle x, x' \rangle|} d\mu_{\Psi_n}(x) d\pi'_\alpha(\mu_{\phi_0})(x') \\
&\quad + \int_{(\pi'_n)^{-1}\rho'_n(E'_n)^c} 1 d\mu_{\phi_0}(x') \\
&= \int_{E_n} \int_{\pi'_\alpha(E') \cap \rho'_n(E'_n)} \frac{|\langle x, x' \rangle|}{R + |\langle x, x' \rangle|} d\pi'_\alpha(\mu_{\phi_0})(x') d\mu_{\Psi_n}(x) \\
&\quad + \int_{(\pi'_\alpha)^{-1}\rho'_n(E'_n)^c} 1 d\mu_{\phi_0}(x') \\
&= \int_{E_n} \int_{(\pi'_n)^{-1}\rho'_n(E'_n)} \frac{|\langle x, \pi'_\alpha x' \rangle|}{R + |\langle x, \pi'_\alpha x' \rangle|} d\mu_{\phi_0}(x') d\mu_{\Psi_n}(x) \\
&\quad + \int_{E_n} \int_{(\pi'_\alpha)^{-1}\rho'_n(E'_n)^c} 1 d\mu_{\phi_0}(x') d\mu_{\Psi_n}(x) \\
&= \int_{E_n} \int_{E'} \frac{|\langle x, \pi'_\alpha x' \rangle|}{R + |\langle x, \pi'_\alpha x' \rangle|} d\mu_{\phi_0}(x') d\mu_{\Psi_n}(x).
\end{aligned}$$

We put

$$G_\alpha^n = \left\{ x \in E_n : \int_{E'} \frac{|\langle x, \pi'_\alpha x' \rangle|}{R + |\langle x, \pi'_\alpha x' \rangle|} d\mu_{\phi_0}(x') < \delta \right\}.$$

Then G_α^n belongs to $Z(E_n, E'_n)$. For every $\lambda \in A$, we have $\mu_{\Psi_n}((G_\alpha^n)^c) < \varepsilon/2$ by the same argument as in theorem 4-4. For $x \in G_\alpha^n$ we have

$$\int_{E'} \frac{|\langle x, \pi'_\alpha x' \rangle|}{R + |\langle x, \pi'_\alpha x' \rangle|} d\mu_{\phi_0}(x') < \frac{\varepsilon}{2}.$$

Consequently we have

$$\begin{aligned}
&\int_{E_n} \int_{E'} \frac{|\langle x, \pi'_\alpha x' \rangle|}{R + |\langle x, \pi'_\alpha x' \rangle|} d\mu_{\phi_0}(x') d\mu_{\Psi_n}(x) \\
&\leq \varepsilon,
\end{aligned}$$

for every $\alpha \in A$, every $n \geq N$ and every $R \geq R_0$, by the same proof as in theorem 4-4. Thus by theorem 7-12, μ_ϕ is a $\sigma(E', E)$ -Radon measure.

Let $E = \varprojlim_n E_n$ be the projective limit of locally convex spaces $\{E_n\}$ such that $\rho_n(E)$ is dense in E_n for every n , where $\rho_n: E \rightarrow E_n$ is the natural projection. Now assume each E_n has the S-topology τ_n . Since the S-topology τ_n is determined by

the family of all $\sigma(E'_n, E_n)$ -Radon random linear functionals, τ_{n+1} is stronger than τ_n , that is, the mapping $\rho_{n,n+1}: (E_{n+1}, \tau_{n+1}) \rightarrow (E_n, \tau_n)$ is continuous. In fact, for the $\sigma(E'_n, E_n)$ -Radon measure μ , the image $\nu = \rho'_{n,n+1}(\mu)$ is also a $\sigma(E'_{n+1}, E_{n+1})$ -Radon and we have

$$\begin{aligned} \rho_{n,n+1} \left(\left\{ x \in E_{n+1}; \int_{E'_{n+1}} \frac{|\langle x, x' \rangle|}{1 + |\langle x, x' \rangle|} d\nu(x') < \varepsilon \right\} \right) \\ \subset \left\{ x \in E_n; \int_{E'_n} \frac{|\langle x, x' \rangle|}{1 + |\langle x, x' \rangle|} d\mu(x') < \varepsilon \right\}. \end{aligned}$$

Thus we can consider the projective limit topology τ on E of $\{\tau_n\}_{n=1}^\infty$.

We show this projective limit topology is the S-topology in the case of theorem 7-13.

THEOREM 7-14. Let $E = \varprojlim_n E_n$ be the projective limit of barrelled locally convex spaces $\{E_n\}$ such that $\rho_n(E)$ is dense in E_n for each n . We assume each E_n has the S-topology τ_n and E also has the S-topology τ_0 . Then the S-topology τ_0 must be the projective limit topology τ of $\{\tau_n\}_{n=1}^\infty$.

PROOF. Since the projective limit τ is determined by $\sigma(E'_n, E_n)$ -Radon measures ($n=1, 2, \dots$) which are also Radon in $\sigma(E', E)$ by the natural embedding, τ is weaker than the S-topology (theorem 4-1). To see the converse, we prove for every random linear functional $\phi: E \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ which is $\sigma(E', E)$ -Radon, ϕ is also continuous in τ .

For every $\varepsilon > 0$, we take a $\sigma(E', E)$ -compact set K such that $\mu_\phi(K) \geq 1 - \varepsilon/2$. The set K is contained in $\rho'_n(E'_n)$ for some n . We shall identify $\rho'_n(E'_n)$ with E'_n , so assume K is $\sigma(E'_n, E_n)$ -compact set in E'_n . We put

$$\nu(A) = \frac{1}{\mu_\phi(K)} \mu_\phi(A \cap K)$$

for every $A \in \mathcal{Z}(E'_n, E_n)$. Then ν is a $\sigma(E'_n, E_n)$ -Radon measure. Hence the random linear functional $\phi_0: E_n \rightarrow L^0(\mathfrak{B}(E'_n, \sigma(E'_n, E_n)), \nu)$ defined by $\phi_0(x) = \langle x, \cdot \rangle$ is τ_n -continuous. Thus for every $\varepsilon > 0$, there is a τ_n -neighborhood W_n of E_n such that

$$\int_{E'_n} \frac{|\langle x, x' \rangle|}{1 + |\langle x, x' \rangle|} d\nu(x') < \frac{1}{\mu_\phi(K)} \frac{\varepsilon}{2},$$

for every $x \in W_n$. We put $U_n = \rho_n^{-1}(W_n)$. Then U_n is a neighborhood of 0 in τ

and for every $x \in U_n$ we have

$$\begin{aligned} |\phi(x)|_{L^0(\Omega_\phi)} &= \int_{E'} \frac{|\langle x, x' \rangle|}{1 + |\langle x, x' \rangle|} d\mu_\phi(x') \\ &\leq \int_K \frac{|\langle x, x' \rangle|}{1 + |\langle x, x' \rangle|} d\mu_\phi(x') + \frac{\varepsilon}{2} \\ &\leq \int_{E'_n} \frac{|\langle x, x' \rangle|}{1 + |\langle x, x' \rangle|} d\mu(x') \mu_\phi(K) + \frac{\varepsilon}{2} \\ &< \varepsilon, \end{aligned}$$

which shows the τ -continuity of ϕ .

This proves the theorem.

COROLLARY 7-15. In theorem 7-13, the S-topology τ is the projective limit of $\{\tau_n\}_{n=1}^\infty$.

REMARK 7-16. In a general setting of theorem 7-14, it is an open problem if there is really the Sazonov topology τ_0 on E .

Let $\phi: E \rightarrow L^0(\Omega_\phi, \mathfrak{A}_\phi, \mathbf{P}_\phi)$ be a random linear functional. Then ϕ is said to be of weakly p -th order, $0 < p < +\infty$, if $\phi(E) \subset L^p(\Omega_\phi)$. For example if $\phi(x)$ is a Gaussian random variable for every $x \in E$, then ϕ is of weakly p -th order for every p .

THEOREM 7-17. Let $E = \varprojlim_n E_n$ be the projective limit of locally convex spaces E_n such that $\rho_n(E)$ is dense in E_n for every n . We assume there is the Sazonov topology τ_n on E_n and let τ be the projective limit of $\{\tau_n\}_{n=1}^\infty$. Then for every random linear functional $\phi: E \rightarrow L^0(\Omega_\phi)$ of weakly p -th order, $0 < p < +\infty$, if ϕ is τ -continuous into $L^p(\Omega_\phi)$ the cylinder set measure μ_ϕ is $\sigma(E', E)$ -Radon. Moreover in this case we have $\mu_\phi(\rho'_n(E'_n)) = 1$ for sufficiently large n .

PROOF. Since ϕ is continuous from (E, τ) into $L^p(\Omega_\phi)$, there is a neighborhood V of 0 with respect to τ_n for some n such that it holds

$$\left(\int_{\Omega_\phi} |\phi(x)(\omega)|^p d\mathbf{P}_\phi(\omega) \right)^{1/p} < 1$$

for every $x \in \rho_n^{-1}(V)$. Since the L^p -quasi-norm is homogeneous, we can regard as ϕ is continuous from (E_n, τ_n) into $L^p(\Omega_\phi)$. Thus μ_ϕ is the image of a $\sigma(E'_n, E_n)$ -Radon measure on E'_n under the mapping $\rho'_n: E'_n \rightarrow E'$.

This proves the theorem.

The above theorem can be slightly generalized as follows. Let $\phi: E \rightarrow L^0(\Omega_\phi)$ be a τ -continuous random linear functional and assume that $L^0(\Omega_\phi)$ induce a topology on $\phi(E)$ defined by a homogeneous function (for example quasi-norm). Then μ_ϕ is $\sigma(E', E)$ -Radon and there is an n such that $\mu_\phi(\rho'_n(E'_n)) = 1$.

In general, even if a random linear functional is of weakly p -th order and there is the S -topology, $\phi: (E, \tau) \rightarrow L^p(\Omega_\phi)$ is not continuous. For example let $E = l^p (2 < p < +\infty)$, then $E' = l^{p'}$, $1 < p' < 2$. Since E' is not of type p' -stable, there is a Radon probability measure μ of p' -th order such that the cylinder set measure ν defined by

$$\nu^\wedge(x) = \exp \left[- \int_{E'} | \langle x, x' \rangle |^{p'} d\mu(x') \right], \quad x \in E,$$

is not Radon on E (see Okazaki [13]). Hence in this case for the random linear functional ϕ derived by μ , $\phi: (E, \tau) \rightarrow L^{p'}(\Omega_\phi)$ is not continuous. In fact, if continuous, then the cylinder set measure ν must be Radon on E' . We remark in this case ϕ is continuous into $L^q(\Omega_\phi)$ for every $q < p'$.

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