REMARKS ON SEMI-PROPER EQUIVALENCE RELATIONS ON COMPLEX SPACES

By

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Introduction

Let $(X, \mathcal{O})$ be a (maximal) complex space, and R a semi-proper equivalence relation on X . Then the author [4] and Wiegmann [5] showed that the quotient space $(X/R, \mathcal{O}/R)$ is a (maximal) complex space if and only if it is locally- $\mathcal{O}/R$ -separable. Further, Wiegmann [5] showed that if $(X, \mathcal{O})$ is a reduced complex space, then the space $(X/R, \mathcal{O}/R)$ is a reduced complex space if and only if it is locally- $\mathcal{O}/R$ -separable and locally- $\mathcal{O}/R$ -immersible. Now, in this article, we shall give some remarks on semi-proper equivalence relations and show the following theorem;

THEOREM. *Let $(X, \mathcal{O})$ be a (maximal) complex space, and R a semi-proper equivalence relation on X . If the quotient space X/R is locally compact, then $(X/R, \mathcal{O}/R)$ is a (maximal) complex space if and only if it is locally- $\mathcal{O}/R$ -regular.*

1. Preliminaries

Let X and Y be topological spaces. A continuous mapping φ of X in Y is said to be *semi-proper* if for every point y of Y , there exists an open neighbourhood V of y in Y and a compact subset K of X such that $\varphi^{-1}(y') \cap K \neq \emptyset$ for every point y' of $V \cap \varphi(X)$. If X and Y are locally compact, then the mapping φ is semi-proper if and only if for any compact subset L of Y , there exists a compact subset K of X such that $\varphi(X) \cap L = L \cap \varphi(X)$. An equivalence relation R on X is said to be *semi-proper* if the natural projection $\pi: X \rightarrow X/R$ is a semi-proper mapping. For a subset A of X , the set $R(A) := \pi^{-1}(\pi(A))$ is called R -saturated hull of A in X . Then we have the following lemma.

LEMMA 1. *Let X and Y be locally compact topological spaces. Then a*

continuous mapping $\varphi: X \rightarrow Y$ is semi-proper if and only if for any point y of Y , there exists an open neighbourhood U of y in Y and an open set V in X such that $\varphi(V) = U \cap \varphi(X)$ and the restriction $\varphi|_V: V \rightarrow U$ is semi-proper mapping.

PROOF. Let U be an open neighbourhood of a point y of Y such that the closure $\bar{U}$ is compact in Y . If the mapping φ is semi-proper, then there exists a compact subset K of X such that $\varphi(K) = \bar{U} \cap \varphi(X)$. Let W be a relatively compact open neighbourhood of K in X , and we put $\varphi^{-1}(U) \cap W =: V$. Then $\varphi(V) = U \cap \varphi(X)$ and the restriction $\varphi|_V: V \rightarrow U$ is semi-proper mapping. We can see that the converse follows from Lemma 7.12 in [1].

LEMMA 2. Let X be a topological space, R an equivalence relation on X , and $\pi: X \rightarrow X/R$ the natural projection. Then the following statements are equivalent.

- (1) R is semi-proper,
- (2) for any point x of X , there exists a R -saturated open neighbourhood U of x in X and a compact subset K of X such that $R(x') \cap K \neq \emptyset$ for every point x' of U , namely, $R(K) \supset U$.
- (3) for any point y of X/R , there exists an open neighbourhood U of y in X/R and a compact subset K of X such that $\pi(K) \supset U$.

Let (X, θ) be a reduced complex space, R an equivalence relation on X , and $\pi: X \rightarrow X/R$ the natural projection. R is said to be analytic if the graph $\text{gr}(R) := \{(x, y) \in X \times X; \pi(x) = \pi(y)\}$ of R is an analytic subset of $X \times X$. Let $p_i: R \rightarrow X$ ($i=1, 2$) be the natural projections. For subsets A and B of X , we have $A \cap R(B) = p_i(\text{gr}(R) \cap (A \times B))$ and $R(A) = p_i(\text{gr}(R) \cap (X \times A)) = p_2^{-1}(p_1(A))$.

PROPOSITION 1. Let (X, θ) be a reduced complex space and R an analytic equivalence relation on X . Then p_i 's are semi-proper holomorphic mappings.

PROOF. Let p_i ($i=1, 2$) be semi-proper holomorphic mappings. Then for any point x of X , there exists an open neighbourhood V of x in X and a compact subset K of R such that $p_i(K) \supset V$. We put $K_i := p_i(K)$. Then the set $L := p_1(\text{gr}(R) \cap (K_1 \times K_2)) = K_1 \cap R(K_2)$ is compact in X . The set $U := R(V)$ is a R -saturated open neighbourhood of x in X . Since $L \cap R(x') \neq \emptyset$ for every point x' of U , $R(L) \supset U$. Therefore R is semi-proper by Lemma 2. Conversely, if R is a semi-proper analytic equivalence relation, then the projections p_i are semi-proper, because the graph $\text{gr}(R)$ of R is closed in $X \times X$.

Let (X, θ) be a reduced complex space, and U an open subset of X . A complex valued function f in U is said to be c -holomorphic if f is continuous in U and

holomorphic in the set of all the regular points of U . For a point x of X , f is said to be *c-holomorphic* at x if the point x has an open neighbourhood U in X such that f is a *c-holomorphic* function in U . Let $\tilde{\mathcal{O}}$ be the sheaf of germs of *c-holomorphic* functions in X . A reduced complex space $(X, \mathcal{O})$ is said to be *maximal* if $\mathcal{O} = \tilde{\mathcal{O}}$. A $\mathbb{C}$ -local ringed space $(X, \mathcal{O}_X)$ is said to be *locally- $\mathcal{O}_X$ -separable* if any point of X has an open neighbourhood V such that the sections of $\mathcal{O}_X$ over V separate the points of V . Then the author [4] and Wiegmann [5] showed that following theorem;

THEOREM 1. *Let $(X, \mathcal{O})$ be a (maximal) complex space and R a semi-proper equivalence relation on X . Then the quotient space $(X/R, \mathcal{O}/R)$ is a (maximal) complex space if and only if each of the following conditions is satisfied.*

- (1) *X/R is locally- $\mathcal{O}/R$ -separable.*
- (2) *any point of X/R has an open neighbourhood V such that the ringed mappings $V \rightarrow Z$ separate the points of V , where Z is any (reduced) complex space.*

PROOF. We shall prove that the condition (2) is sufficient. Let y be an arbitrary point of X/R . Then the point y has an open neighbourhood V such that the ringed mappings $\varphi_i: V \rightarrow Z_i$ ($i \in I$) separate the points of V . Since the quotient space X/R is locally compact [5], there exist a relatively compact open neighbourhood U of y such that the closure $\bar{U}$ in X/R is contained in V . By the lemma of Cartan [2], there exists a finite set of ringed mappings $\varphi_i: V \rightarrow Z_i$ ($i = 1, \dots, n$) such that the equivalence relation defined by φ_i 's coincides with R on $\bar{U}$. We put $Z := Z_1 \times \dots \times Z_n$ and $\varphi := (\varphi_1, \dots, \varphi_n)$. Then the space Z is a complex space and the mapping $\varphi: \bar{U} \rightarrow Z$ is continuous and one to one. Therefore the restriction $\varphi|_{\bar{U}}: \bar{U} \rightarrow \varphi(\bar{U})$ is homeomorphism. There exists an open subset Y of Z such that $\varphi(U) = \varphi(\bar{U}) \cap Y$, and $\varphi(U)$ is closed in Y . The mapping $\varphi|_U: U \rightarrow Y$ is semi-proper [4]. Since the natural projection $\pi: X \rightarrow X/R$ is semi-proper, there exists an open set W in X such that $\pi(W) = U$ and the restriction $\pi|_W: W \rightarrow U$ is semi-proper mapping by Lemma 1. Since φ is a ringed mapping, φ induces a (semi-proper) holomorphic mapping F of W in Y . We denote by $R(F)$ the equivalence relation on W defined by the mapping F . Then the quotient space $W/R(F)$ is isomorphic to U as a ringed space. Therefore U is a (maximal) complex space by Theorem 1 in [4].

COROLLARY. *Let $(X, \mathcal{O})$ be a (maximal) complex space and R an open equivalence relation on X . Then X/R is a (maximal) complex space if and only if X/R is locally- $\mathcal{O}/R$ -separable.*

2. Local- $\mathcal{O}$ / R -regularity

Let $(X, \mathcal{O}_X)$ be a $\mathbf{C}$ -local ringed space, and $M_{x,x}$ a maximal ideal of $\mathbf{C}$ -local ring $\mathcal{O}_{X,x}$ for every point x of X . Then $\mathcal{O}_{X,x}/M_{x,x} \cong \mathbf{C}$. The space $(X, \mathcal{O}_X)$ is said to be *locally- $\mathcal{O}_X$ -immersible* if any point of X has an open neighbourhood V in X such that every mapping $\Gamma(V, \mathcal{O}_X) \rightarrow M_{x,x}/M_{x,x}^2$, $f \mapsto ((f)_x - f(x)) \bmod M_{x,x}^2$, is surjective for every point x of V , where $(f)_x$ is germ of f at x . Then Wiegmann [5] showed the following theorem;

THEOREM 2. *Let $(X, \mathcal{O})$ be a reduced complex space and R a semi-proper equivalence relation on X . Then the quotient space $(X/R, \mathcal{O}/R)$ is a reduced complex space if and only if X/R is locally- $\mathcal{O}/R$ -separable and locally- $\mathcal{O}/R$ -immersible.*

Further, the $\mathbf{C}$ -local ringed space $(X, \mathcal{O}_X)$ is said to be *locally- $\mathcal{O}_X$ -regular* if there exist sections $f_1, \dots, f_n$ of $\mathcal{O}_X$ over V such that $((f_j)_x - f_j(x)) \bmod M_{x,x}^2$ ($1 \leq j \leq n$) are basis of $\mathbf{C}$ -vector space $M_{x,x}/M_{x,x}^2$ for every point x of V . Then the germs $(f_j)_x - f_j(x)$ ($1 \leq j \leq n$) generate the maximal ideal $M_{x,x}$ of $\mathcal{O}_{X,x}$ for every point x of V by Nakayama's lemma. If $(X, \mathcal{O}_X)$ is a complex space, then $(X, \mathcal{O}_X)$ is locally- $\mathcal{O}_X$ -regular. Thus, we have the following theorem;

THEOREM 3. *Let $(X, \mathcal{O})$ be a (maximal) complex space, and R a semi-proper equivalence relation on X . If the quotient space X/R is locally compact, then $(X/R, \mathcal{O}/R)$ is a (maximal) complex space if X/R is locally- $\mathcal{O}/R$ -regular.*

PROOF. We shall prove the sufficiency. Any point of X has an open neighbourhood V in X and a finite set of sections $f_1, \dots, f_k$ of $\mathcal{O}/R$ over V such that $((f_1)_x - f_1(x), (f_k)_x - f_k(x)) \cdot \mathcal{O}_{X,x} = M_{x,x}$ for every point x of V . We put $F := (f_1, \dots, f_k): V \rightarrow \mathbf{C}^k$. Then the mapping F is continuous and one to one. Let U be an open neighbourhood of x in X/R such that the closure $\bar{U}$ is contained in V . Then there exists an open set D in $\mathbf{C}^k$ such that $F(U) = F(\bar{U}) \cap D$. From the proof of Theorem 1, it follows that U is a (maximal) complex space.

COROLLARY. *Let $(X, \mathcal{O})$ be a (maximal) complex space, and R an open equivalence relation on X . If the quotient space X/R is a Hausdorff space, then $(X/R, \mathcal{O}/R)$ is a (maximal) complex space if and only if X/R is locally- $\mathcal{O}/R$ -regular.*

References

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