**NUMERICAL SOLUTION OF TIME OPTIMAL
CONTROL PROBLEMS BY A FINITE
CHEBYSHEV SERIES, PART 2;
NUMERICAL SOLVABILITY**

By

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Time optimal control problems are proved to be numerically solvable by computing the optimal solutions in connection with different Chebyshev series tied at the time of discontinuity of piecewise constant optimal controls.

§ 1. Reduction to a boundary value problem with nonlinear boundary conditions

The control system is given by

$$(1.1) \quad \dot{x} = a(x, t) + B(x, t)u,$$

where x is an n dimensional state vector, u is a K dimensional control, a is an n -vector and B an $n \times K$ matrix. The cubic control restraint is given by

$$(1.2) \quad \Omega: |u_i| \leq 1, \quad i=1, \dots, K.$$

We seek the optimal control which steers the initial point x_0 to the continuously moving compact target θ_t in minimum time subject to (1.1) and (1.2). Here

$$\theta_t = \{x: \Psi(x, t) = 0\},$$

where $\Psi(x, t)$ is a q -vector ($q \leq n$) function. We assume that the functions $a(x, t)$, $B(x, t)$ and $\Psi(x, t)$ are three times continuously differentiable with respect to (x, t) .

From the maximal principle [4], the optimal control is piecewise continuous and of the form

$$(1.3) \quad u(t) = \operatorname{sgn} B^T(x(t), t) \eta^T(t),$$

where T denotes transpose and

$$(1.4) \quad \operatorname{sgn} b = (\operatorname{sgn} b_1, \dots, \operatorname{sgn} b_K), \quad \text{for a } K\text{-vector } b = (b_1, \dots, b_K).$$

The adjoint vector function η satisfies the system of differential equations

$$(1.5) \quad \dot{\eta} = -\eta(a_x(x, t) + B_x(x, t)u),$$

with boundary conditions

$$(1.6) \quad \eta(t_r) = \gamma \cdot \Psi_x(x(t_r), t_r),$$

where t_r is a final optimal time, γ a q -row vector, Ψ_x a $q \times n$ matrix and

$$a_x(x, t) = \partial a(x, t) / \partial x, \quad B_x(x, t)u = \partial B(x, t)u / \partial x,$$

$$\Psi_x(x, t) = \partial \Psi(x, t) / \partial x.$$

In addition, final values $\eta(t_r)$ and $x(t_r)$ satisfy

$$(1.7) \quad \eta(t_r) \{a(x(t_r), t_r) + B(x(t_r), t_r)u(t_r)\} + \gamma \Psi_t(x(t_r), t_r) \geq 0.$$

In ordinary numerical examples the situation “=” hardly happens for (1.7). If we eliminate “=” from (1.7), then (1.7) is replaced by

$$(1.8) \quad \eta(t_r) \{a(x(t_r), t_r) + B(x(t_r), t_r)u(t_r)\} + \gamma \Psi_t(x(t_r), t_r) = 1,$$

since $\lambda \eta$ can be also an adjoint vector function for any scalar value λ . The final state vector $x(t_r)$ and initial state vector $x(t_0)$ satisfy

$$(1.9) \quad \Psi(x(t_r), t_r) = 0,$$

$$(1.10) \quad x(t_0) = x_0.$$

The optimal control is determined by (1.3). The $2n$ differential equations (1.1) and (1.5) with the $2n$ boundary conditions (1.6) and (1.10) form a two point boundary value problem with parameters γ to be found in (1.6) so that the condition (1.9) is satisfied. The final optimal time t_r is determined from (1.8).

We are concerned with the control such that the number of switching times is finite namely $r-1$. Since the optimal control is of the form (1.3), control takes the values only on vertexes of Ω . We call the time instant when the values of control moves from a vertex of Ω to another “a switching time” and the function $B^T(x, \cdot) \eta^T$ “a switching function”. At the switching time t_j , let the control u

changes its signs at $k_1, k_2, \dots, k_s$ -th components, then switching time t_j satisfies

$$b_{k_i}(x(t_j), t_j)\eta^T(t_j)=0, \quad i=1, \dots, s$$

at the same time, where b_{k_i} is k_i -th row vector of B^T . In the present paper, we assume that the equations are dependent at time t_j [4]. The switching time t_j is determined by any one of above equations. We take only one of equations and reject the other equations. Let the control u changes its sign at k_j -th component at least for time t_j , then switching times $t_j, j=1, \dots, r-1$ satisfy

$$(1.11) \quad b_{k_j}(x(t_j), t_j)\eta^T(t_j)=0,$$

where

$$t_0 < t_1 < t_2 < \dots < t_{r-1} < t_r.$$

We regard $z(t) = \begin{bmatrix} x \\ \eta^T \end{bmatrix}(t)$ on subintervals $[t_{j-1}, t_j], j=1, \dots, r$ as independent unknowns. Let $z^j = \begin{bmatrix} x^j \\ \eta^{jT} \end{bmatrix}$ be the solution $\begin{bmatrix} x \\ \eta^T \end{bmatrix}(t)$ restricted on $[t_{j-1}, t_j], j=1, \dots, r$. Since the control is constant on $[t_{j-1}, t_j]$, z^j is a solution obtained by the constant control on $[t_{j-1}, t_j]$. Each $z^j(t)$ satisfies the continuous differential system

$$(1.12) \quad dz^j/dt = Y^j(z^j, t) \quad \text{on} \quad [t_{j-1}, t_j],$$

where

$$(1.13) \quad Y^j(z, t) = \begin{bmatrix} a(x, t) + B(x, t) \operatorname{sgn} B^T(x, t)\eta^T \\ -\{a_x(x, t) + B_x(x, t) \operatorname{sgn} B^T(x, t)\eta^T\}^T \eta^T \end{bmatrix}.$$

The control $\operatorname{sgn} B^T(x, t)\eta^T$ takes the constant values in each $Y^j(z, t)$. Since the functions z^{j-1} and z^j are connected, we have

$$(1.14) \quad z^{j+1}(t_j) = z^j(t_j), \quad j=1, 2, \dots, r-1.$$

The initial condition (1.10) is expressed as linear conditions

$$(1.15) \quad L_0 z(t_0) + L_1 z(t_r) = l.$$

Matrices L_0 and L_1 and vector l are given as

$$L_0 = [\langle n \rangle, 0_{n,n}], \quad L_1 = [0_{n,2n}], \quad l = x_0,$$

($\langle n \rangle$: an $n \times n$ unit matrix, $0_{i,j}$: an $i \times j$ zero matrix).

The conditions (1.6), (1.8) and (1.9) are expressed as nonlinear boundary conditions at final time t_r ,

$$(1.16) \quad f^r(z(t_r), t_r, \gamma) = 0,$$

where $f^r(z, t, \gamma)$ is a $q+n+1$ vector function such that

$$(1.17) \quad f^r(z, t, \gamma) = \begin{bmatrix} \Psi(x, t) \\ \eta^T - \Psi_x^T(x, t)\gamma^T \\ \eta\{a(x, t) + B(x, t) \operatorname{sgn} B^T(x, t)\eta^T(t)\} + \gamma\Psi_t(x(t_r), t_r) - 1 \end{bmatrix}.$$

with $f^j(z(t_j), t_j) = 0$ with $f^j(z, t) = b_{k_j}(x, t)\eta^T$, $j = 1, \dots, r-1$.

Now on $[t_{j-1}, t_j]$, we consider the time variable transformation from t to τ :

$$t = \frac{t_j - t_{j-1}}{2}\tau + \frac{t_j + t_{j-1}}{2}, \quad j = 1, \dots, r.$$

Then $t = t_{j-1}$ and t_j correspond respectively to $\tau = -1$ and 1 . We put the transformation by $t_j(\tau)$. We put

$$z^j(\tau) = z^j(t_j(\tau)), \text{ that is } \begin{bmatrix} x^j(\tau) \\ \eta^j(\tau) \end{bmatrix} = \begin{bmatrix} x^j(t_j(\tau)) \\ \eta^j(t_j(\tau)) \end{bmatrix}.$$

From (1.12)~(1.17), the control problem is reduced to seek the independent solutions $z^j(\tau)$ and unknown parameters t_j , $j = 1, \dots, r$ and γ satisfying

$$(1.19) \quad dz^j/d\tau = Y^j(z^j, t_j(\tau))(t_j - t_{j-1})/2, \quad j = 1, \dots, r,$$

$$(1.20) \quad L_0 z^1(-1) + L_1 z^r(1) = l$$

$$(1.21) \quad z^{j+1}(-1) = z^j(1), \quad j = 1, \dots, r-1,$$

$$(1.22) \quad f^j(z^j(1), t_j) = 0, \quad j = 1, \dots, r-1,$$

$$(1.23) \quad f^r(z^r(1), t_r, \gamma) = 0.$$

§2. The construction of matrix G_1 by the variations in initial values $z^1(-1)$, in times t_j and in parameters γ

We guess initial values $z^1(-1)$ and times t_j , $j = 1, \dots, r$ and determine z^1 by (1.19) for $j=1$ starting from $z^1(-1)$, z^2 by (1.19) for $j=2$ starting from $z^1(1)$,

that is, by (1.19) for $j=2$ and (1.21) for $j=1$, z^3 by (1.19) for $j=3$ starting from $z^2(1)$ and so on. The solution $z^j(j=1, \dots, r)$ is determined by (1.19) and (1.21) with respect to initial values $z^1(-1)$ and times t_i , $i=1, \dots, j$. For a given initial variations v_1 in $z^1(-1)$ and given variation $\mathcal{T}_i$ in t_i for $i=1, \dots, j$, we can construct variations y^j in path z^j , since z^j is determined by $[z^1(-1), t_i, i=1, \dots, j]$. The variations v_1 of $z^1(-1)$ and variations y^1 of z^1 clearly satisfy

$$v_1 = y^1(-1).$$

Variations y^j of z^j and $\mathcal{T}_j$ of t_j satisfy the first variation equations (2.1) from (1.19) and the conditions (2.2) from (1.21) which connects y^j and y^{j+1} .

$$(2.1) \quad dy^j/d\tau = Y_z^j(z^j, t_j(\cdot))y^j(t_j - t_{j-1})/2 + Y_{t_j}^j(z^j, t_j(\cdot))\mathcal{T}_j(\cdot)(t_j - t_{j-1})/2 \\ + Y^j(z^j, t_j(\cdot))(\mathcal{T}_j - \mathcal{T}_{j-1})/2,$$

$$(\mathcal{T}_j(\tau) = (\mathcal{T}_j - \mathcal{T}_{j-1})\tau/2 + (\mathcal{T}_j + \mathcal{T}_{j-1})/2), \quad j=1, \dots, r,$$

$$(2.2) \quad y^{j+1}(-1) = y^j(1), \quad j=1, \dots, r-1.$$

We put

$$Y^j(\tau) = Y^j(z^j(\tau), t_j(\tau)), \quad j=1, \dots, r.$$

Let $\Phi_j(\tau)$ be fundamental matrices for

$$(2.3) \quad \dot{y} = Y_z^j(z^j, t_j(\cdot))y \quad \text{with} \quad \Phi_j(-1) = \langle 2n \rangle, \quad j=1, \dots, r.$$

The solution y^1 of (2.1) for $j=1$ is expressed as, by later lemma 8.2 in § 8,

$$(2.4) \quad y^1(\tau) = \Phi_1(\tau)v_1 + Y^1(\tau)(\tau+1)/2 \cdot \mathcal{T}_1.$$

The solution y^2 of (2.1) for $j=2$ is given, by later lemma 8.2 in § 8, as

$$(2.5) \quad y^2(\tau) = \Phi_2(\tau)y^2(-1) + Y^2(\tau)(\tau+1)/2 \cdot \mathcal{T}_2 + Y^2(\tau)(1-\tau)/2 \cdot \mathcal{T}_1 \\ - \Phi_2(\tau)Y^2(-1)\mathcal{T}_1.$$

Substituting $y^1(1)$ into $y^2(-1)$, we have, by (2.4),

$$(2.6) \quad y^2(\tau) = \Phi_2(\tau)(\Phi_1(1)v_1 + Y^1(1)\mathcal{T}_1) + Y^2(\tau)(\tau+1)/2 \cdot \mathcal{T}_2 \\ + Y^2(\tau)(1-\tau)/2 \cdot \mathcal{T}_1 - \Phi_2(\tau)Y^2(-1)\mathcal{T}_1 = \Phi_2(\tau)(\Phi_1(1)v_1 \\ + D^1\mathcal{T}_1) + Y^2(\tau)(\tau+1)/2 \cdot \mathcal{T}_2 + Y^2(\tau)(1-\tau)/2 \cdot \mathcal{T}_1.$$

Here D^j , $j=1, \dots, r$ are jumps between $Y^j(1)$ and $Y^{j+1}(-1)$ given as

$$(2.7) \quad D^j = Y^j(1) - Y^{j+1}(-1).$$

Finally by induction, variations y^j are expressed with respect to v_1 and $\mathcal{T}_i$, $i=1, \dots, j$:

$$(2.8) \quad y^j(\tau) = \Phi_j(\tau) \left\{ \Phi_{j-1,1} v_1 + \sum_{i=1}^{j-1} \Phi_{j-1,i+1} D^i \mathcal{T}_i \right\} \\ + Y^j(\tau)(\tau+1)/2 \cdot \mathcal{T}_j + Y^j(\tau)(1-\tau)/2 \cdot \mathcal{T}_{j-1},$$

where

$$\Phi_{j,i} = \Phi_j(1) \Phi_{j-1}(1) \cdots \Phi_{i+1}(1) \Phi_i(1), \quad \text{for } i \leq j,$$

and put

$$\Phi_{j,i} = \langle 2n \rangle, \quad \text{for } i > j.$$

If we express z^j with respect to $z^1(-1)$ and t_i , $i=1, \dots, j$ and substitute z^j into (1.20), (1.22) and (1.23), then the initial values $z^1(-1)$, times t_j , $j=1, \dots, r$ and parameters γ , that is, $2n+r+q$ unknown parameters are solved from the $2n+r+q$ boundary conditions (1.20), (1.22) and (1.23). In other words, conditions (1.20), (1.22) and (1.23) are expressed by parameters $[z^1(-1), t_j, j=1, \dots, r, \gamma]$. We shall show that variations of (1.20), (1.22) and (1.23) due to variations in the parameters $[z^1(-1), t_j, j=1, \dots, r, \gamma]$ are expressed in linear form with respect to the variations $[v_1, \mathcal{T}_j, j=1, \dots, r, \delta]$, where δ is variations of γ . Then the Jacobian matrix of those conditions with respect to $[z^1(-1), t_j, j=1, \dots, r, \gamma]$ will be given as the coefficient matrix of $[v_1, \mathcal{T}_j, j=1, \dots, r, \delta]$ in the linear form. The variations of the left hand side formulas in equalities (1.20), (1.22) and (1.23) are given as

$$L_0 y^1(-1) + L_1 y^r(1), f_z^j(1) y^j(1) + f_t^j(1) \mathcal{T}_j, \quad j=1, \dots, r-1, \\ f_z^r(1) y^r(1) + f_t^r(1) \mathcal{T}_r + f_\gamma^r(1) \delta,$$

where

$$f_z^j(1) = f_z^j(z^j(1), t_j), f_t^j(1) = f_t^j(z^j(1), t_j), \quad j=1, \dots, r, \\ f_\gamma^r(1) = f_\gamma^r(z^r(1), t_r).$$

Substituting (2.8) into those variations, we have

$$(2.9) \quad L_0 v_1 + L_1 \left\{ \Phi_{r,1} v_1 + \sum_{i=1}^{r-1} \Phi_{r,i+1} D^i \mathcal{T}_i + Y^r(1) \mathcal{T}_r \right\},$$

$$(2.10) \quad f_z^j(1) \{ \Phi_{j,1} v_1 + \sum_{i=1}^{j-1} \Phi_{j,i+1} D^i \mathcal{T}_i \} + \{ f_z^j(1) Y^j(1) + f_z^j(1) \} \mathcal{T}_j,$$

$$(2.11) \quad f_z^r(1) \{ \Phi_{r,1} v_1 + \sum_{i=1}^{r-1} \Phi_{r,i+1} D^i \mathcal{T}_i \} + \{ f_z^r(1) Y^r(1) + f_z^r(1) \} \mathcal{T}_r + f_\gamma^r(1) \delta.$$

(2.9)~(2.11) are expressed in matrix form as

$$(2.12) \quad G_1 [v_1 \ \mathcal{T} \ \delta]^T$$

where $\mathcal{T} = [\mathcal{T}_1, \dots, \mathcal{T}_r]$ and G_1 is a $(2n+r+q) \times (2n+r+q)$ matrix given in the following (2.13).

$$(2.13) \quad \begin{bmatrix} L_0 + L_1 \Phi_{r,1} & L_1 \Phi_{r,2} D^1 & & L_1 \Phi_{r,j} D^{j-1} & \dots \\ f_z^1(1) \Phi_{1,1} & A_1 & & & \\ \vdots & & \ddots & & \\ f_z^j(1) \Phi_{j,1} & f_z^j(1) \Phi_{j,2} D^1 \dots f_z^j(1) \Phi_{j,j} D^{j-2} & & A_j & \dots \\ f_z^{r-1}(1) \Phi_{r-1,1} & f_z^{r-1}(1) \Phi_{r-1,2} D^1 & \dots & f_z^{r-1}(1) \Phi_{r-1,j} D^{j-1} & \\ f_z^r(1) \Phi_{r,1} & f_z^r(1) \Phi_{r,2} D^1 & \dots & f_z^r(1) \Phi_{r,j} D^{j-1} & \\ & & & L_1 \Phi_{r,r} D^{r-1} & L_1 Y^r(1) \\ & & & \dots & \\ & & & \dots f_z^{r-1}(1) \Phi_{r-1,r-1} D^{r-2} & A_{r-1} \\ & & & \dots & f_z^r(1) \Phi_{r,r} D^{r-1} & f_z^r(1) Y^r(1) + f_z^r(1), f_\gamma^r(1) \end{bmatrix}$$

with

$$(2.14) \quad A_j = f_z^j(1) Y^j(1) + f_z^j(1), \quad j = 1, \dots, r-1.$$

Thus the variations of the formulas in (1.20), (1.22) and (1.23) are expressed with respect to variations $[v_1, \mathcal{T}, \delta]$ of unknown parameters $[z^1(-1), t_j, j=1, \dots, r, \gamma]$ in (2.12). Matrix G_1 is the Jacobian matrix of (1.20), (1.22) and (1.23) with respect to $[z^1(-1), t_j, j=1, \dots, r, \gamma]$. If G_1 is nonsingular and if the solution $[z^1(-1), t_j, j=1, \dots, r, \gamma]$ of equations (1.20), (1.22) and (1.23) exists, then the solution $[z^1(-1), t_j, j=1, \dots, r, \gamma]$ is uniquely determined in its neighborhood. Since z^j are uniquely determined by $[z^1(-1), t_i, i=1, \dots, j]$, thus $z^j, j=1, \dots, r$ are also

uniquely determined if G_1 is nonsingular.

If both A_j and D^j are zero for some j in G_1 , then matrix G_1 becomes singular, since $2n+j$ column vector is zero. In order that G_1 becomes nonsingular, it must hold that $A_j \neq 0$ or $D^j \neq 0$ for every $j=1, \dots, r$.

In [3], the integral quadratic cost minimization problems are treated. The boundary value problem (1.19)~(1.23) is similar in appearance to the boundary value problem treated in [3] but different in character: in [3], the control (1.3) is continuous and piecewise smooth in t , thus $D^j=0$, t_r is a given fixed value and only linear boundary conditions (1.20) are considered, while in (1.19)~(1.23), $D^j \neq 0$, $j=1, \dots, r-1$, t_r is a variable and both the linear conditions and the nonlinear conditions (1.23) are considered. If we put $D^j=0$ and eliminate the terms $L_1 Y^r(1)$, $f_z^r(1)Y^r(1)+f_t^r(1)$ concerning t_r and the terms concerning f^r in G_1 , then G_1 matrix becomes

$$\begin{bmatrix} L_0 + L_1 \Phi_{r,1} & & & & \\ f_z^1(1) \Phi_{1,1} & A_1 & & & \\ \vdots & \vdots & \ddots & & \\ f_z^j(1) \Phi_{j,1} & & A_j & & \\ \vdots & & \vdots & \ddots & \\ f_z^{r-1}(1) \Phi_{r-1,1} & & & & A_{r-1} \end{bmatrix}.$$

G_1 is nonsingular if and only if $A_j \neq 0$, $j=1, \dots, r$ and the matrix $L_0 + L_1 \Phi_{r,1}$ is nonsingular, since A_j , $j=1, \dots, r-1$ are at diagonals. Those conditions coincide with the isolatedness conditions in [3].

§3. Conditions of isolatedness for optimal solutions

The jump D^j is calculated by (1.13) as

$$\begin{aligned} (3.1) \quad D^j &= \begin{bmatrix} B|_{t=t_j}(u(t_j-) - u(t_j+)) \\ -\{B_x|_{t=t_j}(u(t_j-) - u(t_j+))\}^T \eta^T(t_j) \end{bmatrix} \\ &= 2u_{k_j}(t_j-) \begin{bmatrix} b_{k_j}^T \\ -((b_{k_j}^T)_x)^T \eta^T \end{bmatrix}_{t=t_j}, \end{aligned}$$

since only k_j -th component of $u(t_j-) - u(t_j+)$ is

$$u_{k_j}(t_j-) - u_{k_j}(t_j+) = 2u_{k_j}(t_j-), \quad (u_{k_j}(t_j-) = 1 \quad \text{or} \quad -1)$$

and other components are zero. The jump D^j is nonzero if and only if the

vector $[b_{k_j}, -(b_{k_j}\eta^T)_{x}]_{t=t_j}$ is nonzero.

In the matrix G_1 in (2.13) even if $A_j=0$ for some j , it is highly probable that G_1 becomes nonsingular at least for $D^j \neq 0$. The equations (1.11) (that is (1.17)) determine unknown parameters t_j , if the derivative of (1.11) with respect to t_j is nonzero for $j=1, \dots, r-1$. The derivative is calculated, by (1.13), (1.17) and (2.14), as

$$\begin{aligned} db_{k_j}(x(t_j), t_j)\eta^T(t_j)/dt_j &= [(b_{k_j}\eta^T)_x, b_{k_j}] \begin{bmatrix} \dot{x} \\ \dot{\eta}^T \end{bmatrix} + \eta b_{k_j}^T|_{t=t_j} \\ &= f_z^j(1)Y^j(1) + f_t^j(1) = A_j. \end{aligned}$$

The condition $A_j \neq 0$ implies that the derivative of (1.11) is nonzero. Geometrically, $A_j \neq 0$ implies that the switching curve $B^T(t)\eta^T(t)$ transverses the hyperplane $u_{k_j}=0$ and $A_j=0$ implies that the switching curve is tangent to $u_{k_j}=0$. Assume that $A_j=0$. The switching function $B^T(t)\eta^T(t)$ must tangent at one point in $u_{k_j}=0$ and yet pierces through at the point not turning back to make $D^j \neq 0$. The switching function must pierce through at one point in the hyperplane transversing ($A_j \neq 0$) or tangent ($A_j=0$) to it to make G_1 nonsingular. Assume that $B(x, t)$ is independent of t , then $b_{k_j t}=0$. If $D^j=0$, then $A_j=0$ from (3.1) and G_1 is singular. D^j must be nonzero to make G_1 nonsingular.

§4. Numerical solvability for the method of Chebyshev series

Now for the boundary value problem (1.19)~(1.23), we seek the Chebyshev approximate solution $z_m^j(\tau)$, times t_j , $j=1, \dots, r$ and parameters γ which satisfy the differential system

$$(4.1) \quad dz_m^j(\tau)/d\tau - P_{m-1}Y^j(z_m^j(\tau), \tau)(t_j - t_{j-1})/2 = 0, \quad j=1, \dots, r,$$

with boundary conditions

$$(4.2) \quad L_0 z_m^1(-1) + L_1 z_m^r(1) = 0, \quad z_m^{j+1}(-1) = z_m^j(1), \quad j=1, \dots, r-1,$$

$$(4.3) \quad f^j(z_m^j(1), t_j) = 0, \quad j=1, \dots, r-1, \quad f^r(z_m^r(1), t_r, \gamma) = 0.$$

Here $P_{m-1}f$ is the truncated Chebyshev series expression of a function f , discarding the terms of the order higher than $m-1$. Chebyshev series $z_m^j(\tau)$ are expressed as

$$z_m^j(\tau) = a_0^j + \sqrt{2} \sum_{i=1}^m a_i^j T_i(\tau) = \sum_{i=0}^m a_i^j T_i'(\tau), \quad (T_0' = T_0, T_i' = \sqrt{2} T_i),$$

where a_i^j are N -vector ($N=2n$) coefficients of $z_m^j(\tau)$ and $T_i(\tau)$ i -th order Chebyshev polynomial such that

$$T_i(\cos \theta) = \cos i\theta \quad \text{for } \tau = \cos \theta: T_i(\tau) = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{i}{2k} \tau^{i-2k} (\tau^2 - 1)^k.$$

Coefficients $a_{i-1}^{j'}$ of the derivative $\dot{z}_m^j$ is expressed as a linear combination of coefficients a_s^j [5]:

$$(4.4) \quad \dot{z}_m^j = \sum_{i=1}^m a_{i-1}^{j'} T'_{i-1} = \sum_{i=1}^m (1/e_{i-1} \cdot \sum_{s=0}^m v_{s-(i-1)} s a_s^j) T'_{i-1},$$

where

$$e_k = \begin{cases} \sqrt{2} & \text{for } k=0, \\ 1 & \text{otherwise,} \end{cases} \quad v_k = \begin{cases} 0, & \text{for } k < 0, \\ 1 - (-1)^k, & \text{for } k \geq 0. \end{cases}$$

The truncated polynomial $P_{m-1} Y^j$ is expressed as

$$(4.5) \quad P_{m-1} Y^j = \sum_{i=1}^m \{ (\sqrt{2}/(\pi e_{i-1})) \cdot \int_0^\pi Y^j(z_m^j(\cos \theta), t_j(\cos \theta)) \cos(i-1)\theta d\theta T'_{i-1} \}.$$

The $(i-1)$ -th Chebyshev coefficient of (4.1) is, by (4.4) and (4.5),

$$\begin{aligned} Z_i^j(a_i^j, \quad i=0, \dots, m, t_{j-1}, t_j) \\ = a_{i-1}^{j'} - \sqrt{2}/(\pi e_{i-1}) \cdot \int_0^\pi Y^j(z_m^j(\cos \theta), t_j(\cos \theta)) \cos(i-1)\theta d\theta (t_j - t_{j-1})/2, \\ i=1, \dots, m, \quad j=1, \dots, r. \end{aligned}$$

Note that Z_i^j is nonlinear with respect to a_i^j , $i=0, \dots, m$, t_{j-1} and t_j , since $z_m^j(\cos \theta)$ is expressed in a_i^j , $i=0, \dots, m$ and Y^j is nonlinear with respect to z_m^j , t_{j-1} and t_j . Let α be an $(m+1)N + r + q$ vector such that

$$\alpha = [a_i^j, \quad i=0, \dots, m, t_j, \quad j=1, \dots, r, \gamma].$$

Writing $Z_i^j(a_i^j, i=0, \dots, m, t_{j-1}, t_j)$ by $Z_i^j(\alpha)$, we regard Z_i^j as a function of α . (4.1) is expressed as

$$(4.6) \quad \sum_{i=1}^m Z_i^j(\alpha) T'_i = 0.$$

We put

$$g(\alpha) = [f^j(z_m^j(1), t_j), \quad j=1, \dots, r-1, f^r(z_m^r(1), t_r, \gamma)],$$

where g is a $(r-1) + q + n + 1$ vector function. We put

$$\begin{aligned} L(\alpha) &= L(a_i^j, \quad i=0, \dots, m, \quad j=1, \dots, r) \\ &= [L_0 z_m^1(-1) + L_1 z_m^r(1) - l, z_m^{j+1}(-1) - z_m^j(1), \quad j=1, \dots, r-1], \end{aligned}$$

where L is a $n + N(r-1)$ vector function. Then we have a system of algebraic equations

$$(4.7) \quad Z_i^j(\alpha) = 0, \quad i=1, \dots, m, \quad j=1, \dots, r,$$

$$(4.8) \quad g(\alpha) = 0,$$

$$(4.9) \quad L(\alpha) = 0,$$

with α as unknowns, that is, Chebyshev coefficients $(a_i^j, \quad i=0, \dots, m, \quad j=1, \dots, r)$ and switching times $t_j, j=1, \dots, r$ and parameters γ as unknowns. We call equations (4.7) ~ (4.9) "a determining equation" since it determines α .

The following theorem 4.1 shows that if there exists an isolated solution satisfying (1.19)~(1.23) then we can find an approximate solution $(z_m^j, t_j, j=1, \dots, r, \gamma)$ which converges uniformly to the isolated solution, from the determining equation (4.7)~(4.9). We assume that the functions $Y^j(z, t), j=1, \dots, r$ and $f^j(z, t), j=0, \dots, r$ are twice continuously differentiable with respect to z and t . This assumption is satisfied also for the boundary value problem derived from our control problems, since a, B and Ψ are three times continuously differentiable with respect to x and t .

THEOREM 4.1. *Suppose that for the exact solution $(\hat{z}^j(\tau), \hat{t}_j, j=1, \dots, r, \hat{\gamma})$ satisfying (1.19)~(1.23), the matrix G_1 is nonsingular. Then for sufficiently large m , there are some finite Chebyshev series $z_m^j(\tau), j=1, \dots, r$ of any order $m \geq m_0$, approximate times $t_j, j=1, \dots, r$ and parameters γ such that*

$$z_m^j(\tau) \longrightarrow \hat{z}^j(\tau), \quad \text{uniformly as } m \rightarrow \infty,$$

$$t_j \longrightarrow \hat{t}_j, \quad \gamma \rightarrow \hat{\gamma} \quad \text{as } m \rightarrow \infty, \quad j=1, \dots, r.$$

§ 5. The proof of theorem 4.1

The proof of theorem 4.1 is based on the following proposition 5.1 which has been presented by the author in [1]. In the proposition, the error bound of

the solution is expressed in the sum of the errors depending on the residual errors for linear part and those for nonlinear part in algebraic equations.

PROPOSITION 5.1. *Consider a real system of algebraic equations*

$$(5.1) \quad \begin{cases} E\alpha - l = 0 \\ F(\alpha) = 0, \end{cases}$$

where α is a s -vector, E is a $k \times s$ constant matrix and $F(\alpha)$ is a $(s-k)$ -vector function defined in some region Ω of the α -space which is continuously differentiable. Assume that (5.1) has an approximate solution $\alpha = \hat{\alpha}$ for which the determinant of $J(\hat{\alpha}) = \begin{bmatrix} E \\ F_{\alpha}(\hat{\alpha}) \end{bmatrix}$ does not vanish. Further assume that there are a positive constant ρ and a non-negative number $\kappa < 1$ such that

- (a) $\Omega_{\rho} = \{\alpha \mid \|\alpha - \hat{\alpha}\| \leq \rho\} \subset \Omega,$
- (b) $\|F_{\alpha}(\alpha) - F_{\alpha}(\hat{\alpha})\| \leq \kappa/M_1, \quad \text{for any } \alpha \in \Omega_{\rho},$
- (c) $(M_1 r + M_2 \varepsilon)/(1 - \kappa) \leq \rho,$

where r is a bound of residual errors for nonlinear equations and ε is a bound of residual errors for linear equations such that

$$\|F(\hat{\alpha})\| \leq r, \quad \|E\hat{\alpha} - l\| \leq \varepsilon.$$

M_1 and M_2 are numbers concerning the inverse of Jacobian matrix such that

$$(5.2) \quad \begin{aligned} \left\| \begin{bmatrix} E \\ F_{\alpha}(\hat{\alpha}) \end{bmatrix}^{-1} \begin{bmatrix} 0_{k,k} & 0_{k,s-k} \\ 0_{s-k,k} & \langle s-k \rangle \end{bmatrix} \right\| &\leq M_1, \\ \left\| \begin{bmatrix} E \\ F_{\alpha}(\hat{\alpha}) \end{bmatrix}^{-1} \begin{bmatrix} \langle k \rangle & 0_{k,s-k} \\ 0_{s-k,k} & 0_{s-k,s-k} \end{bmatrix} \right\| &\leq M_2, \end{aligned}$$

where k is the number of the linear algebraic equations and $s-k$ is the number of nonlinear algebraic equations.

By symbol $\|\cdot\|$, we denote the Euclidean norm for a vector and $(\sum_{i,j} a_{ij}^2)^{1/2}$ for a matrix with elements a_{ij} .

Then the system of algebraic equations has a unique solution $\alpha = \bar{\alpha}$ in Ω_{ρ} , and for $\alpha = \bar{\alpha}$ it holds that

$$(5.3) \quad \det J(\bar{\alpha}) \neq 0, \quad \|\hat{\alpha} - \bar{\alpha}\| \leq (M_1 r + M_2 \varepsilon)/(1 - \kappa).$$

The proof is similar to that of proposition 1 in [2] discussed for operator equations in normed spaces. We apply proposition 5.1 to the determining equation (4.7)~(4.9): as the nonlinear equations $F(\alpha)=0$ in (5.1) we take equations (4.7) and (4.8) and as the linear equations $E\alpha - l = 0$ we take (4.9).

Let the truncated Chebyshev series of the exact solution $\hat{z}^j$ is expressed as

$$P_m \hat{z}^j = \hat{a}_0^j + \sqrt{2} \sum_{i=1}^m \hat{a}_i^j T_i.$$

First as the approximate solution $\hat{\alpha}$ in proposition 1, we take a vector which is constituted from the Chebyshev coefficients $\hat{a}_i^j$ of $P_m \hat{z}^j$, the exact times $\hat{t}_j$ and the exact parameters $\hat{\gamma}$:

$$\hat{\alpha} = \text{col.} (\hat{a}_i^j, \hat{t}_j, \quad i=0, \dots, m, \quad j=1, \dots, r, \hat{\gamma}).$$

Next in the following sections § 6~§ 8 we shall prove (i)~(iii):

- (i) We calculate the bound r of residual errors for the nonlinear algebraic equations (4.7) and (4.8) and the bound ε of residual errors for the linear algebraic equations (4.9) and prove that r and ε are the order $O(m^{-3/2})$ for large m , since Y^j and f^j are twice continuously differentiable for $j=1, \dots, r$.
- (ii) The difference of Jacobian matrix satisfies

$$\|F_\alpha(\alpha) - F_\alpha(\hat{\alpha})\| \leq (2m+1)M_3 \|\alpha - \hat{\alpha}\|, \quad \text{for some constant } M_3.$$

- (iii) Numbers M_1 and M_2 in (5.2) concerning the inverse of Jacobian matrix of (4.7)~(4.9) are bounded for large m , since G_1 is nonsingular.

Note that if nonlinear boundary conditions (1.22) and (1.23) are reduced to linear boundary conditions and so the boundary value problem has only linear boundary conditions, then $2m+1$ is replaced by $\sqrt{2m+1}$ in the formula in (ii), since the later formula (7.4) does not appear. In [5], the corresponding value was $\sqrt{2m+1}$, since only linear boundary conditions are treated.

For the Chebyshev series

$$f(\tau) = a_0 + \sqrt{2} \sum_{i=1}^{\infty} a_i T_i(\tau)$$

with vector Chebyshev coefficients a_i , the norm $\|\cdot\|_q$ is defined by

$$\|f\|_q = [1/\pi \cdot \int_0^\pi \|f(\cos \theta)\|^2 d\theta]^{1/2}$$

and proved in [5]

$$(5.4) \quad \|f\|_q^2 = \sum_{i=1}^{\infty} \|a_i\|^2.$$

The symbol $\|f\|_n$ denotes the maximum of the Euclidean norm for a continuous function f on $I = [-1, 1]$:

$$\|f\|_n = \max_{\tau \in I} \|f(\tau)\|.$$

For the norm $\|\cdot\|_n$ and the norm $\|\cdot\|_q$, it holds that

$$(5.5) \quad \|P_m f\|_n \leq a_0 + \sqrt{2} \sum_{n=1}^m \|a_n\| \leq \sqrt{2m+1} (\|a_0\|^2 + \sum_{i=1}^m \|a_i\|^2) \\ \leq \sqrt{2m+1} \|P_m f\|_q,$$

$$(5.6) \quad \|f\|_q = \left[\frac{1}{\pi} \int_0^\pi \|f(\cos \theta)\|^2 d\theta \right]^{1/2} \leq \left[\frac{1}{\pi} \int_0^\pi \|f\|_n^2 d\theta \right]^{1/2} = \|f\|_n,$$

The function $(\langle N \rangle - P_{m-1})f$ satisfies

$$(5.7) \quad \|(\langle N \rangle - P_{m-1})f\|_q \leq \|(\langle N \rangle - P_{m-2})\dot{f}\|_q / m, \quad (\dot{f} = \frac{df}{d\tau}),$$

$$(5.8) \quad \|d/d\tau (\langle N \rangle - P_m)f\|_q \\ \leq \sqrt{m+2}/\sqrt{2} \cdot \|(\langle N \rangle - P_{m-1})\dot{f}\|_q + \|(\langle N \rangle - P_{m+1})\dot{f}\|_q,$$

$$(5.9) \quad \|(\langle N \rangle - P_m)f\|_n \leq \sqrt{2}/\sqrt{m} \cdot \|(\langle N \rangle - P_{m-2})\dot{f}\|_q \\ \leq \sqrt{2}/(\sqrt{m}m) \cdot \|(\langle N \rangle - P_{m-1})\ddot{f}\|_q \\ \leq \sqrt{2}/\{\sqrt{m}m(m-1)\} \cdot \|(\langle N \rangle - P_{m-1})\ddot{f}\|_q \\ \leq \sqrt{2}/\{\sqrt{m}m(m-1)\} \cdot \|\ddot{f}\|_q, \quad (\ddot{f} = d^2 f/d\tau^2, \ddot{f} = d^3 f/d\tau^3).$$

$d^3 \hat{z}^j/d\tau^3$ is expressed as 2-nd total derivative of $Y^j(\hat{z}^j(\tau), \hat{t}_j(\tau))(\hat{t}_j - \hat{t}_{j-1})/2$ with respect to τ , that is, $d^2/d\tau^2 Y^j(z^j(\tau), \hat{t}_j(\tau))(\hat{t}_j - \hat{t}_{j-1})/2$. $d^3 \hat{z}^j/d\tau^3$ is given by substituting

$$t = \hat{t}_j(\tau) \quad \text{and} \quad z = \hat{z}^j$$

into

$$\frac{(\hat{t}_j - \hat{t}_{j-1})^3}{2^3} \left[\sum_{k,l} \frac{\partial^2 Y^j(z, t)}{\partial z_k \partial z_l} Y_k^j(z, t) Y_l^j(z, t) + \left\{ Y_z^j(z, t)^2 \right. \right. \\ \left. \left. + 2 \frac{\partial Y_z^j(z, t)}{\partial t} \right\} Y^j(z, t) + Y_z^j(z, t) \frac{\partial Y^j(z, t)}{\partial t} + \frac{\partial^2 Y^j(z, t)}{\partial t^2} \right]$$

where $Y^j(z, t)$ is defined in (1.13), Y_k^j is k -th component of Y^j and z_k a k -th component of z . In what follows, we assume that initial time t_0 , all the exact switching times $t_1, \dots, t_{r-1}$, and the exact final time t_r are included in the interval I . Then it holds that

$$|t_j - t_{j-1}|/2 \leq 1, \quad \text{for } j=1, \dots, r.$$

The 2-nd total derivative is bounded, since Y^j is twice continuously differentiable, and $\hat{z}^j$ is also bounded:

$$(5.10) \quad \|\ddot{\hat{z}}^j\|_n = \|d^2/d\tau^2 Y^j(\hat{z}^j, t_j(\cdot))\|_n \leq K_{17}^j, \quad (K_{17}^j: \text{ a positive value}).$$

From (5.9) and (5.10) follows

$$(5.11) \quad \left(\sum_{j=1}^r \|\hat{z}_m^j - \hat{z}^j\|_n^2 \right)^{1/2} \leq \sqrt{2} / \{ \sqrt{m} m(m-1) \} \cdot \left(\sum_{j=1}^r \|\ddot{\hat{z}}^j\|_n^2 \right)^{1/2} \\ \leq \sqrt{2} K_{17} / \{ \sqrt{m} m(m-1) \},$$

where $\hat{z}_m^j$ denotes the truncated Chebyshev series $\hat{P}_m z^j$ and

$$K_{17} = \left(\sum_{j=1}^r (K_{17}^j)^2 \right)^{1/2}.$$

Since we are interested in a neighboring solution of (1.19)~(1.23), we may restrict the domain of the solution to the neighbourhood of the exact solution $[\hat{z}^j, t_j, j=1, \dots, r, \hat{\gamma}]$

$$U = \{ [z^j, t_j, j=1, \dots, r, \gamma] : \|z^j - \hat{z}^j\|_n \leq \delta_0, |t_j - \hat{t}_j| \leq \delta_0,$$

$$j=1, \dots, r, \|\gamma - \hat{\gamma}\| \leq \delta_0 \}, \quad (\delta_0: \text{ a positive value}).$$

The pair of the truncated Chebyshev series $\hat{z}_m^j$ and $[t_j, j=1, \dots, r, \hat{\gamma}]$ lies in U by (5.11) for any $m \geq m_0$ provided that m_0 satisfies

$$\sqrt{2} K_{17} / \{ \sqrt{m_0} m_0(m_0-1) \} \leq \delta_0.$$

For such m , we shall calculate residual error bounds r and ε and Jacobian matrix. Let (i), (ii) and (iii) hold for such $m \geq m_1 \geq m_0$. Let α be a vector such that

$$\alpha = \text{col.} (a_i^j, t_j, i=0, \dots, m, j=1, \dots, r, \gamma)$$

and z_m^j be a Chebyshev series defined by

$$z_m^j = \sum_{i=0}^m a_i^j T_i.$$

If α satisfies

$$(5.12) \quad \left\{ \sum_{j=1}^r \sum_{i=0}^m \|a_i^j - \hat{a}_i^j\|^2 \right\}^{1/2} \leq 1/\sqrt{2m+1} [\delta_0 - K_{17} \sqrt{2}/\{\sqrt{m_1} m_1(m_1-1)\}],$$

$$|t_j - \bar{t}_j| \leq \delta_0 \quad \text{and} \quad \|\gamma - \bar{\gamma}\| \leq \delta_0$$

for given δ_0 , then we have, from (5.5) and (5.11),

$$(5.13) \quad \|z_m^j - \hat{z}^j\|_n \leq \|z_m^j - \hat{z}_m^j\|_n + \|\hat{z}_m^j - \hat{z}^j\|_n \leq \sqrt{2m+1} \|z_m^j - \hat{z}_m^j\|_q$$

$$+ \max_j K_{17} \sqrt{2}/\{\sqrt{m} m(m-1)\} \leq \delta_0 - \max_j K_{17} \sqrt{2}/\{\sqrt{m_1} m_1(m_1-1)\}$$

$$+ \max_j K_{17} \sqrt{2}/\{\sqrt{m} m(m-1)\} \leq \delta_0, \quad \text{for } m \geq m_1.$$

Let Ω be a set of α satisfying (5.12). The determining equation is defined on Ω , since the corresponding $[z_m^j, t_j, j=1, \dots, r, \gamma]$ lies in U for any $\alpha \in \Omega$ by (5.13). We take the set Ω as the domain of definition of the determining equation. (ii) will be proved on Ω .

We shall prove theorem 4.1, supposing that (i)~(iii) hold. First we take $\kappa=0.5$. If we take ρ such that

$$(5.14) \quad \rho = \rho_m = \frac{1}{2m+1} \min. \left[\frac{0.5}{M_3 M_1}, \delta_0 - K_{17} \frac{\sqrt{2}}{\sqrt{m_1} m_1(m_1-1)} \right]$$

Then (a) in proposition 5.1 is satisfied since $\Omega_\rho \subset \Omega$ and (b) is satisfied from (ii). Since the bounds of the residual errors are the order $O(m^{-3/2})$ by (i), we substitute $O(m^{-3/2})$ into r and ε in (c), then (c) is written as

$$(5.15) \quad (M_1 + M_2)/0.5 \cdot O(m^{-3/2}) \leq \rho_m.$$

The order of the left hand side of (5.15) is $O(m^{-3/2})$ and the right hand is $O(m^{-1})$. Thus (5.15) is satisfied for sufficiently large m . The condition (c) can be satisfied for such large m satisfying (5.15). Since (a), (b) and (c) are satisfied for large m , thus, by proposition 5.1, there exists a solution $\bar{\alpha}$ of determining equation satisfying (5.3). Let $\bar{\alpha}$ be expressed as

$$\bar{\alpha} = [\bar{a}_i^j, \bar{t}_j, \quad i=0, \dots, m, \quad j=1, \dots, r, \bar{\gamma}]$$

and $\bar{z}_m^j$ be a Chebyshev series with Chebyshev coefficients $\bar{a}_i^j$, $i=0, \dots, m$. Then $[\bar{z}_m^j, \bar{t}_j, j=1, \dots, r, \bar{\gamma}]$ satisfies, by (5.4) and (5.3) and (i),

$$(5.16) \quad \left[\sum_{j=1}^r \{ \|\bar{z}_m^j - P_m \hat{z}^j\|_q^2 + |\bar{t}_j - \hat{t}_j|^2 \} + \|\bar{\gamma} - \hat{\gamma}\|^2 \right]^{1/2} = \|\bar{\alpha} - \hat{\alpha}\| \\ \leq (M_1 r + M_2 \varepsilon)/0.5 \leq (M_1 + M_2) O(m^{-3/2})/0.5.$$

(5.16) implies that $\bar{t}_j$, $j=1, \dots, r$ and $\bar{\gamma}$ converge to the exact t_j and $\hat{\gamma}$ in the order $O(m^{-3/2})$ as $m \rightarrow \infty$. We have, by (5.5), (5.11) and (5.16),

$$(5.17) \quad \left(\sum_{j=1}^r \|\bar{z}_m^j - \hat{z}^j\|_n^2 \right)^{1/2} \leq \left(\sum_{j=1}^r \|\bar{z}_m^j - P_m \hat{z}^j\|_n^2 \right)^{1/2} + \left(\sum_{j=1}^r \|P_m \hat{z}^j - \hat{z}^j\|_n^2 \right)^{1/2} \\ \leq \sqrt{2m+1} \left(\sum_{j=1}^r \|\bar{z}_m^j - P_m \hat{z}^j\|_q^2 \right)^{1/2} + \sqrt{2} K_{17} / \{ \sqrt{m} m(m-1) \} \\ \leq \sqrt{2m+1} (M_1 + M_2) O(m^{-3/2})/0.5 + \sqrt{2} K_{17} / \{ \sqrt{m} m(m-1) \}.$$

(5.17) implies that the Chebyshev approximate solution $\bar{z}_m^j$ converges uniformly to the exact solution $\hat{z}^j$ in the order $O(m^{-1})$. Therefore theorem 4.1 is proved under the condition that (i), (ii) and (iii) hold.

We shall prove (i), (ii) and (iii) in the following sections § 6, § 7 and § 8 respectively.

§ 6. Residual errors

The residual error for nonlinear equations (4.7) and (4.8) is

$$\left\{ \sum_{j=1}^r \sum_{i=1}^m \|Z_i^j(\hat{\alpha})\|^2 + \sum_{j=1}^{r-1} \|f^j(\hat{z}_m^j(1), \hat{t}_j)\|^2 + \|f^r(\hat{z}_m^r(1), \hat{t}_r, \hat{\gamma})\|^2 \right\}^{1/2},$$

where $\alpha^j = [\hat{\alpha}_i^j, i=0, \dots, m, \hat{t}_{j-1}, \hat{t}_j]$.

The residual error for linear equations (4.9) is

$$\left\{ \|L_0 \hat{z}_m^j(-1) + L_1 \hat{z}_m^r(1)\|^2 + \sum_{j=1}^{r-1} \|\hat{z}_m^{j+1}(-1) - \hat{z}_m^j(1)\|^2 \right\}^{1/2}.$$

We have from (5.11)

$$(6.1) \quad \|L_0 \hat{z}_m^1(-1) + L_1 \hat{z}_m^r(1) - l\| = \|L_0(\hat{z}_m^1(-1) - \hat{z}^1(-1))\| \\ + \|L_1(\hat{z}_m^r(1) - \hat{z}^r(1))\| \leq (\|L_0\| K_{17}^1 + \|L_1\| K_{17}^1) \sqrt{2} / \{ \sqrt{m} m(m-1) \},$$

$$(6.2) \quad \|\hat{z}_m^{j+1}(-1) - \hat{z}_m^j(1)\| \leq \|\hat{z}_m^{j+1}(-1) - \hat{z}^{j+1}(-1)\| \\ + \|\hat{z}_m^j(1) - \hat{z}^j(1)\| \leq (K_{17}^{j+1} + K_{17}^j) \sqrt{2} / \{ \sqrt{m} m(m-1) \}.$$

(6.1) and (6.2) imply that the bound ε of the residual error for linear equations (4.9) is

$$(6.3) \quad \varepsilon = O(m^{-5/2}).$$

Let U' be a set of the values $(z^j(\tau), t_j(\tau), \gamma)$ in t - z - γ space for functions $[z^j, t_j, j=1, \dots, r, \gamma]$ in the set of functions U :

$$U' = \{(z^j(\tau), t_j(\tau), \gamma), \quad j=1, \dots, r, \tau \in I: [z^j, t_j, j=1, \dots, r, \gamma] \in U\}.$$

Since $Y^j(z, t), j=1, \dots, r, f^j(z, t), j=1, \dots, r-1$ and $f^r(z, t, \gamma)$ are twice continuously differentiable with respect to z, t , and γ , there exist following bounds $K_1 \sim K_{14}, K_{18} \sim K_{21}$:

$$\|Y^j\|_{U'} = \max_{(z,t) \in U'} \|Y^j(z, t)\| \leq \max_{\substack{(z,t) \in U' \\ j=1, \dots, r}} \|Y^j(z, t)\| = K_1,$$

where the symbol $\|Y^j\|_{U'}$ denotes $\max_{(z,t) \in U'} \|Y^j(z, t)\|$, and

$$\|Y_z^j\|_{U'} \leq K_2, \|Y_t^j\|_{U'} \leq K_3, \|Y_{zz}^j\|_{U'} \leq K_4, \|Y_{zt}^j\|_{U'} \leq K_5,$$

$$\|Y_{tt}^j\|_{U'} \leq K_6, j=1, \dots, r,$$

$$\|f_t^j\|_{U'} \leq K_7, \|f_t^j\|_{U'} \leq K_8, \|f_{zz}^j\|_{U'} \leq K_9, \|f_{zt}^j\|_{U'} \leq K_{10},$$

$$\|f_{tt}^j\|_{U'} \leq K_{11}, j=1, \dots, r$$

$$\max_{\substack{\tau \in I \\ j=1, \dots, r}} \left\| \frac{d}{d\tau} Y_z^j(\hat{z}^j(\tau), \hat{t}_j(\tau)) \right\| \leq K_{12},$$

$$\max_{\substack{\tau \in I \\ j=1, \dots, r}} \left\| \frac{d}{d\tau} Y_t^j(\hat{z}^j(\tau), \hat{t}_j(\tau)) \right\| \leq K_{13},$$

$$\max_{\substack{\tau \in I \\ j=1, \dots, r}} \left\| \frac{d}{d\tau} Y^j(\hat{z}^j(\tau), \hat{t}_j(\tau)) \right\| \leq K_{14}, \|f_\gamma^r\|_{U'} \leq K_{18},$$

$$\|f_{z\gamma}^r\|_{U'} \leq K_{19}, \|f_{\gamma t}^r\|_{U'} \leq K_{20}, \|f_{\gamma\gamma}^r\|_{U'} \leq K_{21}.$$

Here $\|Y_{zz}^j\|, \|f_{zz}^j\|, \|f_{z\gamma}^r\|$ and $\|f_{\gamma\gamma}^r\|$ are evaluated by the norm for 3 dimensional array: the norm for the 3 dimensional array $(a_{i,j,k})$ is defined by $(\sum_{i,j,k} a_{i,j,k}^2)^{1/2}$.

We have by (5.11),

$$(6.4) \quad \|f^j(\hat{z}_m^j(1), \hat{t}_j)\| = \|f^j(\hat{z}_m^j(1), \hat{t}_j) - f^j(\hat{z}^j(1), \hat{t}_j)\| \\ \leq K_8 K_{17} \sqrt{2} \{\sqrt{m} m(m-1)\}, \quad j=1, \dots, r-1,$$

and similarly

$$(6.5) \quad \|f^r(\hat{z}_m^r(1), \hat{t}_r, \hat{\gamma})\| \leq K_8 K_{17}^r \sqrt{2}/\{\sqrt{m} m(m-1)\}.$$

$\sum_{j=1}^r \sum_{i=1}^m \|Z_i^j(\hat{\alpha}^j)\|^2$ is calculated, by (4.6) and (5.4), as

$$(6.6) \quad \sum_{j=1}^r \sum_{i=1}^m \|Z_i^j(\hat{\alpha}^j)\|^2 = \sum_{j=1}^r \|\dot{\hat{z}}_m^j - P_{m-1} Y^j(\hat{z}_m^j, \hat{t}_j(\tau))(\hat{t}_j - \hat{t}_{j-1})/2\|_q^2,$$

$$(6.7) \quad \begin{aligned} \dot{\hat{z}}_m^j - P_{m-1} Y^j(\hat{z}_m^j, \hat{t}_j(\tau))(\hat{t}_j - \hat{t}_{j-1})/2 &= (\dot{\hat{z}}_m^j - \dot{\hat{z}}^j) \\ &+ \{Y^j(\hat{z}^j, \hat{t}_j(\tau)) - P_{m-1} Y^j(\hat{z}_m^j, \hat{t}_j(\tau))\}(\hat{t}_j - \hat{t}_{j-1})/2 \\ &+ P_{m-1} \{Y^j(\hat{z}^j, \hat{t}_j(\tau)) - Y^j(\hat{z}_m^j, \hat{t}_j(\tau))\}(\hat{t}_j - \hat{t}_{j-1})/2. \end{aligned}$$

We have, by (5.8) and applying (5.7) twice,

$$(6.8) \quad \begin{aligned} \|\dot{\hat{z}}_m^j - \dot{\hat{z}}^j\|_q &\leq \frac{\sqrt{m+2}}{\sqrt{2}} \|(\langle N \rangle - P_{m-1})\dot{\hat{z}}^j\|_q + \|(\langle N \rangle - P_{m+1})\dot{\hat{z}}^j\|_q \\ &\leq \frac{\sqrt{m+2}}{\sqrt{2} m} \|(\langle N \rangle - P_{m-2})\ddot{\hat{z}}^j\|_q + \frac{1}{m+2} \|(\langle N \rangle - P_m)\ddot{\hat{z}}^j\|_q \\ &\leq K_{17}^j \left[\frac{\sqrt{m+2}}{\sqrt{2} m(m-1)} + \frac{1}{(m+2)(m+1)} \right], \end{aligned}$$

applying (5.7) twice

$$(6.9) \quad \|Y^j(\hat{z}^j, \hat{t}_j(\tau)) - P_{m-1} Y^j(\hat{z}^j, \hat{t}_j(\tau))\|_q \leq \frac{1}{m(m-1)} \|\ddot{Y}^j\|_q \leq \frac{K_{17}^j}{m(m-1)},$$

and applying (5.7) three times

$$(6.10) \quad \begin{aligned} \|P_{m-1} \{Y^j(\hat{z}^j, \hat{t}_j(\tau)) - Y^j(\hat{z}_m^j, \hat{t}_j(\tau))\}\|_q &\leq K_2^j \|\hat{z}^j - \hat{z}_m^j\|_q \\ &\leq K_2 K_{17}^j / \{(m+1)m(m-1)\}. \end{aligned}$$

(6.6)~(6.10) imply that

$$(6.11) \quad \begin{aligned} \left(\sum_{j=1}^r \sum_{i=1}^{m-1} \|Z_i^j(\alpha)\|^2 \right)^{1/2} &\leq K_{17} \left\{ \frac{\sqrt{m+2}}{\sqrt{2} m(m-1)} + \frac{1}{(m+2)(m+1)} \right. \\ &\quad \left. + \frac{1}{m(m-1)} + \frac{K_2}{(m+1)m(m-1)} \right\}. \end{aligned}$$

(6.4), (6.5) and (6.11) imply that the bound r of the residual error for nonlinear equations (4.7) and (4.8) is

$$(6.12) \quad \|f\| = O(m^{-3/2}).$$

(6.3) and (6.12) prove (i) in § 5.

§7. The discrepancy of Jacobian matrix at two different points for nonlinear equations

The Jacobian matrix $g_\alpha(\alpha)$ of $g(\alpha)$ with respect to α is evaluated from

$$dg(\alpha + \mu h)/d\mu|_{\mu=0} = g_\alpha(\alpha)h,$$

where h is an $N(m+1)r+r+q$ -vector which is the same dimensional with α and constituted from N -vector h_i^j corresponding to a_i^j , scalar $\mathcal{T}_j$ corresponding to t_j and q -vector δ corresponding to γ :

$$(7.1) \quad h = \text{col.} (h_i^j, \quad i=0, \dots, m, \quad \mathcal{T}_j, \quad j=1, \dots, r, \delta).$$

Let $y^j(\tau)$ be a Chebyshev series with coefficients h_i^j :

$$(7.2) \quad y^j(\tau) = \sum_{i=0}^m h_i^j T_i'(\tau).$$

The vector $g_\alpha(\alpha)h$ expresses the variation of $g(\alpha)$ due to the variation h in α . The variation $g_\alpha(\alpha)h$ is calculated as the $(r-1)+q+n+1$ column vector:

$$(7.3) \quad dg(\alpha + \mu h)/d\mu|_{\mu=0} = \text{col.} [f_z^j(z_m^j(1), t_j) y^j(1) + f_i^j(z_m^j(1), t_j) \mathcal{T}_j, \\ j=1, \dots, r-1, \\ f_z^r(z_m^r(1), t_r, \gamma) y^r(1) + f_i^r(z_m^r(1), t_r, \gamma) \mathcal{T}_r + f_\gamma^r(z_m^r(1), t_r, \gamma) \delta].$$

We put

$$\alpha' = [a_i^{j'}, t_j', \quad i=0, \dots, m, \quad j=1, \dots, r, \gamma'], \\ \alpha'' = [a_i^{j''}, t_j'', \quad i=0, \dots, m, \quad j=1, \dots, r, \gamma''], \\ z_m^{j'} = \sum_{i=0}^m a_i^{j'} T_i', \quad z_m^{j''} = \sum_{i=0}^m a_i^{j''} T_i'.$$

Let α' and α'' be arbitrary vectors such that $[z_m^{j'}, t_j', j=1, \dots, r, \gamma']$ and $[z_m^{j''}, t_j'', j=1, \dots, r, \gamma'']$ lies in U together with $[\theta z_m^{j'} + (1-\theta) z_m^{j''}, \theta t_j' + (1-\theta) t_j'', \theta \gamma']$

$+(1-\theta)\gamma''$], $(0 \leq \theta \leq 1)$. $\{g(\alpha') - g(\alpha'')\}h$ is calculated as

$$\begin{aligned}
 (7.4) \quad \text{col. } & [\{f_z^j(z^{j'}(1), t'_j) - f_z^j(z^{j''}(1), t''_j)\}y^j(1) \\
 & + \{f_t^j(z^{j'}(1), t'_j) - f_t^j(z^{j''}(1), t''_j)\}\mathcal{T}_j, j=1, \dots, r-1, \\
 & \{f_z^r(z^{r'}(1), t'_r, \gamma') - f_z^r(z^{r''}(1), t''_r, \gamma'')\}y^r(1) \\
 & + \{f_t^r(z^{r'}(1), t'_r, \gamma') - f_t^r(z^{r''}(1), t''_r, \gamma'')\}\mathcal{T}_r \\
 & + \{f_\gamma^r(z^{r'}(1), t'_r, \gamma') - f_\gamma^r(z^{r''}(1), t''_r, \gamma'')\}\delta] = \text{col.} \\
 & \{[z^{j'}(1) - z^{j''}(1), t'_j - t''_j] \int_0^1 \begin{bmatrix} f_{zz}^j(\theta) & f_{tz}^j(\theta) \\ f_{zt}^j(\theta) & f_{tt}^j(\theta) \end{bmatrix} d\theta \begin{bmatrix} y_j(1) \\ \mathcal{T}_j \end{bmatrix}, \\
 & j=1, \dots, r-1, \\
 & [z^{r'}(1) - z^{r''}(1), t'_r - t''_r, \gamma' - \gamma''] \int_0^1 \begin{bmatrix} f_{zz}^r(\theta) & f_{tz}^r(\theta) & f_{\gamma z}^r(\theta) \\ f_{zt}^r(\theta) & f_{tt}^r(\theta) & f_{\gamma t}^r(\theta) \\ f_{z\gamma}^r(\theta) & f_{t\gamma}^r(\theta) & f_{\gamma\gamma}^r(\theta) \end{bmatrix} d\theta \begin{bmatrix} Y^r(1) \\ \mathcal{T}_r \\ \delta \end{bmatrix} \},
 \end{aligned}$$

where $f_{z\gamma}^r(z, t, \gamma)$ is defined as the function for q -vector $\tilde{\gamma}$ and N -vector $\tilde{y}$ such that $\tilde{\gamma} f_{z\gamma}^r(z, t, \gamma) \tilde{y}$ expresses $q+n+1$ column vector $\partial f_z^r(z, t, \tilde{\gamma})y/\partial \gamma \cdot \tilde{\gamma}$ and in the above expression,

$$[\gamma' - \gamma''] f_{z\gamma}^r(z, t, \gamma) y^r(1) = [\partial f_z^r(z, t, \gamma) y^r(1) / \partial \gamma] [\gamma' - \gamma''].$$

In addition $f_{zz}^j, f_{tz}^j, f_{zt}^j, f_{tt}^j, f_{\gamma z}^j, f_{t\gamma}^j, f_{\gamma t}^j$ and $f_{\gamma\gamma}^j$ are defined similarly as $f_{z\gamma}^j$. $f_{z\gamma}^r(\theta)$ is defined as

$$f_{z\gamma}^r(\theta) = f_{z\gamma}^r(\theta z^{r'}(1) + (1-\theta)z^{r''}(1), \theta t'_r + (1-\theta)t''_r, \theta \gamma' + (1-\theta)\gamma'')$$

and $f_{zz}^j(\theta), f_{tz}^j(\theta), f_{zt}^j(\theta), f_{tt}^j(\theta), f_{\gamma z}^j(\theta), f_{t\gamma}^j(\theta), f_{\gamma t}^j(\theta)$ and $f_{\gamma\gamma}^j(\theta)$ are defined similarly as $f_{z\gamma}^j(\theta)$. The norm of $\|g_\alpha(\alpha') - g_\alpha(\alpha'')\|$ is calculated as

$$\begin{aligned}
 \| \{g_\alpha(\alpha') - g_\alpha(\alpha'')\}h \|^2 & \leq \sum_{j=1}^{r-1} (\|z_m^{j'} - z_m^{j''}\|_n^2 + (t'_j - t''_j)^2)(K_9^2 + 2K_{10}^2 + K_{11}^2) \\
 & (\|y^j\|_n^2 + \mathcal{T}_j^2) + (\|z_m^{r'} - z_m^{r''}\|_n^2 + (t'_r - t''_r)^2 + \|\gamma' - \gamma''\|^2)(K_9^2 + 2K_{10}^2 + K_{11}^2 + 2K_{19}^2 \\
 & + 2K_{20}^2 + K_{21}^2)(\|y^r\|_n^2 + \mathcal{T}_r^2 + \|\delta\|^2) \leq (2m+1)^2(K_9^2 + 2K_{10}^2 + K_{11}^2 + 2K_{19}^2 \\
 & + 2K_{20}^2 + K_{21}^2)(\|y\|_q^2 + \|\mathcal{T}\|^2 + \|\delta\|^2) \left(\sum_{j=1}^r \{ \|z_m^{j'} - z_m^{j''}\|_q^2 + (t'_j - t''_j)^2 + \|\gamma' - \gamma''\|^2 \} \right) \\
 & \leq (2m+1)^2(K_9^2 + 2K_{10}^2 + K_{11}^2 + 2K_{19}^2 + 2K_{20}^2 + K_{21}^2)\|\alpha' - \alpha''\|^2 \|h\|^2:
 \end{aligned}$$

$$(7.5) \quad \|g_\alpha(\alpha') - g_\alpha(\alpha'')\| \leq (2m+1) \sqrt{K_9^2 + 2K_{10}^2 + K_{11}^2 + 2K_{19}^2 + 2K_{20}^2 + K_{21}^2} \|\alpha' - \alpha''\|.$$

In [5], only linear boundary conditions are treated, and formula (7.5) did not appear.

We digress from the main subject to evaluate S_m^j : S_m^j will be appear in (8.6) and be necessary to (8.6). We put in (7.4)

$$z_m^{j'} = \hat{z}^j, \quad z_m^{j''} = \hat{z}_m^j, \quad t_j' = t_j'' = \hat{t}_j, \quad \gamma' = \gamma'' = \hat{\gamma}$$

then we have, by (5.11),

$$(7.6) \quad \sum_{j=1}^{r-1} \left\| \{f_z^j(\hat{z}^j(1), \hat{t}_j) - f_z(\hat{z}_m^j(1), \hat{t}_j)\} y^j(1) + \{f_t^j(\hat{z}^j(1), \hat{t}_j) - f_t^j(\hat{z}_m^j(1), \hat{t}_j)\} \mathcal{T}_j \right\|^2 + \left\| \{f_z^r(\hat{z}^r(1), \hat{t}_r, \hat{\gamma}) - f_z^r(\hat{z}_m^r(1), \hat{t}_r, \hat{\gamma})\} y^r(1) + \{f_t^r(\hat{z}^r(1), \hat{t}_r, \hat{\gamma}) - f_t^r(\hat{z}_m^r(1), \hat{t}_r, \hat{\gamma})\} \mathcal{T}_r + \{f_\gamma^r(\hat{z}^r(1), \hat{t}_r, \hat{\gamma}) - f_\gamma^r(\hat{z}_m^r(1), \hat{t}_r, \hat{\gamma})\} \delta \right\|^2 = \text{col.} \{[\hat{z}^j(1) - \hat{z}_m^j(1)] \int_0^1 [f_{zz}^j(\theta), f_{tz}^j(\theta)] d\theta \left[\begin{array}{c} y^j(1) \\ \mathcal{T}_j \end{array} \right], \quad j=1, \dots, r-1, \\ [\hat{z}^r(1) - \hat{z}_m^r(1)] \int_0^1 [f_{zz}^r(\theta), f_{tz}^r(\theta), f_{\gamma z}^r(\theta)] d \left[\begin{array}{c} y^r(1) \\ \mathcal{T}_r \\ \delta \end{array} \right] \} \|^2 \\ \leq \sum_{j=1}^r \|\hat{z}^j - \hat{z}_m^j\|_n^2 (K_9^2 + K_{10}^2 + K_{19}^2) (2m+1) (\|y\|_q^2 + \|\mathcal{T}\|^2 + \|\delta\|^2) \\ \leq K_{17} \frac{2(2m+1)}{m m^2 (m-1)^2} (K_9^2 + K_{10}^2 + K_{19}^2) (\|y\|_q^2 + \|\mathcal{T}\|^2 + \|\delta\|^2), \\ (\mathcal{T} = [\mathcal{T}_1, \dots, \mathcal{T}_r]).$$

Let S_m^j be discrepancy between the values of the function $f_z^j(z, \hat{t}_j) y^j(1) + f_t^j(z, \hat{t}_j) \mathcal{T}_j$ at $z = \hat{z}^j(1)$ and that at $z = \hat{z}_m^j(1)$ and S_m^r be discrepancy between the values of $f_z^r(z, \hat{t}_r, \hat{\gamma}) y^r(1) + f_t^r(z, \hat{t}_r, \hat{\gamma}) \mathcal{T}_r + f_\gamma^r(z, \hat{t}_r, \hat{\gamma}) \delta$ at $z = \hat{z}^r(1)$ and that at $z = \hat{z}_m^r(1)$:

$$\begin{aligned}
S_m^j &= \{f_z^j(\hat{z}^j(1), \hat{t}_j) - f_z^j(\hat{z}_m^j(1), \hat{t}_j)\} y^j(1) + \{f_t^j(\hat{z}^j(1), \hat{t}_j) \\
&\quad - f_t^j(\hat{z}_m^j(1), \hat{t}_j)\} \mathcal{T}_j, \\
S_m^r &= \{f_z^r(\hat{z}^r(1), \hat{t}_r, \hat{\gamma}) - f_z^r(\hat{z}_m^r(1), t_j, \hat{\gamma})\} y^r(1) + \{f_t^r(\hat{z}^r(1), t_r, \hat{\gamma}) \\
&\quad - f_t^r(\hat{z}_m^r(1), \hat{t}_r, \hat{\gamma})\} \mathcal{T}_r + \{f^r(\hat{z}^r(1), \hat{t}_r, \hat{\gamma}) - f^r(\hat{z}_m^r(1), \hat{t}_r, \hat{\gamma})\} \delta.
\end{aligned}
\tag{7.7}$$

Then, by (7.6), we have

$$\|S_m\| = \left(\sum_{j=1}^r \|S_m^j\|^2 \right)^{1/2} \leq O(m^{-2}) (\|y\|_q^2 + \|\mathcal{T}\|^2 + \|\delta\|^2)^{1/2},
\tag{7.8}$$

where

$$S_m = [S_m^1, \dots, S_m^r].$$

We return to our subject. The Chebyshev series with Chebyshev coefficients $Z_{i\alpha}^j(\alpha)h$ is calculated as, by (4.1), (4.4)~(4.6),

$$\begin{aligned}
(7.9) \quad \sum_{i=1}^m Z_{i\alpha}^j(\alpha)hT'_i &= \sum_{i=1}^m dZ_i^j(\alpha + \mu h)/d\mu|_{\mu=0}T'_i = \sum_{i=1}^m T'_{i-1} [1/ \\
&\quad e_{i-1} \cdot \sum_{s=0}^m v_{s-(i-1)} s h_s^j - \sqrt{2}/(\pi e_{i-1}) \cdot \int_0^\pi \{Y_z^j(z_m^j(\cos \theta), t_j(\cos \theta)) y^j \\
&\quad (\cos \theta) + Y_t^j(z_m^j(\cos \theta), t_j(\cos \theta)) \mathcal{T}_j(\cos \theta)\} \cos(i-1)\theta d\theta \\
&\quad (t_j - t_{j-1})/2 - \sqrt{2}/(\pi e_{i-1}) \cdot \int_0^\pi Y^j(z_m^j(\cos \theta), t_j(\cos \theta)) \cos(i-1) \\
&\quad \theta d\theta (\mathcal{T}_j - \mathcal{T}_{j-1})/2] = \dot{y}^j - P_{m-1} [\{Y_z^j(z_m^j, t_j(\tau)) y^j + Y_t^j(z_m^j, t_j(\tau)) \\
&\quad \mathcal{T}_j(\tau)\} (t_j - t_{j-1})/2 + Y^j(z_m^j, t_j(\tau)) (\mathcal{T}_j - \mathcal{T}_{j-1})/2].
\end{aligned}$$

$(Z_{i\alpha}^j(\alpha') - Z_{i\alpha}^j(\alpha''))h$ is calculated successively as follows:

$$\begin{aligned}
(7.10) \quad \sum_{i=0}^m \{Z_{i\alpha}^j(\alpha') - Z_{i\alpha}^j(\alpha'')\} h T'_i &= P_{m-1} \left[\left\{ Y_z^j(z_m^{j'}, t'_j(\tau)) \frac{t'_j - t'_{j-1}}{2} \right. \right. \\
&\quad \left. \left. - Y_z^j(z_m^{j''}, t''_j(\tau)) \frac{t''_j - t''_{j-1}}{2} \right\} y^j + \left\{ Y_t^j(z_m^{j'}, t'_j(\tau)) \frac{t'_j - t'_{j-1}}{2} \right. \right. \\
&\quad \left. \left. - Y_t^j(z_m^{j''}, t''_j(\tau)) \frac{t''_j - t''_{j-1}}{2} \right\} \mathcal{T}_j(\tau) + \{Y^j(z_m^{j'}, t'_j(\tau)) \right. \\
&\quad \left. - Y^j(z_m^{j''}, t''_j(\tau))\} \frac{\mathcal{T}_j - \mathcal{T}_{j-1}}{2} \right] = P_{m-1} \int_0^1 \frac{dg^j(\theta)}{d\theta} d\theta.
\end{aligned}$$

Here we put

$$g^j(\theta) = \{Y_z^j(\theta)y^j + Y_t^j(\theta)\mathcal{T}_j(\tau)\}\left(\theta\frac{t'_j - t'_{j-1}}{2} + (1-\theta)\frac{t''_j - t''_{j-1}}{2}\right) \\ + Y^j(\theta)\frac{\mathcal{T}_j - \mathcal{T}_{j-1}}{2},$$

where

$$Y_z^j(\theta) = Y_z^j(\theta z_m^{j'} + (1-\theta)z_m^{j''}, \theta t'_j(\tau) + (1-\theta)t''_j(\tau))$$

and $Y_t^j(\theta)$, $Y^j(\theta)$, $Y_{zz}^j(\theta)$, $Y_{tz}^j(\theta)$ and $Y_{tt}^j(\theta)$ are defined similarly as $Y_z^j(\theta)$. $dg^j(\theta)/d\theta$ is calculated as

$$\begin{aligned} & [\{z_m^{j'} - z_m^{j''}\}, (t'_j - t''_j)(\tau)] \begin{bmatrix} Y_{zz}^j(\theta) & Y_{tz}^j(\theta) \\ Y_{zt}^j(\theta) & Y_{tt}^j(\theta) \end{bmatrix} \begin{bmatrix} y^j \\ \mathcal{T}_j(\tau) \end{bmatrix} \\ & \left(\theta\frac{t'_j - t'_{j-1}}{2} + (1-\theta)\frac{t''_j - t''_{j-1}}{2}\right) + [Y_z^j(\theta), Y_t^j(\theta)] \begin{bmatrix} y^j \\ \mathcal{T}_j(\tau) \end{bmatrix} \\ & \left(\frac{t'_j - t'_{j-1}}{2} - \frac{t''_j - t''_{j-1}}{2}\right) + [\{z_m^{j'} - z_m^{j''}\}, (t'_j - t''_j)(\tau)] \\ & \begin{bmatrix} Y_z^j(\theta) \\ Y_t^j(\theta) \end{bmatrix} \frac{\mathcal{T}_j - \mathcal{T}_{j-1}}{2} = [\{z_m^{j'} - z_m^{j''}\}, (t'_j - t''_j)\frac{\tau+1}{2}] \\ & + (t'_{j-1} - t''_{j-1})\frac{1-\tau}{2} \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix} \begin{bmatrix} y^j \\ \mathcal{T}_j\frac{\tau+1}{2} + \mathcal{T}_{j-1}\frac{1-\tau}{2} \end{bmatrix} \\ & + [A_4, A_5] \begin{bmatrix} y^j \\ \mathcal{T}_j\frac{\tau+1}{2} + \mathcal{T}_{j-1}\frac{1-\tau}{2} \end{bmatrix} \left(\frac{t'_j - t''_j}{2} - \frac{t'_{j-1} - t''_{j-1}}{2}\right) \\ & + \left[\{z_m^{j'} - z_m^{j''}\}, (t'_j - t''_j)\frac{\tau+1}{2} + (t'_{j-1} - t''_{j-1})\frac{1-\tau}{2}\right] \begin{bmatrix} A_4 \\ A_5 \end{bmatrix} \frac{\mathcal{T}_j - \mathcal{T}_{j-1}}{2} \\ & = \frac{1}{4} \{[\{z_m^{j'} - z_m^{j''}\}, t'_j - t''_j] B_1 \begin{bmatrix} y^j \\ \mathcal{T}_j \end{bmatrix} + [\{z_m^{j'} - z_m^{j''}\}, t'_{j-1} - t''_{j-1}] B_2 \end{aligned}$$

$$\begin{aligned} & \begin{bmatrix} y^j \\ \mathcal{T}_j \end{bmatrix} + [\{z_m^{j'} - z_m^{j''}\}, t'_j - t''_j] B_3 \begin{bmatrix} y^j \\ \mathcal{T}_{j-1} \end{bmatrix} + [\{z_m^{j'} - z_m^{j''}\}, \\ & t'_{j-1} - t''_{j-1}] B_4 \begin{bmatrix} y^j \\ \mathcal{T}_{j-1} \end{bmatrix} \}, \end{aligned}$$

where $(t'_j - t''_j)(\tau)$ expresses the function such that

$$\begin{aligned} (t'_j - t''_j)(\tau) &= \frac{(t'_j - t''_j) - (t'_{j-1} - t''_{j-1})}{2} \tau + \frac{(t'_j - t''_j) + (t'_{j-1} - t''_{j-1})}{2} \\ &= (t'_j - t''_j) \frac{\tau + 1}{2} + (t'_{j-1} - t''_{j-1}) \frac{1 - \tau}{2} \end{aligned}$$

and

$$A_1 = Y_{zz}^j(\theta) \left(\theta \frac{t'_j - t'_{j-1}}{2} + (1 - \theta) \frac{t''_j - t''_{j-1}}{2} \right), \quad A_4 = Y_z^j(\theta),$$

$$A_2 = Y_{iz}^j(\theta) \left(\theta \frac{t'_j - t'_{j-1}}{2} + (1 - \theta) \frac{t''_j - t''_{j-1}}{2} \right), \quad A_5 = Y_i^j(\theta),$$

$$A_2^T = Y_{zi}^j(\theta) \left(\theta \frac{t'_j - t'_{j-1}}{2} + (1 - \theta) \frac{t''_j - t''_{j-1}}{2} \right),$$

$$A_3 = Y_{it}^j(\theta) \left(\theta \frac{t'_j - t'_{j-1}}{2} + (1 - \theta) \frac{t''_j - t''_{j-1}}{2} \right),$$

$$B_1 = \begin{bmatrix} A_1, & (1 + \tau)A_2 + A_4 \\ (1 + \tau)A_2^T + A_4, & (1 + \tau)^2 A_3 + 2(1 + \tau)A_5 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} A_1, & (1 + \tau)A_2 + A_4 \\ (1 - \tau)A_2^T - A_4, & (1 - \tau^2)A_3 - 2\tau A_5 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} A_1, & (1 - \tau)A_2 - A_4 \\ (1 + \tau)A_2^T + A_4, & (1 - \tau^2)A_3 - 2\tau A_5 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} A_1, & (1 - \tau)A_2 - A_4 \\ (1 - \tau)A_2^T - A_4, & (1 - \tau)^2 A_3 - 2(1 - \tau)A_5 \end{bmatrix}.$$

The norm $\|B_p\|$ is bounded:

$$\begin{aligned} \|B_p\| \leq \|A_1\| + 2(2\|A_2\| + \|A_4\|) + 4\|A_3\| + 4\|A_5\| \leq \max_{0 \leq \theta \leq 1} \{ \|Y_{zz}^j(\theta)\|_n \\ + 2(2\|Y_{tz}^j(\theta)\|_n + \|Y_z^j(\theta)\|_n) + 4\|Y_{tr}^j(\theta)\|_n + 4\|Y_t^j(\theta)\|_n \} \leq \{K_4 + 2(2K_5 \\ + K_2) + 4(K_6 + K_3)\}, \text{ for } \theta \in [0, 1], \tau \in [-1, 1], p = 1, \dots, 4; \end{aligned}$$

$$\max_{\substack{p=1, \dots, 4, \\ \theta \in [0, 1] \\ \tau \in [-1, 1]}} \|B_p\| \leq \{K_4 + 2(2K_5 + K_2) + 4(K_6 + K_3)\}$$

($\sum_{j=1}^r \sum_{i=0}^m \|Z_{i\alpha}^j(\alpha') - Z_{i\alpha}^j(\alpha'')\|^2$)^{1/2} is calculated as follows:

$$\begin{aligned} \left\| \sum_{i=0}^m (Z_{i\alpha}^j(\alpha') - Z_{i\alpha}^j(\alpha'')) h \right\| &\leq \left\| \int_0^1 \frac{dg^j(\theta)}{d\theta} d\theta \right\|_q \leq \frac{1}{4} \max_{p, \theta, \tau} \|B_p\| \\ &\{ \sqrt{\|z_m^{j'} - z_m^{j''}\|_n^2 + (t_j' - t_j'')^2} + \sqrt{\|z_m^{j'} - z_m^{j''}\|^2 + (t_{j-1}' - t_{j-1}'')^2} \} \{ \sqrt{\|y^j\|_q^2 + \mathcal{T}_j^2} \\ &+ \sqrt{\|y^j\|_q^2 + \mathcal{T}_{j-1}^2} \} \leq \sqrt{\frac{2}{2}} \cdot \max_{p, \theta, \tau} \|B_p\| \{ \|z_m^{j'} - z_m^{j''}\|_n^2 + (t_j' - t_j'')^2 + \|z_m^{j'} - z_m^{j''}\|_n^2 \\ &+ (t_{j-1}' - t_{j-1}'')^2 \}^{1/2} \sqrt{\|y\|_q^2 + \|\mathcal{T}\|^2}; \\ \sum_{i=0}^m \|Z_{i\alpha}^j(\alpha') - Z_{i\alpha}^j(\alpha'')\|^2 &\leq \left(\sqrt{\frac{2}{2}} \max_{p, \theta, \tau} \|B_p\| \right)^2 \{ 2\|z_m^{j'} - z_m^{j''}\|_n^2 + (t_j' - t_j'')^2 \\ &+ (t_{j-1}' - t_{j-1}'')^2 \}; \\ (7.11) \quad \left(\sum_{j=1}^r \sum_{i=0}^m \|Z_{i\alpha}^j(\alpha') - Z_{i\alpha}^j(\alpha'')\|^2 \right)^{1/2} &\leq \sqrt{\frac{2}{2}} \max_{p, \theta, \tau} \|B_p\| \left[\sum_{j=1}^r \{ 2\|z_m^{j'} - z_m^{j''}\|_n^2 \right. \\ &\left. + (t_j' - t_j'')^2 + (t_{j-1}' - t_{j-1}'')^2 \} \right]^{1/2} \leq \max_{p, \theta, \tau} \|B_p\| \sqrt{2m+1} \|\alpha' - \alpha''\|. \end{aligned}$$

Thus (ii) in § 5 is proved in (7.5) and (7.11).

We digress from the main subject to evaluate R_m^j which will appear in (8.9). We put

$$z_m^{j'} = \hat{z}_m^j, \quad z_m^{j''} = \hat{z}_m^j, \quad t_j' = t_j'' = \hat{t}_j$$

in (7.10) and in $dg^j(\theta)/d\theta$ of § 7. Then

$$\begin{aligned} (7.12) \quad &\left\{ \sum_{j=1}^r \left\| P_{m-1} \left[Y_z^j(\hat{z}_m^j, \hat{t}_j(\tau)) y^j + Y_t^j(\hat{z}_m^j, \hat{t}_j(\tau)) \mathcal{T}_j(\tau) \right] \frac{\hat{t}_j - \hat{t}_{j-1}}{2} \right. \right. \\ &\left. \left. + Y^j(\hat{z}_m^j, \hat{t}_j(\tau)) \frac{\mathcal{T}_j - \mathcal{T}_{j-1}}{2} - \{ Y_z^j(\hat{z}_m^j, \hat{t}_j(\tau)) y^j + Y_t^j(\hat{z}_m^j, \hat{t}_j(\tau)) \} \right\| \right\} \end{aligned}$$

$$\begin{aligned}
& \left\| \mathcal{T}_j(\tau) \left\{ \frac{\hat{t}_j - \hat{t}_{j-1}}{2} - Y^j(\hat{z}_m^j, \hat{t}_j(\tau)) \frac{\mathcal{T}_j - \mathcal{T}_{j-1}}{2} \right\} \right\|_q^2 \Bigg\}^{1/2} \\
& \leq (K_4 + 2K_5 + K_2) \left\{ \sum_{j=1}^r \|\hat{z}_m^j - \hat{z}^j\|_n^2 \right\}^{1/2} \sqrt{\|y\|_q^2 + \|\mathcal{T}\|^2} \\
& \leq K_{17}(K_4 + 2K_5 + K_2) / \{\sqrt{m} m(m-1)\} \cdot \sqrt{\|y\|_q^2 + \|\mathcal{T}\|^2}.
\end{aligned}$$

We put

$$\begin{aligned}
X^j(z, \tau) = & \{Y_z^j(z, \hat{t}_j(\tau))y^j + Y_t^j(z, \hat{t}_j(\tau))\mathcal{T}_j(\tau)\}(\hat{t}_j - \hat{t}_{j-1})/2 + Y^j(z, \hat{t}_j(\tau)) \\
& (\mathcal{T}_j - \mathcal{T}_{j-1})/2.
\end{aligned}$$

Let R_m^j be discrepancy between the values of the function $X^j(z, \tau)$ at $z = \hat{z}^j$ and that at $z = \hat{z}_m^j$:

$$\begin{aligned}
(7.13) \quad R_m^j = & -\{X^j(\hat{z}^j, \tau) - P_{m-1}X^j(\hat{z}_m^j, \tau)\} \\
& = -(\langle N \rangle - P_{m-1})X^j(\hat{z}^j, \tau) - P_{m-1}\{X^j(\hat{z}^j, \tau) - X^j(\hat{z}_m^j, \tau)\}.
\end{aligned}$$

The norm of the second term $P_{m-1}\{X^j(\hat{z}^j, \tau) - X^j(\hat{z}_m^j, \tau)\}$ of R_m^j is calculated, by (7.12), as

$$(7.14) \quad \|P_{m-1}\{X^j(\hat{z}^j, \tau) - X^j(\hat{z}_m^j, \tau)\}\|_q \leq O(m^{-5/2})\sqrt{\|y\|_q^2 + \|\mathcal{T}\|^2}.$$

(7.14) will be necessary in (8.9).

§8. The inverse of the Jacobian matrix of the determining equation

For the variations $Z_{i\alpha}^j(\hat{\alpha})h$ and $g_\alpha(\hat{\alpha})h$ of nonlinear formulas, we put

$$(8.1) \quad Z_{i\alpha}^j(\hat{\alpha})h = \zeta_i^j,$$

$$(8.2) \quad g_\alpha(\hat{\alpha})h = \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix},$$

where ζ_i^j is N -vector, c_j a scalar for $j=1, \dots, r-1$ and c_r an $n+q+1$ -vector. (8.2) is calculated, by (7.3), as

$$\begin{aligned}
(8.3) \quad & f_z^j(\hat{z}_m^j(1), \hat{t}_j)y^j(1) + f_t^j(\hat{z}_m^j(1), \hat{t}_j)\mathcal{T}_j = c_j, \quad j=1, \dots, r-1, \\
& f_z^r(\hat{z}_m^r(1), \hat{t}_r, \hat{\gamma})y^r(1) + f_t^r(\hat{z}_m^r(1), \hat{t}_r, \hat{\gamma})\mathcal{T}_r + f_\gamma^r(\hat{z}_m^r(1), \hat{t}_r, \hat{\gamma})\delta = c_r.
\end{aligned}$$

For the variation $L(\hat{\alpha})h$ for linear formulas, we put

$$L(\hat{\alpha})h = \begin{bmatrix} b_r \\ b_1 \\ \vdots \\ b_{r-1} \end{bmatrix}, \text{ that is,}$$

$$(8.4) \quad L_0 y^1(-1) + L_1 y^r(1) = b_r,$$

$$(8.5) \quad y^{j+1}(-1) - y^j(1) = b_j, \quad j=1, \dots, r-1,$$

where b_j an N - and b_r an n -vector. The algebraic equations (8.3) at $\hat{\alpha}$ with unknowns h are regarded as the boundary conditions defined at the truncated Chebyshev series $\hat{z}_m^j$, exact times $\hat{t}_j$, $j=1, \dots, r$ and exact parameters $\hat{\gamma}$, which has y^j , $\mathcal{T}_j$, $j=1, \dots, r$ and δ as unknowns. The boundary conditions (8.3) at $(\hat{z}_m^j, \hat{t}_j, j=1, \dots, r, \hat{\gamma})$ are rewritten as the boundary conditions at the exact solution $(\hat{z}^j, \hat{t}_j, j=1, \dots, r, \gamma)$ as follows:

$$(8.6) \quad \begin{aligned} f_z^j(\hat{z}^j(1), \hat{t}_j) y^j(1) + f_t^j(\hat{z}^j(1), \hat{t}_j) \mathcal{T}_j &= c_j + S_m^j, \\ f_z^r(z^r(1), \hat{t}_r, \hat{\gamma}) y^r(1) + f_t^r(\hat{z}^r(1), \hat{t}_r, \hat{\gamma}) \mathcal{T}_r + f_\gamma^r(\hat{z}^r(1), \hat{t}_r, \hat{\gamma}) \delta &= c_r + S_m^r, \end{aligned}$$

where the differences S_m^j are defined in (7.7) and have small values, with the order $O(m^{-2})$ for large m as shown in (7.8). We put

$$(8.7) \quad \varphi^j = \sum_{i=1}^m \zeta_i^j T'_i, \quad \varphi = (\varphi^1, \dots, \varphi^r).$$

The algebraic equations (8.1) at $\hat{\alpha}$ with unknowns h turn into

$$\sum_{i=0}^m Z_{i\alpha}^j(\hat{\alpha}) h T'_i = \sum_{i=0}^m \zeta_i^j T'_i = \varphi^j, \quad j=1, \dots, r.$$

Note that $Z_{i\alpha}^j(\hat{\alpha})h$ does not depend on $\hat{\gamma}$ and δ by (4.4) and (4.5). The equations (8.1) are written, by (4.1)~(4.6), as the differential equations for the truncated Chebyshev series $\hat{z}_m^j$, and the exact time $\hat{t}_j$, $j=1, \dots, r$ with unknowns $(y^j, \mathcal{T}_j, j=1, \dots, r)$:

$$(8.8) \quad \begin{aligned} \dot{y}^j - P_{m-1} X^j(\hat{z}_m^j, \tau) &= \varphi^j, \quad j=1, \dots, r, \text{ that is,} \\ \dot{y}^j - P_{m-1} \{ Y_z^j(\hat{z}_m^j, \hat{t}_j(\tau)) y^j + Y_t^j(\hat{z}_m^j, \hat{t}_j(\tau)) \mathcal{T}_j(\tau) \} &(\hat{t}_j - \hat{t}_{j-1})/2 \\ + Y^j(\hat{z}_m^j, \hat{t}_j(\tau)) (\mathcal{T}_j - \mathcal{T}_{j-1})/2 &= \varphi^j. \end{aligned}$$

The system (8.8) is rewritten as the differential system to the exact solution $(\hat{z}^j, \hat{t}_j, j=1, \dots, r)$

$$(8.9) \quad \dot{y}^j - X^j(\hat{z}^j, \tau) = \varphi^j + R_m^j,$$

where the differences R_m^j are defined in (7.13) and the norm of the second term of (7.13) is small with the order $O(m^{-5/2})$ for large m as shown in (7.14). The norm of the first term of R_m^j in (7.13) is calculated, by (5.7),

$$\| -(\langle N \rangle - P_{m-1})X^j(\hat{z}^j(\tau), \tau) \|_q \leq \left\| \frac{dX^j}{d\tau}(\hat{z}^j(\tau), \tau) \right\|_q / m,$$

where

$$\begin{aligned} \left\| \frac{d}{d\tau} X^j(\hat{z}^j(\tau), \tau) \right\|_q &\leq \left\| \left\{ \frac{dY_z^j(\hat{z}^j(\tau), \hat{t}_j(\tau))}{d\tau} y^j + \frac{dY_t^j(\hat{z}^j(\tau), \hat{t}_j(\tau))}{d\tau} \mathcal{T}_j(\tau) \right\} \right. \\ &\quad \left. \frac{\hat{t}_j - \hat{t}_{j-1}}{2} + \frac{dY^j(\hat{z}^j, \hat{t}_j(\tau))}{d\tau} \frac{\mathcal{T}_j - \mathcal{T}_{j-1}}{2} + \{Y_z^j \dot{y}^j + Y_t^j(\mathcal{T}_j - \mathcal{T}_{j-1})\} \frac{\hat{t}_j - \hat{t}_{j-1}}{2} \right\|_q \\ &\leq K_{12} \|y^j\|_q + K_{13}(|\mathcal{T}_j| + |\mathcal{T}_{j-1}|) + K_{14}/2 \cdot (|\mathcal{T}_j| + |\mathcal{T}_{j-1}|) + K_2 \|\dot{y}\|_q \\ &\quad + K_3(|\mathcal{T}_j| + |\mathcal{T}_{j-1}|) \end{aligned}$$

with

$$\begin{aligned} \|\dot{y}\|_q &= \|P_{m-1}X^j(\hat{z}_m^j, \tau) + \varphi^j\|_q \leq K_2 \|y^j\|_q + K_3(|\mathcal{T}_j| + |\mathcal{T}_{j-1}|) + K_1/2 \cdot \\ &\quad (|\mathcal{T}_j| + |\mathcal{T}_{j-1}|) + \|\varphi^j\|_q, \end{aligned}$$

We have

$$\begin{aligned} \| -(\langle N \rangle - P_{m-1})X^j(\hat{z}^j, \tau) \|_q &\leq 1/m \cdot [(K_{12} + K_2^2) \|y^j\|_q + \{K_{13} + K_{14}/2 + K_3 \\ &\quad + K_2(K_3 + K_1/2)\} (|\mathcal{T}_j| + |\mathcal{T}_{j-1}|) + K_2 \|\varphi^j\|_q] \leq H/m \cdot (\sqrt{\|y^j\|_q^2 + |\mathcal{T}_j|^2} + \|\varphi^j\|_q^2 \\ &\quad + \sqrt{\|y^j\|_q^2 + |\mathcal{T}_{j-1}|^2} + \|\varphi^j\|_q^2) \leq \sqrt{2H/m} \cdot (2\|y^j\|_q^2 + \mathcal{T}_j^2 + \mathcal{T}_{j-1}^2 + 2\|\varphi^j\|_q^2)^{1/2}, \end{aligned}$$

where H is a constant such that

$$H^2 = \left(\frac{K_{12} + K_2^2}{2} \right)^2 + \left(\frac{K_2}{2} \right)^2 + \left\{ K_{13} + \frac{K_{14}}{2} + K_3 + K_2 \left(K_3 + \frac{K_1}{2} \right) \right\}^2.$$

Finally we have

$$\left\{ \sum_{j=1}^r \| -(\langle N \rangle - P_{m-1})X^j(\hat{z}^j, \tau) \|_q^2 \right\}^{1/2} \leq 2H/m \cdot \sqrt{\|y\|_q^2 + \|\mathcal{T}\|^2 + \|\varphi\|^2}.$$

Thus we have, by (7.14) and the above inequality,

$$\|R_m\|_q = \left(\sum_{j=1}^r \|R_m^j\|_q^2 \right)^{1/2} = O(m^{-1}) \sqrt{\|y\|_q^2 + \|\mathcal{T}\|^2 + \|\varphi\|^2},$$

where

$$R_m = [R_m^1, \dots, R_m^r].$$

We shall find a solution $[y^j, \mathcal{T}_j, j=1, \dots, r, \delta]$ of the differential system (8.9) with the boundary conditions (8.4)~(8.6). The following lemmas need to find the solution.

In what follows, we use symbols $Y^j(\tau)$ in §1 and $\Phi_j(\tau)$, D^j , $\Phi_{j,i}$, $f_z^j(1)$, $f_i^j(1)$, A_j , G_1 in §2 putting

$$z^j = \hat{z}^j, \quad t_j = \hat{t}_j$$

and defining those in the same way.

LEMMA 8.1. *Let y^j , $j=1, \dots, r$ be solutions obtained successively from the system of differential equations*

$$(8.10) \quad \dot{y}^j = Y^j(\hat{z}^j, \hat{t}_j(\tau))(\hat{t}_j - \hat{t}_{j-1})/2 \cdot y_j + \sigma^j$$

satisfying

$$y^1(-1) = b_0,$$

and (8.5), that is,

$$y^j(-1) = b_{j-1} + y^{j-1}(1), \quad j=2, \dots, r.$$

Here σ^j is a continuous function on I . Then y^j are expressed as

$$(8.11) \quad y^j(\tau) = \Phi_j(\tau) \sum_{i=1}^j \Phi_{j-1,i} b_{i-1} + \Psi_j(\tau), \quad j=1, \dots, r,$$

where

$$\Psi_j(\tau) = \Phi_j(\tau) \left\{ \int_{-1}^{\tau} \Phi_j^{-1}(s) \sigma^j(s) ds + \sum_{i=1}^{j-1} \Phi_{j-1,i} \int_{-1}^1 \Phi_i^{-1}(s) \sigma^i(s) ds \right\}, \quad j=1, \dots, r.$$

Further we have

$$\|\Psi_j(1)\| \leq \sum_{i=1}^j K_{15ji} \|\sigma^i\|_q, \quad \|\Psi_j\|_q \leq \sum_{i=1}^j K_{16ji} \|\sigma^i\|_q,$$

where

$$K_{15ji} = \|\Phi_{j,i}\| \sqrt{\int_0^\pi \pi \|\Phi_j^{-1}(\cos \theta)\|^2 \sin^2 \theta d\theta},$$

$$K_{16ji} = \left[\int_0^\pi \|\Phi_j(\cos \theta)\|^2 d\theta \right]^{1/2} \|\Phi_{j-1,i}\| \sqrt{\int_0^\pi \|\Phi_j^{-1}(\cos \theta)\|^2 \sin^2 \theta d\theta},$$

$$i = 1, \dots, j, j = 1, \dots, r.$$

(8.11) is proved by induction.

LEMMA 8.2. The function $Y_t^j(\hat{z}^j, \hat{t}_j(\tau)) \mathcal{T}_j(\tau)(\hat{t}_j - \hat{t}_{j-1})/2 + Y^j(\hat{z}^j, \hat{t}_j(\tau))(\mathcal{T}_j - \mathcal{T}_{j-1})/2$ is rewritten as $q_j \mathcal{T}_{j-1} + g_j \mathcal{T}_j$, where

$$g_j(z, \tau) = Y_t^j(z, \hat{t}_j(\tau)) \frac{\hat{t}_j - \hat{t}_{j-1}}{2} \frac{\tau + 1}{2} + \frac{1}{2} Y^j(z, \hat{t}_j(\tau)), j = 1, 2, \dots, r-1,$$

$$g_r(z, \tau) = 0, q_1(z, \tau) = 0, q_j(z, \tau) = Y_t^j(z, \hat{t}_j(\tau)) \frac{\hat{t}_j - \hat{t}_{j-1}}{2} \frac{1 - \tau}{2}$$

$$- \frac{1}{2} Y^j(z, \hat{t}_j(\tau)), j = 2, \dots, r.$$

For g_j and q_j , we have

$$(8.12) \quad Y^j(\hat{z}^j(\tau), \hat{t}_j(\tau))(\tau + 1)/2 = \Phi_f(\tau) \int_{-1}^\tau \Phi^{-1}(s) g_j(\hat{z}^j(s), s) ds,$$

$$(8.13) \quad Y^j(\hat{z}^j(\tau), \hat{t}_j(\tau))(1 - \tau)/2 - \Phi_f(\tau) Y^j(\hat{z}^j(-1), \hat{t}_{j-1})$$

$$= \Phi_f(\tau) \int_{-1}^\tau \Phi_j^{-1}(s) q_j(\hat{z}^j(s), s) ds.$$

PROOF. Since the function $Y^j(\hat{z}^j(\tau), \hat{t}_j(\tau))(\tau + 1)/2$ is a solution of

$$\dot{y} = Y_z^j(\hat{z}^j, \hat{t}_j(\cdot))(\hat{t}_j - \hat{t}_{j-1})/2 \cdot y + g_j,$$

it is expressed as (8.12). (8.13) is proved similarly as (8.12).

Q. E. D.

We put

$$c_j' = c_j + S_m^j, \varphi^{j'} = \varphi^j + R_m^j, \quad j = 1, \dots, r, c' = [c_1', c_2', \dots, c_r'],$$

$$\varphi' = [\varphi^{1'}, \dots, \varphi^{r'}], c = [c_1, c_2, \dots, c_r].$$

The solution satisfying (8.9) and (8.5) and starting from $y^1(-1) = b_0$, are expressed as

$$(8.14) \quad y^j(\tau) = \Phi_f(\tau) \sum_{i=1}^j \Phi_{j-1,i} b_{i-1} + \Lambda_j(\tau) + \Omega_j(\tau), \quad j = 1, \dots, r,$$

by lemma 8.1. Here

$$(8.15) \quad \begin{aligned} \Lambda_j(\tau) = & \Phi_j(\tau) \left\{ \int_{-1}^{\pi} \Phi_j^{-1}(s) (g_j \mathcal{T}_j + q_j \mathcal{T}_{j-1}) ds \right. \\ & \left. + \sum_{i=1}^{j-1} \Phi_{j-1,i} \int_{-1}^1 \Phi_i^{-1}(s) (g_i \mathcal{T}_i + q_i \mathcal{T}_{j-1}) ds \right\}, \end{aligned}$$

$$(8.16) \quad \Omega_j(\tau) = \Phi_j(\tau) \left\{ \int_{-1}^{\tau} \Phi_j^{-1}(s) \varphi^{j'}(s) ds + \sum_{i=1}^{j-1} \Phi_{j-1,i} \int_{-1}^1 \Phi_i(s) \varphi^i(s) ds \right\}.$$

Function $\Lambda_j(\tau)$ is rewritten, by lemma 8.2, as

$$\begin{aligned} \Lambda_j(\tau) = & Y^j(\tau) \{(\tau+1)/2 \cdot \mathcal{T}_j + (1-\tau)/2 \cdot \mathcal{T}_{j-1}\} - \Phi_j(\tau) Y^j(-1) \mathcal{T}_{j-1} \\ & + \Phi_j(\tau) \sum_{i=1}^{j-1} \Phi_{j-1,i+1} (Y^i(1) \mathcal{T}_i - \Phi_i(1) Y^i(-1) \mathcal{T}_{i-1}) \\ = & Y^j(\tau) \{(\tau+1)/2 \cdot \mathcal{T}_j + (1-\tau)/2 \cdot \mathcal{T}_{j-1}\} + \Phi_j(\tau) \sum_{i=1}^{j-1} \Phi_{j-1,i+1} D^i \mathcal{T}_i. \end{aligned}$$

Substituting y^j into (8.4) and (8.6), we have

$$\begin{aligned} L_0 b_0 + L_1 \left[\sum_{i=1}^r \Phi_{r,i} b_{i-1} + Y^r(1) \mathcal{T}_r + \sum_{i=1}^{r-1} \Phi_{r,i+1} D^i \mathcal{T}_i + \Omega_r(1) \right] &= b_r, \\ f_z^j(1) \left(\sum_{i=1}^j \Phi_{j,i} b_{i-1} + Y^j(1) \mathcal{T}_j + \sum_{i=1}^{j-1} \Phi_{j,i+1} D^i \mathcal{T}_i + \Omega_j(1) \right) &+ f_z^j(1) \mathcal{T}_j = c'_j, \\ &j=1, \dots, r, \end{aligned}$$

$$f_z^r(1) \left(\sum_{i=1}^r \Phi_{r,i} b_{i-1} + Y^r(1) \mathcal{T}_r + \sum_{i=1}^{r-1} \Phi_{r,i+1} D^i \mathcal{T}_i + \Omega_r(1) \right) + f_z^r(1) \mathcal{T}_r + f_z^r(1) \delta = c'_r,$$

and those are rewritten as

$$G_1 \begin{bmatrix} b_0 \\ \mathcal{T} \\ \delta \end{bmatrix} = \begin{bmatrix} b_r - L_1 \left(\sum_{i=2}^r \Phi_{r,i} b_{i-1} + \Omega_r(1) \right) \\ c'_j - f_z^j(1) \left(\sum_{i=2}^j \Phi_{j,i} b_{i-1} + \Omega_j(1) \right), j=1, \dots, r \\ \end{bmatrix}$$

If G_1 is nonsingular, $[b_0, \mathcal{T}, \delta]$ is solved as

$$(8.17) \quad \begin{bmatrix} b_0 \\ \mathcal{T} \\ \delta \end{bmatrix} = G_1^{-1} \begin{bmatrix} b_r - L_1 \left(\sum_{i=2}^r \Phi_{r,i} b_{i-1} + \Omega_r(1) \right) \\ c'_j - f_z^j(1) \left(\sum_{i=2}^j \Phi_{j,i} b_{i-1} + \Omega_j(1) \right), j=1, \dots, r \\ \end{bmatrix}.$$

$y^j(\tau)$ is solved with respect to $\varphi^{j'}$, c_j and b_j , substituting b_0 and $\mathcal{T}_j$ into (8.14).

We shall calculate M_1 and M_2 in (5.2). The number of linear formulas is $\beta_1 = N(r-1) + n$ and that of nonlinear formulas is $\beta_2 = rmN + n + 1 + q + (r-1)$. Thus k and $s-k$ is replaced by β_1 and β_2 respectively in (5.2). For the Jacobian

matrix $J_m = \begin{bmatrix} L \\ g_a \\ Z_{ia}^j \end{bmatrix}$, we shall evaluate $\left\| J_m^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \langle \beta_2 \rangle \end{bmatrix} \right\|$ and $\left\| J_m^{-1} \begin{bmatrix} \langle \beta_1 \rangle & 0 \\ 0 & 0 \end{bmatrix} \right\|$:

$$J_m^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \langle \beta_2 \rangle \end{bmatrix} = \sup_a \frac{\left\| J_m^{-1} \begin{bmatrix} 0 \\ a \end{bmatrix} \right\|}{\|a\|}, \quad \left\| J_m^{-1} \begin{bmatrix} \langle \beta_1 \rangle & 0 \\ 0 & 0 \end{bmatrix} \right\| \leq \sup_b \frac{\left\| J_m^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|}{\|b\|},$$

where b is a β_1 vector constituted from b_j , $j=1, \dots, r$ and a is a β_2 vector constituted from ζ_i^j , $i=0, \dots, m$, $j=1, \dots, r$, c_j , $j=1, \dots, r$. We put $b=(b_1, \dots, b_r)=0$ in (8.4) and (8.5). Then the solution h satisfying (8.1), (8.2), (8.4) and (8.5) for $b=0$ is expressed as

$$h = J_m^{-1} \begin{bmatrix} 0 \\ a \end{bmatrix}.$$

Corresponding to h , we consider a solution $[y^j, \mathcal{T}, \delta]$ satisfying (8.8), (8.4)~(8.6) for $b=0$. The solution $[y^j, \mathcal{T}, \delta]$ is given in (8.14) and (8.17) putting $b=0$. For the solution $[y^j, \mathcal{T}, \delta]$, we have, by (7.1), (7.2) and (5.4),

$$(8.18) \quad \left(\sum_{j=1}^r \|y^j\|_q^2 + \|\mathcal{T}\|^2 + \|\delta\|^2 \right)^{1/2} = \left\| J_m^{-1} \begin{bmatrix} 0 \\ a \end{bmatrix} \right\|.$$

Substituting $b=0$ into (8.17), we have

$$(8.19) \quad \begin{aligned} \| [b_0, \mathcal{T}] \| &\leq \| [\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \| \{ \|c'\| + (\|L_1\| + \|f_z'(1)\|) \\ &\quad \| \Omega_r(1) \| + \sum_{j=1}^{r-1} \| f_z^j(1) \| \| \Omega_j(1) \| \} \\ &\leq \| [\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \| \{ \|c'\| + \sum_{j=1}^r \sum_{i=1}^j H_{15ji} \| \varphi^{i'} \|_q \}, \end{aligned}$$

where

$$H_{15ji} = \begin{cases} \|f_z^j(1)\| K_{15ji}, & j=1, \dots, r-1, \\ (\|L_1\| + \|f_z^r(1)\|) K_{15ji}, & j=r. \end{cases}$$

Putting $b=0$ in (8.14), we have

$$y^j(\tau) = \Phi_j(\tau) (\Phi_{j-1,1} b_0 + \sum_{i=1}^{j-1} \Phi_{j-1,i+1} D^i \mathcal{T}_i) + Y^j(\tau) \{(\tau+1)/2 \cdot \mathcal{T}_j + (1-\tau)/2 \cdot \mathcal{T}_{j-1}\} + \Omega_j(\tau)$$

and

$$(8.20) \quad \|y^j\|_q \leq \|\Phi_j\|_n \|\Phi_{j-1,1}\| \|b_0\| + \|\Phi_j\|_n \sum_{i=1}^{j-1} \|\Phi_{j-1,i+1} D^i\| \|\mathcal{T}_i\| + \|Y^j\|_n |\mathcal{T}_j| + \|Y^j\|_n |\mathcal{T}_{j-1}| + \|\Omega_j\|_q \leq H_j \|b_0, \mathcal{T}\| + \sum_{i=1}^j K_{16ji} \|\varphi^{i'}\|_q,$$

where H_j is a positive constant such that

$$H_j^2 = \|\Phi_j\|_n^2 \|\Phi_{j-1,1}\|^2 + \sum_{i=1}^{j-2} \|\Phi_j\|_n^2 \|\Phi_{j-1,i+1} D^i\|^2 + (\|\Phi_j\|_n \|D^{j-1}\| + \|Y^j\|_n)^2 + \|Y^j\|_n^2.$$

Substituting (8.19) into (8.20), we have

$$(8.21) \quad \|y^j\|_q \leq H_j \|[\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \|c'\| + \sum_{k=1}^r \sum_{i=1}^k H_{15ki} \|\varphi^{i'}\|_q + \sum_{i=1}^j K_{16ji} \|\varphi^{i'}\|_q \leq H_j \|[\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \|c'\| + \sum_{k=1}^r \sum_{i=1}^k H_{16ki}^j \|\varphi^{i'}\|_q$$

where

$$H_{16ki}^j = \begin{cases} H_j \|[\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \|H_{15ki}, & k \neq j, \\ H_j \|[\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \|H_{15ji} + K_{16ji}, & k = j. \end{cases}$$

We have, from (8.17),

$$(8.22) \quad \|[\mathcal{T}, \delta]\| \leq \| [0_{r+q,N} \langle r+q \rangle] G_1^{-1} \|c'\| + \sum_{j=1}^r \sum_{i=1}^j H_{15ji} \|\varphi^{i'}\|_q,$$

similarly as (8.19). We put

$$H_1^2 = \sum_{j=1}^r \{H_j^2 \|[\langle N+r \rangle, 0_{N+r,q}] G_1^{-1}\|^2 + \sum_{i=1}^r (\sum_{k=i}^r H_{16ki}^j)^2\} + \|[0_{r+q,N}, \\ \langle r+q \rangle] G_1^{-1}\|^2 \{1 + \sum_{i=1}^r (\sum_{k=i}^r H_{15ji})^2\},$$

then from (8.21) and (8.22), we have

$$\begin{aligned} \sum_{j=1}^r \|y^j\|_q^2 + \|\mathcal{T}\|^2 + \|\delta\|^2 &\leq H_1^2 (\|c'\|^2 + \|\varphi'\|_q^2) \leq H_1^2 (\|c + S_m\|^2 + \|\varphi + R_m\|_q^2) \\ &\leq 2H_1^2 (\|c\|^2 + \|\varphi\|_q^2 + \|S_m\|^2 + \|R_m\|_q^2) : \sum_{j=1}^r \|y^j\|_q^2 + \|\mathcal{T}\|^2 + \|\delta\|^2 \\ &\leq \frac{2H_1^2(1+O(m^{-2}))}{1-2H_1^2O(m^{-2})} (\|c\|^2 + \|\varphi\|_q^2) \leq M_1^2 \|a\|_2, \text{ with} \\ M_1 &= \left\{ \frac{2H_1^2(1+O(m^{-2}))}{1-2H_1^2O(m^{-2})} \right\}^{1/2} \text{ for some large } m_1. \end{aligned}$$

We have, by (8.18), a constant M_1 such that

$$(8.23) \quad \left\| J_m^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \langle \beta_2 \rangle \end{bmatrix} \right\| \leq M_1.$$

Similarly as M_1 , we shall calculate a constant M_2 such that

$$(8.24) \quad \left\| J_m^{-1} \begin{bmatrix} \langle \beta_1 \rangle & 0 \\ 0 & 0 \end{bmatrix} \right\| \leq M_2.$$

We put $a=0$ ($\varphi=0$, $c_j=0$) in (8.17), that is, put $\varphi^{i'}=R_m^i$, $\varphi^{j'}=R_m^j$ in (8.16) and $c'_j=S_m^j$ in (8.17):

$$\begin{aligned} (8.25) \quad \|[b_0, \mathcal{T}]\| &\leq \|[\langle N+r \rangle, 0_{N+r,q}] G_1^{-1}\| \{\|b_r\| + \|L_1\| \sum_{i=2}^r \|\Phi_{r,i}\| \|b_{i-1}\| \\ &\quad + \sum_{j=2}^{r-1} \|f_z^j(1)\| \sum_{i=2}^j \|\Phi_{j,i}\| \|b_{i-1}\| + \|f_z^r(1)\| \sum_{i=2}^r \|\Phi_{r,i}\| \|b_{i-1}\| \\ &\quad + \|S_m\| + \sum_{j=1}^r \sum_{i=1}^j H_{15ji} \|R_m^i\|_q\} = \|[\langle N+r \rangle, 0_{N+r,q}] G_1^{-1}\| (\|b_r\| \\ &\quad + \sum_{j=2}^r \sum_{i=2}^j H_{12ji} \|b_{i-1}\| + \|S_m\| + \sum_{j=1}^r \sum_{i=1}^j H_{15ji} \|R_m^i\|_q), \end{aligned}$$

where

$$H_{12ji} = \begin{cases} \|f_z^j(1)\| \|\Phi_{j,i}\|, & \text{for } j=2, \dots, r-1 \\ (\|L_1\| + \|f_z^r(1)\|) \|\Phi_{r,i}\|, & \text{for } j=r. \end{cases}$$

We put $a=0$ ($\varphi=0$) in (8.14), that is, put $\varphi^{i'}=R_m^i$, $\varphi^{j'}=R_m^j$ in (8.16), then we have

$$(8.26) \quad \|y^j\|_q \leq H_j \| [b_0, \mathcal{T}] \| + \|\Phi_j\|_n \sum_{i=2}^j \|\Phi_{j-1,i}\| \|b_{i-1}\| + \sum_{i=1}^j K_{16ji} \|R_m^i\|_q.$$

Substituting (8.25) into (8.26), we have

$$(8.27) \quad \|y^j\|_q \leq H_j \| [\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \| (\|b_r\| + \sum_{j=2}^r \sum_{i=1}^j H_{12ji} \|b_{i-1}\| \\ + \|S_m\| + \sum_{j=1}^r \sum_{i=1}^j H_{15ji} \|R_m^i\|_q) + \|\Phi_j\|_n \sum_{i=2}^j \|\Phi_{j-1,i}\| \|b_{i-1}\| \\ + \sum_{i=1}^j K_{16ji} \|R_m^i\|_q \leq H_j \| [\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \| (\|b_r\| + \|S_m\|) \\ + \sum_{k=2}^r \sum_{i=2}^k H_{13ki}^j \|b_{i-1}\| + \sum_{k=1}^r \sum_{i=1}^k H_{16ki}^j \|R_m^i\|_q,$$

where

$$H_{13ki}^j = \begin{cases} H_j \| [\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \| H_{12ki}^j, & k \neq j \\ H_j \| [\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \| H_{12ji}^j + \|\Phi_j\|_n \|\Phi_{j-1,i}\|, & k=j. \end{cases}$$

We have, by (8.17),

$$(8.28) \quad \| [\mathcal{T}, \delta] \| = \| [0_{r+q,N} \langle r+q \rangle] G_1^{-1} \| (\|b_r\| + \sum_{j=2}^r \sum_{i=2}^j H_{12ji} \|b_{i-1}\| \\ + \|S_m\| + \sum_{j=1}^r \sum_{i=2}^j H_{15ji} \|R_m^i\|_q).$$

We put

$$H_2^2 = \sum_{j=1}^r \{ H_j^2 \| [\langle N+r \rangle, 0_{N+r,q}] G_1^{-1} \|^2 + \sum_{i=2}^r (\sum_{k=i}^r H_{13ki}^j)^2 \} \\ + \| [0_{r+q,N} \langle r+q \rangle] G_1^{-1} \|^2 + \sum_{i=2}^r (\sum_{j=i}^r H_{12ji})^2,$$

then we have

$$\sum_{j=1}^r \|y^j\|_q^2 + \|\mathcal{T}\|^2 + \|\delta\|^2 \leq H_2^2 \|b\|^2 + H_1^2 (\|S_m\|^2 + \|R_m\|^2),$$

that is,

$$\sum_{j=1}^r \|y^j\|_q^2 + \|\mathcal{T}\|^2 + \|\delta\|^2 \leq M_2^2 \|b\|^2,$$

where M_2 is a positive constant such that

$$M_2^2 = \frac{H_2^2}{1 - O(m_1^{-2})H_1^2}, \text{ for sufficiently large } m_1.$$

Thus we have a constant satisfying (8.24). (8.23) and (8.24) prove (iii) in §5. Thus theorem 4.1 is proved.

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