ADMISSIBLE TRANSLATES OF MEASURES ON A TOPOLOGICAL GROUP

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Abstract

The algebraical and the measure theoretical properties of admissible (singular) translates on a topological group are studied. It is shown the sets of all admissible (singular) translates such as $RA(\mu)$, $LA(\mu)$, $TA(\mu)$, $RE(\mu)$, $LE(\mu)$, $TE(\mu)$, $RS(\mu)$, $LS(\mu)$, $TS(\mu)$, are Borel subsets of the topological group. If μ_1 is right quasi-invariant and μ_2 is left quasi-invariant then μ_1, μ_2 are equivalent. In a locally compact case each right (or left) quasi-invariant measure is equivalent to the Haar measure. For a right quasi-invariant measure μ on a separable group, a right translation invariant measure μ_0 which is equivalent to μ is constructed. Using this fact the following result is proved, which gives some informations about the measure theoretical size of the set of admissible translates.

THEOREM 1. For each Radon probability measure μ on a separable group G , $\mu(RE(\mu))=0$, or $RE(\mu)$ is a locally compact group and the restriction $\mu|RE(\mu)$ is equivalent to the Haar measure on $RE(\mu)$.

1. Introduction

Let G be a topological group, $\mathcal{B}(G)$ be the Borel field and $M(G)$ be the set of all probability measures on $\mathcal{B}(G)$. For an $a \in G$ let R_a (resp. L_a) be the right (resp. left) translation, that is,

$$R_a(x) = xa \quad (\text{resp. } L_a(x) = ax) \quad \text{for } x \in G.$$

For $\mu, \nu \in M(G)$, we write $\mu \ll \nu$ (resp. $\mu \perp \nu$) when μ is absolutely continuous (resp. singular) with respect to ν . If μ and ν are mutually absolutely continuous (=equivalent) we denote $\mu \sim \nu$.

NOTATIONS. $RA(\mu) = \{a \in G; R_a(\mu) \ll \mu\}$, $LA(\mu) = \{a \in G; L_a(\mu) \ll \mu\}$, $TA(\mu) = RA(\mu) \cap LA(\mu)$, $RE(\mu) = \{a \in G; R_a(\mu) \sim \mu\}$, $LE(\mu) = \{a \in G; L_a(\mu) \sim \mu\}$, $TE(\mu) = RE(\mu) \cap LE(\mu)$, $RS(\mu) = \{a \in G; R_a(\mu) \perp \mu\}$, $LS(\mu) = \{a \in G; L_a(\mu) \perp \mu\}$, $TS(\mu) = RS(\mu) \cap LS(\mu)$, $(R_a(\mu), L_a(\mu))$ are the image measures of μ by R_a, L_a .

The elements of $RA(\mu)$ (resp. $LA(\mu)$, $TA(\mu)$) is called the right (resp. left, two sided) admissible translates of μ . The elements of $RS(\mu)$ (resp. $LS(\mu)$, $TS(\mu)$) is called the right (resp. left, two sided) singular translates of μ .

In this paper we study the algebraical and the measure theoretical properties of admissible (singular) translates.

In section 2, we show the sets $RE(\mu)$, $LE(\mu)$ are subgroups. The semigroup properties are also proved such as $TS(\mu) \cdot TE(\mu) \subset TS(\mu)$, $TE(\mu) \cdot TS(\mu) \subset TS(\mu)$.

In section 3, we investigate the image of admissible (singular) translates by a Borel homomorphism $\psi: G \rightarrow H$. We show, for example, $\psi(RA(\mu)) \subset RA(\psi(\mu))$, $\psi(RE(\mu)) \subset RE(\psi(\mu))$, and $\psi^{-1}(RS(\psi(\mu))) \subset RS(\mu)$. Applying this result we prove if G is abelian and $\mu = v_1 * v_2 * \dots * v_n$ then it hold $TA(v_1) \cdot TA(v_2) \cdot \dots \cdot TA(v_n) \subset TA(\mu)$ and $TE(v_1) \cdot TE(v_2) \cdot \dots \cdot TE(v_n) \subset TE(\mu)$.

In the first half of section 4, we prove the subsets $RA(\mu)$, $RE(\mu)$, $RS(\mu)$ are Borel subsets when G is separable (=second countable). The work of Dubins and Freedman [4] is essential. In the latter half we investigate the quasi-invariant measures. We show if μ_1 is right quasi-invariant and μ_2 is left quasi-invariant then μ_1 is equivalent to μ_2 . Similar result was given by Umemura [8] in case G is a vector space. If G is a locally compact group then each right quasi-invariant measure is equivalent to the Haar measure.

In section 5 we show the "size" of the set of admissible translates is small in the following sense. For a separable group G and a Radon measure $\mu \in M(G)$ we have $(RE(\mu)) = 0$, or $RE(\mu)$ is a locally compact group and the restriction $\mu|_{RE(\mu)}$ is equivalent to the Haar measure on $RE(\mu)$. The equivalence with the Haar measure was investigated by Brockett [3]. Similar result for $TA(\mu)$ is also given in case G is abelian and μ is symmetric. If G is a separable vector space and $RE(\mu)$ is a linear subspace, then $RE(\mu)$ is finite dimensional or $\mu(RE(\mu)) = 0$. This was proved by Umemura [8], Zinn [10].

2. Algebraical properties of admissible (singular) translates

PROPOSITION 1. (1) The sets $RA(\mu)$, $LA(\mu)$ and $TA(\mu)$ are semigroups.
 (2) The sets $RE(\mu)$, $LE(\mu)$ and $TE(\mu)$ are groups.

PROOF. The first assertion is easy. To see $RE(\mu)$ is a group, we show $x^{-1} \in RE(\mu)$ for every $x \in RE(\mu)$. For every $A \in \mathcal{B}(G)$, $\mu(A)=0$ is equivalent to $\mu(Ax^{-1})=0$, hence $\mu(Ax)=0$ is equivalent to $\mu(A)=0$ since $\mathcal{B}(G)$ is translation invariant (that is, $\mathcal{B}(G)=\{Ax; A \in \mathcal{B}(G)\}$). This shows $R_{x^{-1}}(\mu) \sim \mu$.

PROPOSITION 2. Suppose $a \in RA(\mu)$ and $a^n=e$ for some n . Then it hold $a \in RE(\mu)$ and each two measures in $\{\mu, R_a(\mu), R_{a^2}(\mu), \dots, R_{a^{n-1}}(\mu)\}$ are equivalent.

PROOF. Let A be $R_a(\mu)(A)=\mu(Aa^{-1})=0$. Then we have $\mu(Aa^{-2})=\mu(Aa^{-3})=\dots=\mu(Aa^{-n})=\mu(A)=0$.

PROPOSITION 3.

- (1) $RA(\mu) \cdot RS(\mu) \subset RS(\mu)$
- (2) $LS(\mu) \cdot LA(\mu) \subset LS(\mu)$

PROOF. (1) Let $a \in RA(\mu)$ and $b \in RS(\mu)$. Since $R_b(\mu) \perp \mu$, there is an $A \in \mathcal{B}(G)$ such that $\mu(A)=0$, $\mu(A^c b^{-1})=0$. By $R_a(\mu) \ll \mu$, we have $\mu(A^c b^{-1} a^{-1})=R_{ab}(\mu)(A^c)=0$. Thus it holds $ab \in RS(\mu)$.

(2) Let $a \in LS(\mu)$ and $b \in LA(\mu)$. Take an $A \in \mathcal{B}(G)$ so that $\mu(A)=0$, $\mu(a^{-1} A^c)=0$. Since $b \in LA(\mu)$, we have $\mu(b^{-1} a^{-1} A^c)=L_{ab}(\mu)(A^c)=0$, that is, $ab \in LS(\mu)$.

PROPOSITION 4.

- (1) $RS(\mu) \cdot RE(\mu) \subset RS(\mu)$
- (2) $LE(\mu) \cdot LS(\mu) \subset LS(\mu)$

PROOF. (1) Let $a \in RS(\mu)$ and $b \in RE(\mu)$. By $R_a(\mu) \perp \mu$, there is an $A \in \mathcal{B}(G)$ such that $\mu(A)=0$ and $\mu(A^c a^{-1})=0$. Since $R_b(\mu) \sim \mu$ and $0=\mu(Ab b^{-1})=R_b(\mu)(Ab)$, we have $\mu(Ab)=0$. On the other hand we have $R_{ab}(\mu)(A^c b)=\mu(A^c b b^{-1} a^{-1})=\mu(A^c a^{-1})=0$. This shows $R_{ab}(\mu) \perp \mu$, that is, $ab \in RS(\mu)$.

(2) Let $a \in RE(\mu)$ and $b \in LS(\mu)$. Take $A \in \mathcal{B}(G)$ so that $\mu(A)=0$, $\mu(b^{-1} A^c)=0$. Since $\mu \sim L_a(\mu)$ and $L_a(\mu)(aA)=\mu(A)=0$, we have $\mu(aA)=0$. On the other hand we have $L_{ab}(\mu)(aA^c)=\mu(b^{-1} a^{-1} aA^c)=\mu(b^{-1} A^c)=0$. This

shows $L_{ab}(\mu) \perp \mu$.

PROPOSITION 5.

- (1) $TS(\mu) \cdot TE(\mu) \subset TS(\mu)$
- (2) $TE(\mu) \cdot TS(\mu) \subset TS(\mu)$

PROOF. The assertions follow from propositions 3 and 4.

3. Measure theoretical properties of admissible (singular) translates

PROPOSITION 6. Let $\mu \in M(G)$ and H be a Borel subgroup of G such that $\mu(H)=1$. Then we have $RA(\mu) \cup LA(\mu) \subset H$.

PROOF. If there is an $a \in RA(\mu)$ such that $a \notin H$, then we have $\mu(H^c a^{-1}) = R_a(\mu)(H^c) = 0$ and $H \cap (Ha^{-1}) = \emptyset$. Thus $\mu(G) \geq \mu(H) + \mu(Ha^{-1}) = 2$, which contradicts to $\mu(G) = 1$.

LEMMA 1. Let G, H be two topological groups and $\psi: G \rightarrow H$ be a Borel holomorphism. Then we have

- (1) $\psi(R_a(\mu)) = R_{\psi(a)}(\psi(\mu))$, and
- (2) $\psi(L_a(\mu)) = L_{\psi(a)}(\psi(\mu))$.

PROOF. (1) First we remark $\psi^{-1}(A) \cdot a^{-1} = \psi^{-1}(A \cdot \psi(a)^{-1})$. In fact if $x \in \psi^{-1}(A \cdot \psi(a)^{-1})$ then it holds $\psi(x) \in A \cdot \psi(a)^{-1}$, that is, $\psi(xa) = \psi(x)\psi(a) \in A$. Thus $xa \in \psi^{-1}(A)$ holds, so we have $x \in \psi^{-1}(A) \cdot a^{-1}$. For every $A \in \mathcal{B}(H)$, we have $\psi(R_a(\mu))(A) = R_a(\mu)(\psi^{-1}(A)) = \mu(\psi^{-1}(A) \cdot a^{-1}) = \mu(\psi^{-1}(A \cdot \psi(a)^{-1})) = \psi(\mu)(A \cdot \psi(a)^{-1}) = R_{\psi(a)}(\psi(\mu))(A)$, which shows (1).

(2) is analogous to (1).

LEMMA 2. Let $f: G \rightarrow H$ be a Borel mapping and $\mu, \nu \in M(G)$.

- (1) If $\mu \ll \nu$ then $f(\mu) \ll f(\nu)$ holds.
- (2) If $\mu \sim \nu$ then $f(\mu) \sim f(\nu)$ holds.

PROOF. (1) Let $A \in \mathcal{B}(H)$ be $f(\nu)(A) = 0$, that is $\nu(f^{-1}(A)) = 0$. Then it holds $\mu(f^{-1}(A)) = f(\mu)(A) = 0$. This proves (1).

(2) follows from (1).

We consider the image of admissible translates by a Borel homomorphism.

PROPOSITION 7. Let $\psi: G \rightarrow H$ be a Borel homomorphism and $\mu \in M(G)$. Then we have

(1) $\psi(RA(\mu)) \subset RA(\psi(\mu))$, $\psi(LA(\mu)) \subset LA(\psi(\mu))$ and $\psi(TA(\mu)) \subset TA(\psi(\mu))$.

(2) $\psi(RE(\mu)) \subset RE(\psi(\mu))$, $\psi(LE(\mu)) \subset LE(\psi(\mu))$ and $\psi(TE(\mu)) \subset TE(\psi(\mu))$.

If ψ is a Borel isomorphism, then equalities hold.

PROOF. The assertion follow from lemmas 1 and 2.

For the singular translates we have the following result (see also Zinn [10]).

PROPOSITION 8. Let $\psi: G \rightarrow H$ be a Borel homomorphism and $\mu \in M(G)$. Then we have $\psi^{-1}(RS(\psi(\mu))) \subset RS(\mu)$, $\psi^{-1}(LS(\psi(\mu))) \subset LS(\mu)$ and $\psi^{-1}(TS(\psi(\mu))) \subset TS(\mu)$. If ψ is a Borel isomorphism, the equalities hold.

PROOF. For example we prove $\psi^{-1}(RS(\psi(\mu))) \subset RS(\mu)$. Let $x \in \psi^{-1}(RS(\psi(\mu)))$, that is, $\psi(x) \in RS(\psi(\mu))$. By lemma 1 we have $\psi(R_x(\mu)) \perp \psi(\mu)$. Take a $D \in \mathcal{B}(H)$ such that $\psi(\mu)(D) = 0$ and $\psi(R_x(\mu))(D^c) = 0$. Then for $A = \psi^{-1}(D)$ we have $\mu(A) = 0$ and $R_x(\mu)(A^c) = 0$. Thus it hold $R_x(\mu) \perp \mu$, hence we have $x \in RS(\mu)$.

COROLLARY 1. We have the following inclusions; $RA(\mu) \subset \psi^{-1}(RS(\psi(\mu))^c)$, $LA(\mu) \subset \psi^{-1}(LS(\psi(\mu))^c)$ and $TA(\mu) \subset \psi^{-1}(TS(\psi(\mu))^c)$.

Now we consider the convolution of measures when G is abelian. For $\mu, \nu \in M(G)$, we denote by $\mu * \nu$ the convolution of μ and ν .

PROPOSITION 9. Let G be abelian and $\mu = \nu_1 * \nu_2 * \dots * \nu_n$, $\mu, \nu_1, \nu_2, \dots, \nu_n \in M(G)$. Then we have

(1) $TA(\nu_1) \cdot TA(\nu_2) \cdot \dots \cdot TA(\nu_n) \subset TA(\mu)$, and

(2) $TE(\nu_1) \cdot TE(\nu_2) \cdot \dots \cdot TE(\nu_n) \subset TE(\mu)$.

PROOF. We remark $RA(\mu) = LA(\mu) = TA(\mu)$ and $RE(\mu) = LE(\mu) = TE(\mu)$ since G is abelian. Let ψ be the continuous homomorphism

$$\psi: G \times G \times \dots \times G \longrightarrow G, \quad \psi(x_1, x_2, \dots, x_n) = x_1 \cdot x_2 \cdot \dots \cdot x_n.$$

Now let $\nu_1 \otimes \nu_2 \otimes \dots \otimes \nu_n$ be the product measure on $G \times G \times \dots \times G$, then the image $\psi(\nu_1 \otimes \nu_2 \otimes \dots \otimes \nu_n)$ is the convolution $\nu_1 * \nu_2 * \dots * \nu_n = \mu$. For every $a_i \in TA(\nu_i)$, the n -tuple $(a_1, a_2, \dots, a_n)$ belongs to $TA(\nu_1 \otimes \nu_2 \otimes \dots \otimes \nu_n)$ on $G \times G \times \dots \times G$. By proposition 7 we have $a_1 \cdot a_2 \cdot \dots \cdot a_n = \psi((a_1, a_2, \dots, a_n)) \in TA(\mu)$.

Let G be abelian. We say $\nu \in M(G)$ is a factor of $\mu \in M(G)$ if there exists a $\lambda \in M(G)$ such that $\nu * \lambda = \mu$.

COROLLARY 2. Let G be abelian and v be a factor of μ . Then it holds $TA(v) \subset TA(\mu)$, also $TE(v) \subset TE(\mu)$.

We say $\mu \in M(G)$ is symmetric if it holds $\mu(A) = \mu(A^{-1})$ for every $A \in \mathcal{B}(G)$.

PROPOSITION 10. If μ is symmetric, we have $RA(\mu)^{-1} = LA(\mu)$ and $RE(\mu)^{-1} = LE(\mu)$.

PROOF. Let $x \in RA(\mu)$, then $\mu(A) = 0$ implies $\mu(Ax^{-1}) = 0$. By the symmetry of μ , $\mu(A^{-1}) = 0$ implies $\mu(xA^{-1}) = 0$. Since $\mathcal{B}(G) = \{A^{-1}; A \in \mathcal{B}(G)\}$, we have $L_{x^{-1}}(\mu) \leq \mu$. This shows $x^{-1} \in LA(\mu)$.

COROLLARY 3. Let G be abelian and $\mu \in M(G)$ be symmetric, then $TA(\mu)$ is a group.

4. Quasi-invariant measure

First we prove the Borel measurability of $RA(\mu)$, $RE(\mu)$, ... etc. In this section we assume G is separable (=second countable).

LEMMA 3. Let $\mu \in M(G)$ and U be an open subset of G . Then the functions

(1) $x \longrightarrow \mu(Ux^{-1})$, and (2) $x \longrightarrow \mu(x^{-1}U)$ are lower semi-continuous.

PROOF. Since G is separable μ is a τ -smooth measure (see Amemiya, Okada and Okazaki [1]). Hence by the way similar to Amemiya, Okada and Okazaki [1], Lemma 9.4 and Bourbaki [2], § 2, n° 6, Proposition 11 the assertion follows. In fact remark the following equality;

$$\mu(x^{-1}U) = \int_G \chi_{\psi^{-1}(U)}(x, y) d\mu(y),$$

where $\psi(x, y) = xy$ and χ_A is the characteristic function of the Borel set A .

This proves the lemma.

Following Dubins and Freedman [4], we put the following Borel structure Σ on $M(G)$. The σ -algebra Σ on $M(G)$ is the one generated by the functions

$$M(G) \ni \mu \longrightarrow \mu(A) \in [0, 1]$$

where A varies over all Borel subsets of G . Since G is separable there is a counta-

ble generator $\{A_n\}$ of $\mathcal{B}(G)$. We may assume $\{A_n\}_{n=1}^\infty$ is an algebra. So Σ is generated by the countable functions $\mu \rightarrow \mu(A_n)$ $n=1, 2, \dots$.

PROPOSITION 11. Let Σ_0 be the σ -algebra on $M(G)$ generated by the functions

$$M(G) \ni \mu \longrightarrow \int_G f d\mu, \quad f \in C^b(G),$$

where $C^b(G)$ be the space of all bounded continuous functions on G . Then it holds $\Sigma_0 = \Sigma$.

PROOF. First we prove $\Sigma \subset \Sigma_0$. It is sufficient to show the function $\mu \rightarrow \mu(U)$ is Σ_0 -measurable for every open subset U since G is separable. We can take a sequence of closed subsets F_n such that $F_n \uparrow U$ by the metrizability of G . For each F_n , let $f_n \in C^b(G)$ be such that $f_n = 1$ on F_n , $f_n = 0$ on U^c and $f_n \geq 0$. Then for every μ in $M(G)$ we have

$$\mu(U) = \lim_n \int_G f_n d\mu,$$

which shows $\mu \rightarrow \mu(U)$ is Σ_0 -measurable.

Conversely let f be any bounded continuous function on G . Then by Bourbaki [2], § 2, n° 6, Lemma 3, there is a sequence f_n of step functions which are the linear combinations of characteristic functions of open subsets such that $f_n \uparrow f$ in G . For every μ in $M(G)$, we have

$$\int_G f d\mu = \lim_n \int_G f_n d\mu.$$

Since $\mu \rightarrow \int_G f_n d\mu$ is Σ -measurable, $\mu \rightarrow \int_G f d\mu$ is also Σ -measurable, which shows $\Sigma_0 \subset \Sigma$.

This completes the proof.

REMARK 1. $M(G)$ is a separable metric space with respect to the weak *-topology $\sigma(M(G), C^b(G))$ since G is separable. So the σ -field Σ coincides with the Borel field for the weak *-topology

LEMMA 4. For every Borel set $A \in \mathcal{B}(G)$ the functions

$$x \longrightarrow \mu(Ax^{-1}), \quad x \longrightarrow \mu(x^{-1}A)$$

are $\mathcal{B}(G)$ -measurable.

PROOF. Let $\mathcal{A}$ be the algebra generated by an open base. Then for every B in $\mathcal{A}$, the functions

$$x \longrightarrow \mu(Bx^{-1}), \quad x \longrightarrow \mu(x^{-1}B)$$

are $\mathcal{B}(G)$ -measurable by lemma 3. Since the family $\mathcal{A}$ generates $\mathcal{B}(G)$, the assertion follows.

PROPOSITION 12. The following subsets are Borel sets; $RA(\mu)$, $LA(\mu)$, $TA(\mu)$, $RE(\mu)$, $LE(\mu)$, $TE(\mu)$, $RS(\mu)$, $LS(\mu)$ and $TS(\mu)$, where $\mu \in M(G)$.

PROOF. We show $RA(\mu)$ is a Borel set. Let $\psi_\mu: G \rightarrow M(G) \times M(G)$ be defined by

$$\psi_\mu(x) = (R_x(\mu), \mu).$$

Then ψ is $\mathcal{B} - \Sigma \otimes \Sigma$ -measurable, since

$$x \longrightarrow (R_x(\mu)(A), \mu(B)) = (\mu(Ax^{-1}), \mu(B))$$

is measurable for every $A, B \in \mathcal{B}(G)$. By Dubins and Freedman [4], 2.11, the set $D = \{(\lambda, \nu); \lambda \ll \nu\}$ is in $\Sigma \otimes \Sigma$, hence $\psi^{-1}(D) = RA(\mu)$ is in $\mathcal{B}(G)$.

The other cases are analogously proved by Dubins and Freedman [4], 2.1.

Let E be a subset of G . A measure $\mu \in M(G)$ is said to be right E -quasi-invariant (resp. left E -quasi-invariant) if it holds $E \subset RA(\mu)$ (resp. $E \subset LA(\mu)$). In case $E = G$, we say μ is right quasi-invariant (resp. left quasi-invariant).

PROPOSITION 13. Let $\mu_1 \in M(G)$ be right quasi-invariant and $\mu_1(LA(\mu_2)^{-1}) > 0$, then it holds $\mu_2 \ll \mu_1$.

PROOF. We adopt a way similar to Umemura [8], Proposition 4.1. For every $A \in \mathcal{B}(G)$ we have

$$\begin{aligned} \mu_1 * \mu_2(A) &= \int \chi_A(xy) d\mu_1(x) d\mu_2(y) \\ &= \int \mu_1(Ay^{-1}) d\mu_2(y) \\ &= \int \mu_2(x^{-1}A) d\mu_1(x). \end{aligned}$$

Thus $\mu_1(A) = 0$ implies $\mu_1 * \mu_2(A) = 0$, so $\mu_2(x_0^{-1}A) = 0$ for some $x_0 \in LA(\mu_2)^{-1}$.

Since $x_0^{-1} \in LA(\mu_2)$, it holds $\mu_2(A) = \mu_2(x_0 x_0^{-1} A) = L_{x_0^{-1}}(\mu_2)(x_0^{-1} A) = 0$, which shows $\mu_2 \ll \mu_1$.

COROLLARY 4. Let $\mu_1 \in M(G)$ be right quasi-invariant and $\mu_2 \in M(G)$ be left quasi-invariant. Then we have $\mu_1 \sim \mu_2$.

COROLLARY 5. Suppose G is abelian and $\mu \in M(G)$ is right quasi-invariant. Then for every $a \in G$, it holds $R_a(\mu) \sim \mu$, that is $RE(\mu) = G$.

PROOF. We remark $R_a(\mu)$ is also right quasi-invariant for every $a \in G$.

COROLLARY 6. Let G be a locally compact topological group and $\mu \in M(G)$ be right (or left) quasi-invariant. Then μ is equivalent to the Haar measure of G . The above is a result of Brockett [3], Theorem 5.

PROPOSITION 14. Let $\mu \in M(G)$ be a right quasi-invariant measure. Then there is a $\mu_0 \in M(G)$ such that $\mu \sim \mu_0$ and μ_0 is invariant under the translation of G , that is $R_a(\mu_0) = \mu_0$ for every $a \in G$.

PROOF. Since G is separable, the Borel field $\mathcal{B}(G)$ is countably generated and it holds $\mathcal{B}(G) \otimes \mathcal{B}(G) = \mathcal{B}(G \times G)$. Since $dR_a(\mu)/d\mu(x)$ can be taken $\mu \otimes \mu$ -measurable (Mackey [6], Lemma 3.1), the result follows by the way similar to Umemura [8], Proposition 6.2, Mackey [7], Lemma 7.3.

5. Size of the set of admissible translates

THEOREM 1. Let G be a separable topological group. Then the subsets $RE(\mu)$, $LE(\mu)$ and $TE(\mu)$ are Borel subgroups. Moreover if μ is a Radon measure then it hold

(1) $\mu(RE(\mu)) = 0$, or $RE(\mu)$ is a locally compact group and the restriction $\mu|_{RE(\mu)}$ is equivalent to the Haar measure on $RE(\mu)$,

(2) $\mu(LE(\mu)) = 0$, or $LE(\mu)$ is a locally compact group and $\mu|_{LE(\mu)}$ is equivalent to the Haar measure on $LE(\mu)$.

PROOF. (1) Assume $\mu(RE(\mu)) > 0$. Then $\nu = \mu|_{RE(\mu)}$ is a right quasi-invariant measure on $RE(\mu)$. Thus there is a right invariant Radon measure λ on $RE(\mu)$ so that $\nu \sim \lambda$ (Proposition 14). By Gowrisankaran [5], Corollary 1, or Weil [9], Appendix 1, $RE(\mu)$ is locally compact group and λ is equivalent to the Haar measure (Corollary 6). (2) is analogous to (1).

This proves the theorem.

THEOREM 2. Let G be a separable abelian topological group and μ be a symmetric Radon probability measure on G . Then $\mu(TA(\mu))=0$, or $TA(\mu)$ is a locally compact group and $\mu|TA(\mu)$ is equivalent to the Haar measure on $TA(\mu)$.

PROOF. By corollary 3, $TA(\mu)$ is a Borel subgroup. By the way same to theorem 1, the theorem is proved.

REMARK 2. Let G be a complete separable metric group. In theorem 2, if μ is not symmetric we consider the symmetrization ν , $\nu=\mu*\mu^\vee$, where $\mu^\vee(A)=\mu(A^{-1})$. Then by proposition 9 we have $TA(\mu)\subset TA(\nu)$ and $TA(\nu)$ is a Borel subgroup (Corollary 3 and proposition 12). Thus if $\mu(TA(\mu))>0$, we have

$$\begin{aligned}\nu(TA(\nu)) &\geq \nu(TA(\mu)TA(\mu)^{-1}) \geq \int_{TA(\mu)} \mu(TA(\mu)TA(\mu)^{-1}x) d\mu(x) \\ &\geq \mu(TA(\mu))\mu(TA(\mu)) > 0.\end{aligned}$$

By theorem 2, $TA(\nu)$ is a locally compact group and $TA(\mu)\subset TA(\nu)$.

REMARK 3. In theorems 1 and 3, if μ is only a Borel measure we can show $RE(\mu)$, $TA(\mu)$ is a dense subgroup of a locally compact group. It is sufficient to consider the completion $\hat{G}$ of G and use proposition 7.

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