ON PERIODS OF CUSP FORMS

By

Yoshiji KADOTERA

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1. Introduction

In the present note we give an alternative proof of a theorem of Shimura [13], [14] on the rationality of the special values of the zeta functions associated with primitive cusp forms. By making use of the Eichler cohomology with real coefficients and the Shimura isomorphism ([3], [11], [12]), Manin [5] and Razar [9] obtained the similar rationality results for $SL(2, \mathbf{Z})$ and $\Gamma_0(N)$ respectively. Our method here is the same as the one of Manin and Razar, but we make use of the Eichler cohomology with complex coefficients and also certain relations between Fourier coefficients and periods of cusp forms, which include congruences for Fourier coefficients, with a remark on the Ramanujan-Petersson inequality.

2. Periods of cusp forms

Let N be a positive integer and $\Gamma_0(N)$, $\Gamma_1(N)$ the congruence subgroups of $SL(2, \mathbf{Z})$ defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \mid c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \mid a \equiv d \equiv 1 \pmod{N} \right\}.$$

For a fixed positive integer $k \geq 2$ and a Dirichlet character χ modulo N such that $\chi(-1) = (-1)^k$, we denote by $S_k(\Gamma_1(N))$ the vector space of all cusp forms of weight k with respect to $\Gamma_1(N)$, and by $S_k(N, \chi)$ the subspace of $S_k(\Gamma_1(N))$ consisting of all cusp forms $f(z)$ satisfying $f((az+b)/(cz+d)) = \chi(d)(cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Then we have $S_k(\Gamma_1(N)) = \sum_{\chi} S_k(N, \chi)$, where χ runs over all

Dirichlet characters modulo N with $\chi(-1)=(-1)^k$.

Let f be an element of $S_k(N, \chi)$ which is a common eigenfunction of all Hecke operators of level N ,

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

its Fourier expansion, and $a_1=1$. For every automorphism σ of the complex number field $\mathbf{C}$, we can define an element f^σ of $S_k(N, \chi^\sigma)$ which is a common eigenfunction of all Hecke operators of level N by $f^\sigma(z) = \sum_{n=1}^{\infty} a_n^\sigma e^{2\pi i n z}$. Here χ^σ is defined by $\chi^\sigma(n) = \chi(n)^\sigma$. Let K_f denote the field generated over the rational number field $\mathbf{Q}$ by the Fourier coefficients a_n of f for all n . Then we first prove the following

THEOREM 1. *For every injection σ of K_f into $\mathbf{C}$, there exist non-zero complex numbers w_σ^+ and w_σ^- such that*

$$\frac{1}{w_\sigma^\pm} \left\{ \int_{\gamma(i\infty)}^{i\infty} f^\sigma(z) z^l dz \pm (-1)^{l+1} \int_{-\gamma(i\infty)}^{i\infty} f^\sigma(z) z^l dz \right\}$$

belong to K_{f^σ} for all $\gamma \in \Gamma_1(N)$, $l=0, 1, \dots, k-2$, and

$$\begin{aligned} & \left[\frac{1}{w_1^\pm} \left\{ \int_{\gamma(i\infty)}^{i\infty} f(z) z^l dz \pm (-1)^{l+1} \int_{-\gamma(i\infty)}^{i\infty} f(z) z^l dz \right\} \right]^\sigma \\ &= \frac{1}{w_\sigma^\pm} \left\{ \int_{\gamma(i\infty)}^{i\infty} f^\sigma(z) z^l dz \pm (-1)^{l+1} \int_{-\gamma(i\infty)}^{i\infty} f^\sigma(z) z^l dz \right\}, \end{aligned}$$

where $w_1^\pm$ denote the complex numbers corresponding to the identity injection.

First, following Shimura [12], we define a representation $M_{k-2}: GL(2, \mathbf{C}) \rightarrow GL(k-1, \mathbf{C})$ by $M_{k-2}(\alpha) \begin{bmatrix} u \\ v \end{bmatrix}^{k-2} = \left(\alpha \begin{bmatrix} u \\ v \end{bmatrix} \right)^{k-2}$, where $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathbf{C}^2$ and $\begin{bmatrix} u \\ v \end{bmatrix}^{k-2} = {}^t(u^{k-2}, u^{k-3}v, \dots, uv^{k-3}, v^{k-2})$, and consider $\mathbf{C}^{k-1}$ as a $\Gamma_1(N)$ -module by M_{k-2} . If $k=2$, we understand that $\begin{bmatrix} u \\ v \end{bmatrix}^0 = 1$ and $M_{k-2}(\alpha) = 1$ for every $\alpha \in GL(2, \mathbf{C})$. Let $Z^1(\Gamma_1(N), \mathbf{C}^{k-1})$, $Z_P^1(\Gamma_1(N), \mathbf{C}^{k-1})$, and $B^1(\Gamma_1(N), \mathbf{C}^{k-1})$ be the 1-cocycle group of $\Gamma_1(N)$ with coefficients in $\mathbf{C}^{k-1}$, its subgroup consisting of all 1-cocycles Q such that $Q_\gamma \in \{M_{k-2}(\gamma) - 1\} \mathbf{C}^{k-1}$ for every parabolic element γ in $\Gamma_1(N)$, and the 1-coboundary group of $\Gamma_1(N)$ with coefficients in $\mathbf{C}^{k-1}$ respectively. Then we have the Eichler cohomology group $H_P^1(\Gamma_1(N), \mathbf{C}^{k-1}) = Z_P^1(\Gamma_1(N), \mathbf{C}^{k-1}) / B^1(\Gamma_1(N), \mathbf{C}^{k-1})$.

Now we can define a $\mathbf{C}$ -linear map $\delta_0: S_k(\Gamma_1(N)) \rightarrow Z_P^1(\Gamma_1(N), \mathbf{C}^{k-1})$ by $\delta_0(g)_\gamma = \int_{\gamma(i\infty)}^{i\infty} g(z) d\mathfrak{z}_{k-2}$ for every $\gamma \in \Gamma_1(N)$, where $\int_{\gamma(i\infty)}^{i\infty} g(z) d\mathfrak{z}_{k-2}$ denotes the $(k-1)$ -dimensional complex vector $\left(\int_{\gamma(i\infty)}^{i\infty} g(z) z^{k-2} dz, \int_{\gamma(i\infty)}^{i\infty} g(z) z^{k-3} dz, \dots, \int_{\gamma(i\infty)}^{i\infty} g(z) dz \right)$. Let $\delta: S_k(\Gamma_1(N)) \rightarrow H_P^1(\Gamma_1(N), \mathbf{C}^{k-1})$ be the $\mathbf{C}$ -linear map induced by δ_0 . Then δ_0, δ are injective and

$$(2.1) \quad \dim_{\mathbf{C}} H_P^1(\Gamma_1(N), \mathbf{C}^{k-1}) = 2 \dim_{\mathbf{C}} S_k(\Gamma_1(N))$$

([3], [11], [12]).

If we define an automorphism of $Z_P^1(\Gamma_1(N), \mathbf{C}^{k-1})$ by $Q \mapsto \bar{Q}$, $\bar{Q}_\gamma = \overline{Q_\gamma}$, where the bar means the complex conjugation, we obtain the following lemma from the Shimura isomorphism $S_k(\Gamma_1(N)) \simeq H_P^1(\Gamma_1(N), \mathbf{R}^{k-1})$ ([11], [12]) and the formula (2.1).

LEMMA 1. *We have the decomposition*

$$Z_P^1(\Gamma_1(N), \mathbf{C}^{k-1}) = \delta_0(S_k(\Gamma_1(N))) \oplus \overline{\delta_0(S_k(\Gamma_1(N)))} \oplus B^1(\Gamma_1(N), \mathbf{C}^{k-1}).$$

The above automorphism naturally induces an automorphism of $H_P^1(\Gamma_1(N), \mathbf{C}^{k-1})$ and we denote it by the bar as well.

COROLLARY. $H_P^1(\Gamma_1(N), \mathbf{C}^{k-1}) = \delta(S_k(\Gamma_1(N))) \oplus \overline{\delta(S_k(\Gamma_1(N)))}$.

Since $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$, every element β of $\Gamma_0(N)$ acts on $Z_P^1(\Gamma_1(N), \mathbf{C}^{k-1})$ by

$$(Q|\beta)_\gamma = M_{k-2}(\beta^{-1}) Q_{\beta\gamma\beta^{-1}}$$

for every $\gamma \in \Gamma_1(N)$. Every element of $\Gamma_1(N)$ sends $B^1(\Gamma_1(N), \mathbf{C}^{k-1})$ into itself, so that the abelian group $\Gamma_0(N)/\Gamma_1(N)$ acts on $H_P^1(\Gamma_1(N), \mathbf{C}^{k-1})$. Therefore we have the decomposition

$$(2.2) \quad H_P^1(\Gamma_1(N), \mathbf{C}^{k-1}) = \sum_{\chi} H_P^1(\Gamma_1(N), \mathbf{C}^{k-1}, \chi)$$

by means of characters χ of $\Gamma_0(N)/\Gamma_1(N)$. If we identify $\Gamma_0(N)/\Gamma_1(N)$ with $(\mathbf{Z}/N\mathbf{Z})^*$ in a natural way, then χ in (2.2) runs over all Dirichlet characters modulo N with $\chi(-1) = (-1)^k$ and each $H_P^1(\Gamma_1(N), \mathbf{C}^{k-1}, \chi)$ is the maximal subspace of $H_P^1(\Gamma_1(N), \mathbf{C}^{k-1})$ on which the action of $\Gamma_0(N)/\Gamma_1(N)$ is given by χ .

LEMMA 2. *Let χ be a Dirichlet character modulo N with $\chi(-1) = (-1)^k$.*

Then

$$H_p^1(\Gamma_1(N), \mathbf{C}^{k-1}, \chi) = \delta(S_k(N, \chi)) \oplus \overline{\delta(S_k(N, \bar{\chi}))},$$

where $\bar{\chi}$ is defined by $\bar{\chi}(n) = \overline{\chi(n)}$.

PROOF. For every Dirichlet character χ modulo N with $\chi(-1) = (-1)^k$, we see that $\delta(S_k(N, \chi)) + \overline{\delta(S_k(N, \bar{\chi}))}$ is contained in $H_p^1(\Gamma_1(N), \mathbf{C}^{k-1}, \chi)$. Therefore our assertion follows from Corollary of Lemma 1.

Next we consider the Hecke operator T_p on $S_k(\Gamma_1(N))$. For an arbitrary prime p , let

$$\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) = \bigcup_i \Gamma_1(N) \alpha_i$$

be a disjoint coset decomposition. If we set

$$(g|_k \alpha)(z) = \det(\alpha)^{k/2} g((az+b)/(cz+d))(cz+d)^{-k}$$

for every $g \in S_k(\Gamma_1(N))$ and every real matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(\alpha) = ad - bc > 0$, the Hecke operator T_p on $S_k(\Gamma_1(N))$ is defined by

$$(g|T_p)(z) = p^{k/2-1} \sum_i (g|_k \alpha_i)(z).$$

Corresponding to T_p , we define a $\mathbf{C}$ -linear transformation $\tilde{T}_p$ on $Z_p^1(\Gamma_1(N), \mathbf{C}^{k-1})$ by

$$(Q|\tilde{T}_p)_\gamma = \sum_i M_{k-2}(\alpha_i) Q_{\gamma_i}$$

for every $\gamma \in \Gamma_1(N)$, where $\alpha_i \gamma = \gamma_i \alpha_{j(i)}$ with $\gamma_i \in \Gamma_1(N)$ and $\alpha_i = \det(\alpha_i) \alpha_i^{-1}$ ([11], [12]). Then, for a 1-coboundary Q given by $Q_\gamma = \{M_{k-2}(\gamma) - 1\} \mathbf{v}$ with $\mathbf{v} \in \mathbf{C}^{k-1}$, we have

$$(2.3) \quad (Q|\tilde{T}_p)_\gamma = \{M_{k-2}(\gamma) - 1\} \sum_i M_{k-2}(\alpha_i) \mathbf{v}$$

for every $\gamma \in \Gamma_1(N)$. We denote by $\hat{T}_p$ the transformation on $H_p^1(\Gamma_1(N), \mathbf{C}^{k-1})$ induced by $\tilde{T}_p$. Let η_p be any element of $SL(2, \mathbf{Z})$ congruent to $\begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix}$ modulo N . Then we can take a set of representatives $\{\alpha_i\}$ as follows.

$$\left\{ \eta_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix} \right\} \Big| m \bmod p \Big\} \quad (p \nmid N),$$

$$\left\{ \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix} \middle| m \bmod p \right\} \quad (p|N) .$$

If $p \nmid 2N$, we take $\left\{ \eta_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & \pm 1 \\ 0 & p \end{pmatrix}, \dots, \begin{pmatrix} 1 & \pm \frac{p-1}{2} \\ 0 & p \end{pmatrix} \right\}$ as a set of representatives $\{\alpha_i\}$. Then we immediately obtain

LEMMA 3 ([12]). *Suppose $g \in S_k(\Gamma_1(N))$. Then*

$$\delta_0(g|T_p)_\gamma = \begin{cases} (\delta_0(g)|\hat{T}_p)_\gamma + \{M_{k-2}(\gamma) - 1\} \int_{\eta_p(i\infty)}^{i\infty} g(z) d\bar{z}_{k-2} & (p \nmid N) \\ (\delta_0(g)|\hat{T}_p)_\gamma & (p|N) \end{cases}$$

for every $\gamma \in \Gamma_1(N)$, and therefore $\delta(g|T_p) = \delta(g)|\hat{T}_p$ for every prime p .

By Lemma 3 we see that $\hat{T}_p$ sends $\delta(S_k(\Gamma_1(N)))$, $\overline{\delta(S_k(\Gamma_1(N)))}$, and $H_p^1(\Gamma_1(N), \mathbf{C}^{k-1}, \chi)$ into themselves.

Let ε denote the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $\varepsilon\gamma\varepsilon^{-1} = \varepsilon\gamma\varepsilon = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, so that ε normalizes $\Gamma_0(N)$, $\Gamma_1(N)$. We define an $\mathbf{R}$ -linear transformation $[\varepsilon]$ on $S_k(\Gamma_1(N))$ by $(g|[\varepsilon])(z) = \overline{g(-\bar{z})}$, and a $\mathbf{C}$ -linear transformation $\tilde{\varepsilon}$ on $Z_p^1(\Gamma_1(N), \mathbf{C}^{k-1})$ by

$$(Q|\tilde{\varepsilon})_\gamma = -M_{k-2}(\varepsilon)Q_{\varepsilon\gamma\varepsilon}$$

for every $\gamma \in \Gamma_1(N)$ as in Razar [9]. Let $\hat{\varepsilon}$ denote the $\mathbf{C}$ -linear transformation on $H_p^1(\Gamma_1(N), \mathbf{C}^{k-1})$ induced by $\tilde{\varepsilon}$. If $g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$ is an element of $S_k(N, \chi)$, then $(g|[\varepsilon])(z) = \sum_{n=1}^{\infty} \bar{b}_n e^{2\pi i n z}$ and $g|[\varepsilon] \in S_k(N, \bar{\chi})$. We have further $\overline{\delta_0(g|[\varepsilon])} = \delta_0(g)|\tilde{\varepsilon}$ for every $g \in S_k(\Gamma_1(N))$.

Now let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be an element of $S_k(N, \chi)$ such that $f|T_p = a_p f$ for all primes p . Then $\delta(f)|\hat{T}_p = a_p \delta(f)$, $\overline{\delta(f|[\varepsilon])}| \hat{T}_p = a_p \overline{\delta(f|[\varepsilon])}$ for all primes p by Lemma 3. From Lemmas 1, 2, we obtain

LEMMA 4. *The common eigenspace in $H_p^1(\Gamma_1(N), \mathbf{C}^{k-1}, \chi)$ of all $\hat{T}_p$, belonging to the eigenvalues a_p , is spanned by $\delta(f)$ and $\overline{\delta(f|[\varepsilon])}$. Moreover $\delta(f) + \overline{\delta(f|[\varepsilon])}$, $\delta(f) - \overline{\delta(f|[\varepsilon])}$ are eigenvectors of $\hat{\varepsilon}$ belonging to the eigenvalues $+1$, -1 respectively.*

On the other hand, $H_p^1(\Gamma_1(N), \mathbf{C}^{k-1})$ has a basis in $H_p^1(\Gamma_1(N), \mathbf{Q}^{k-1})$, and

$H_p^1(\Gamma_1(N), \mathbf{Q}^{k-1})$ is stable by the action of all $\hat{T}_p$ and $\hat{\varepsilon}$. Therefore there exist two non-trivial 1-cocycles Q^+ and Q^- in $Z_p^1(\Gamma_1(N), K_f^{k-1})$ satisfying

$$(2.4) \quad \begin{aligned} Q^\pm \bmod B^1(\Gamma_1(N), \mathbf{C}^{k-1}) &\in H_p^1(\Gamma_1(N), \mathbf{C}^{k-1}, \chi), \\ Q^\pm | \hat{T}_p &\equiv a_p Q^\pm \bmod B^1(\Gamma_1(N), K_f^{k-1}) \quad \text{for all primes } p, \\ Q^\pm | \hat{\varepsilon} &\equiv \pm Q^\pm \bmod B^1(\Gamma_1(N), K_f^{k-1}), \\ Q_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^\pm &= 0. \end{aligned}$$

Then, by Lemma 4, there exist non-zero complex numbers w^+ , w^- , and $(k-1)$ -dimensional complex vectors $\mathbf{v}^+ = {}^t(v_{k-2}^+, v_{k-3}^+, \dots, v_1^+, v_0^+)$, $\mathbf{v}^- = {}^t(v_{k-2}^-, v_{k-3}^-, \dots, v_1^-, v_0^-)$ such that

$$(2.5) \quad \delta_0(f)_\gamma \pm \overline{\delta_0(f | [\varepsilon])}_\gamma = w^\pm [Q_\gamma^\pm + \{M_{k-2}(\gamma) - 1\} \mathbf{v}^\pm]$$

for all $\gamma \in \Gamma_1(N)$. Note that the terms $\{M_{k-2}(\gamma) - 1\} \mathbf{v}^\pm$ do not occur for the case $k=2$. Suppose $k > 2$. By taking $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ as γ in (2.5), we see $v_0^\pm = v_1^\pm = \dots = v_{k-3}^\pm = 0$. Furthermore, if we operate $\hat{T}_p$ with $p \equiv 1 \pmod N$ and $p \neq 2$ on both sides of the relation (2.5) of cocycles and take $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as η_p , we obtain

$$w^\pm \{M_{k-2}(\gamma) - 1\} \{a_p - \sum_i M_{k-2}(\alpha_i)\} \mathbf{v}^\pm = w^\pm (Q^\pm | \hat{T}_p - a_p Q^\pm)_\gamma$$

from (2.3) and Lemma 3. By taking $\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}$ as γ , we see that $N\{a_p - (1 + p^{k-1})\} v_{k-2}^\pm$ are equal to the second components of $(Q^\pm | \hat{T}_p - a_p Q^\pm)_{\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}}$ respectively, because

$$\sum_i M_{k-2}(\alpha_i) = \begin{pmatrix} 1 + p^{k-1} & 0 & * \dots \end{pmatrix}$$

by our choice of representatives $\{\alpha_i\}$. Since f is a cusp form and $k > 2$, for every sufficiently large prime p , $a_p \neq 1 + p^{k-1}$, so that

$$(2.6) \quad v_{k-2}^\pm = \text{the second component of } (Q^\pm | \hat{T}_p - a_p Q^\pm)_{\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}} / N\{a_p - 1 - p^{k-1}\}.$$

Therefore, for $k \geq 2$,

$$\frac{1}{w^\pm} \{\delta_0(f)_\gamma \pm \overline{\delta_0(f | [\varepsilon])}_\gamma\} = Q_\gamma^\pm + \{M_{k-2}(\gamma) - 1\} \mathbf{v}^\pm$$

belong to K_f^{k-1} for all $\gamma \in \Gamma_1(N)$. Moreover, for every automorphism σ of $\mathbf{C}$,

we have

$$\{\delta_0(f)_\gamma + \overline{\delta_0(f|[\varepsilon])_\gamma}\}^\sigma = (w^\pm)^\sigma [(Q^\pm)_\gamma^\sigma + \{M_{k-2}(\gamma) - 1\}(\mathbf{v}^\pm)^\sigma],$$

where $(Q^+)^\sigma, (Q^-)^\sigma$ are the cocycles in $Z_p^1(\Gamma_1(N), K_{f^\sigma}^{k-1})$ defined by $(Q^\pm)_\gamma^\sigma = (Q^\pm)_\gamma^\sigma$. Since $(Q^\pm)^\sigma$ also satisfy the condition (2.4) for f^σ , there exist non-zero complex numbers w_σ^+, w_σ^- , and $(k-1)$ -dimensional complex vectors $\mathbf{v}_\sigma^+, \mathbf{v}_\sigma^-$ such that

$$\delta_0(f^\sigma)_\gamma \pm \overline{\delta_0(f^\sigma|[\varepsilon])_\gamma} = w_\sigma^\pm [(Q^\pm)_\gamma^\sigma + \{M_{k-2}(\gamma) - 1\} \mathbf{v}_\sigma^\pm].$$

Then, for $k > 2$, the formula (2.6) implies that $\mathbf{v}_\sigma^\pm$ belong to $K_{f^\sigma}^{k-1}$ and $\mathbf{v}_\sigma^\pm = (\mathbf{v}^\pm)^\sigma$. Hence, for $k \geq 2$, we obtain

$$\left[\frac{1}{w^\pm} \{\delta_0(f)_\gamma \pm \overline{\delta_0(f|[\varepsilon])_\gamma}\} \right]^\sigma = \frac{1}{w_\sigma^\pm} \{\delta_0(f^\sigma)_\gamma \pm \overline{\delta_0(f^\sigma|[\varepsilon])_\gamma}\}$$

for all $\gamma \in \Gamma_1(N)$, from which Theorem 1 follows.

3. Fourier coefficients and periods

Let the notation and the assumption be as in the preceding section. We set

$$P_{f,l}^\pm(x) = \int_x^{i\infty} f(z) z^l dz \pm (-1)^{l+1} \int_{-x}^{i\infty} f(z) z^l dz$$

for every rational number x . Suppose p is an odd prime with $p \equiv 1 \pmod{N}$. Then

$$\begin{aligned} a_p \int_x^{i\infty} f(z) d\mathfrak{z}_{k-2} &= \int_x^{i\infty} (f|T_p)(z) d\mathfrak{z}_{k-2} \\ &= \sum_i M_{k-2}(\alpha_i) \int_{\alpha_i(x)}^{i\infty} f(z) d\mathfrak{z}_{k-2} \end{aligned}$$

by our choice of representatives $\{\alpha_i\}$. Furthermore we can see that $\alpha_i(x)$ are $\Gamma_1(N)$ -equivalent cusps to x for all i . If $\alpha'_i \in \Gamma_1(N)$ with $\alpha'_i(x) = \alpha_i(x)$, then

$$\begin{aligned} \int_{\alpha_i(x)}^{i\infty} f(z) d\mathfrak{z}_{k-2} &= \int_{\alpha'_i(i\infty)}^{i\infty} f(z) d\mathfrak{z}_{k-2} + \int_{\alpha'_i(x)}^{\alpha'_i(i\infty)} f(z) d\mathfrak{z}_{k-2} \\ &= \int_{\alpha'_i(i\infty)}^{i\infty} f(z) d\mathfrak{z}_{k-2} + M_{k-2}(\alpha'_i) \int_x^{i\infty} f(z) d\mathfrak{z}_{k-2}. \end{aligned}$$

Therefore we have

$$\{a_p - \sum_i M_{k-2}(\alpha'_i \alpha'_i)\} \int_x^{i\infty} f(z) d\mathfrak{z}_{k-2} = \sum_i M_{k-2}(\alpha'_i) \int_{\alpha'_i(i\infty)}^{i\infty} f(z) d\mathfrak{z}_{k-2}.$$

Since $\{\varepsilon \alpha_i \varepsilon\} = \{\alpha_i\}$ and $\alpha_i(-x) = \varepsilon(\varepsilon \alpha_i \varepsilon)' \varepsilon(-x)$, we obtain

$$(3.1) \quad \{a_p - \sum_i M_{k-2}(\alpha'_i \alpha'_i)\} \left\{ \int_x^{i\infty} f(z) d\mathfrak{z}_{k-2} \pm M_{k-2}(\varepsilon) \int_{-x}^{i\infty} f(z) d\mathfrak{z}_{k-2} \right\} \\ = \sum_i M_{k-2}(\alpha'_i) \left\{ \int_{\alpha'_i(i\infty)}^{i\infty} f(z) d\mathfrak{z}_{k-2} \pm M_{k-2}(\varepsilon) \int_{-\alpha'_i(i\infty)}^{i\infty} f(z) d\mathfrak{z}_{k-2} \right\}.$$

If β is any element of $SL(2, \mathbf{Z})$ such that $\beta(i\infty) = x$, we see that $\beta^{-1}\alpha'_i\alpha'_i\beta$ are upper triangular matrices for all i . More precisely, if $x = c/d$ with $c, d \in \mathbf{Z}$, $(c, d) = 1$, then all $\beta^{-1}\alpha'_i\alpha'_i\beta$ are of the form $\begin{pmatrix} p & * \\ 0 & 1 \end{pmatrix}$ except for $\alpha_i = \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix}$ with $p \nmid d + mc$ or $\alpha_i = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ according as $p \nmid c$ or $p \mid c$, and $\beta^{-1}\alpha'_i\alpha'_i\beta$ is of the form $\begin{pmatrix} 1 & * \\ 0 & p \end{pmatrix}$ for the exceptional α_i . Hence

$$a_p I - \sum_i M_{k-2}(\alpha'_i \alpha'_i) \\ = M_{k-2}(\beta) \begin{pmatrix} a_p - (1 + p^{k-1}) & & & \\ & 0 & a_p - (p + p^{k-2}) & \\ & & \ddots & \\ & 0 & & 0 & a_p - (p^{k-2} + p) \end{pmatrix} M_{k-2}(\beta^{-1}),$$

where I denotes the unit matrix of degree $k-1$. If k is even, the matrix $a_p I - \sum_i M_{k-2}(\alpha'_i \alpha'_i)$ is non-singular for every sufficiently large p by the estimate $a_n = O(n^{k/2-1/5})$ ([8]). If k is odd, the same estimate implies that $a_p - (p^l + p^{k-1-l})$ with $l \neq (k-1)/2$ are non-zero for such a prime p . Moreover, even for $l = (k-1)/2$, there exist infinitely many primes p such that $p \equiv 1 \pmod{N}$ and $a_p - 2p^{(k-1)/2} \neq 0$. We shall prove this in the next section. Therefore, from Theorem 1 and (3.1), we obtain the following theorem, which is equivalent to Theorem 1 of Shimura [13] for all $k \geq 2$.

THEOREM 2. Let $w_\sigma^\pm$ be the complex numbers as in Theorem 1. Then $\frac{1}{w_1^\pm} P_{f,l}^\pm(x)$ belong to K_f for all $x \in \mathbf{Q}$, $l=0, 1, \dots, k-2$, and $\left\{ \frac{1}{w_1^\pm} P_{f,l}^\pm(x) \right\}^\sigma = \frac{1}{w_\sigma^\pm} P_{f\sigma,l}^\pm(x)$ for every injection σ of K_f into $\mathbf{C}$.

The formula (3.1) holds for all primes p with $p \nmid 2N$, if we replace M_{k-2} by $\tilde{\chi} \otimes M_{k-2}$ and the condition $\alpha'_i \in \Gamma_1(N)$ by $\alpha'_i \in \Gamma_0(N)$, where $\tilde{\chi}$ is the character of

the semi-group $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbf{Z}, c \equiv 0 \pmod{N}, (d, N) = 1, ad - bc > 0 \right\}$ defined by $\tilde{\chi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)$. Then, for $x=0$, $a_p - \sum_i (\tilde{\chi} \otimes M_{k-2})(\alpha_i^* \alpha_i')$ is a triangular matrix and each of its diagonal components $a_p - (\chi(p)p^{l+1} + p^{k-2-l})$ ($0 \leq l \leq k-2$) is a linear combination of $P_{f,l}^{\pm}(\gamma(i\infty))/P_{f,l}^{\pm}(0)$ for all γ in a set of generators of $\Gamma_0(N)$, $l'=0, 1, \dots, k-2$ and $P_{f,l''}^{\pm}(0)/P_{f,l}^{\pm}(0)$ for $l''=l+1, \dots, k-2$ with coefficients in $\mathbf{Z}$ provided $P_{f,l}^{\pm}(0) \neq 0$. Therefore the calculation of these quotients of periods yields similar congruences for the Fourier coefficients a_p as in Manin [5].

EXAMPLES. We set $P_l^{\pm}(x) = P_{f,l}^{\pm}(x)$.

$$1) \quad S_2(\Gamma_0(11)) = \mathbf{C}f(z), f(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^2 (1 - e^{22\pi inz})^2.$$

$$P_0^+ \left(-\frac{9}{55} \right) = 0,$$

$$P_0^- \left(-\frac{8}{33} \right) : P_0^- \left(\frac{9}{55} \right) : P_0^-(0) = \frac{5}{2} : 5 : 1,$$

$$a_p \equiv 1 + p \pmod{5} \quad (p \neq 11).$$

$$2) \quad S_4(\Gamma_0(5)) = \mathbf{C}f(z), f(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^4 (1 - e^{10\pi inz})^4.$$

$$P_2^+ \left(-\frac{2}{5} \right) = 0,$$

$$P_1^+ \left(-\frac{2}{5} \right) : P_0^+ \left(-\frac{2}{5} \right) : P_2^+ \left(-\frac{3}{10} \right) : P_1^+ \left(-\frac{3}{10} \right) : P_0^+ \left(-\frac{3}{10} \right) : P_1^+(0) \\ = 5 : -20 : -10 : 35 : -110 : 1,$$

$$P_2^- \left(-\frac{2}{5} \right) : P_1^- \left(\frac{2}{5} \right) : P_0^- \left(-\frac{2}{5} \right) : P_2^- \left(-\frac{3}{10} \right) : P_1^- \left(-\frac{3}{10} \right) : P_0^- \left(-\frac{3}{10} \right) :$$

$$P_2^-(0) : P_0^-(0)$$

$$= -\frac{13}{4} : \frac{117}{16} : -13 : -\frac{13}{8} : \frac{13}{16} : \frac{91}{8} : -\frac{13}{40} : 1,$$

$$a_p \equiv p + p^2 \pmod{10} \quad (p \neq 5),$$

$$a_p \equiv 1 + p^3 \pmod{13} \quad (p \neq 5).$$

$$3) \quad S_6(\Gamma_0(3)) = \mathbf{C}f(z), f(z) = e^{2\pi iz} (1 - e^{2\pi iz})^6 \prod_{n=1}^{\infty} (1 - e^{6\pi inz})^6.$$

$$P_4^+\left(-\frac{1}{3}\right) = P_3^+\left(-\frac{1}{3}\right) = 0,$$

$$P_2^+\left(-\frac{1}{3}\right) : P_1^+\left(-\frac{1}{3}\right) : P_0^+\left(-\frac{1}{3}\right) : P_3^+(0), P_1^+(0) = -2 : 9 : -24 : -\frac{1}{3} : 1,$$

$$P_4^-\left(-\frac{1}{3}\right) = 0,$$

$$P_3^-\left(-\frac{1}{3}\right) : P_2^-\left(-\frac{1}{3}\right) : P_1^-\left(-\frac{1}{3}\right) : P_0^-\left(-\frac{1}{3}\right) : P_4^-(0) : P_2^-(0) : P_0^-(0) \\ = \frac{1}{3} : -1 : \frac{5}{6} : 4 : \frac{1}{9} : \frac{13}{54} : 1,$$

$$a_p \equiv p^2 + p^3 \pmod{18} \quad (p \neq 3).$$

$$4) \quad S_8(\Gamma_0(2)) = \mathbf{C}f(z), f(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^8 (1 - 4\pi inz)^8.$$

$$P_6^+\left(-\frac{1}{2}\right) = P_5^+\left(-\frac{1}{2}\right) = P_0^+\left(-\frac{1}{2}\right) = 0,$$

$$P_4^+\left(-\frac{1}{2}\right) : P_3^+\left(-\frac{1}{2}\right) : P_2^+\left(-\frac{1}{2}\right) : P_1^+\left(-\frac{1}{2}\right) : P_5^+(0) : P_3^+(0) : P_1^+(0) \\ = -\frac{24}{5} : \frac{72}{5} : -24 : 24 : -\frac{6}{5} : \frac{9}{5} : 1,$$

$$P_6^-\left(-\frac{1}{2}\right) = 0,$$

$$P_5^-\left(-\frac{1}{2}\right) : P_4^-\left(-\frac{1}{2}\right) : P_3^-\left(-\frac{1}{2}\right) : P_2^-\left(-\frac{1}{2}\right) : P_1^-\left(-\frac{1}{2}\right) : P_0^-\left(-\frac{1}{2}\right) \\ : P_6^-(0) : P_4^-(0) : P_2^-(0) : P_0^-(0) = -\frac{1}{4} : \frac{1}{2} : -\frac{3}{20} : -\frac{7}{5} : \frac{9}{2} : -9 : \\ -\frac{1}{8} : \frac{17}{120} : \frac{17}{60} : 1,$$

$$a_p \equiv p^3 + p^4 \pmod{120} \quad (p \neq 2).$$

Here we note that the modular groups above have generators as follows ([7]).

$$\Gamma_0(11) : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -8 & -1 \\ 33 & 4 \end{pmatrix}, \begin{pmatrix} -9 & -1 \\ 55 & 6 \end{pmatrix}.$$

$$\Gamma_0(5) : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}, \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix}.$$

$$\Gamma_0(3) : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}.$$

$$\Gamma_0(2) : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

4. The Ramanujan-Petersson inequality

Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a normalized newform in $S_k(N, \chi)$ ($k \geq 2$) ([1], [4], [6]). Then we have the following

THEOREM 3. *The set of primes $P = \{p \mid a_p^2 = 4\chi(p)p^{k-1}\}$ has density zero, i.e. $\#\{p \in P \mid p < x\} = o(x/\log x)$ as $x \rightarrow \infty$.*

This follows from the theory of Deligne and Serre [2] (cf. [15]). Let L be the set of rational primes which split completely in K_f . For each $l \in L$, we denote by $K_{N,l}$ the maximal algebraic extension of $\mathbf{Q}$ unramified outside of Nl and by $K_{N,l}^{ab}$ the maximal subfield of $K_{N,l}$ abelian over $\mathbf{Q}$. Then there exists an l -adic representation $\rho_l: \text{Gal}(K_{N,l}/\mathbf{Q}) \rightarrow GL(2, \mathbf{Z}_l)$ such that

$$(4.1) \quad \text{Tr} \circ \rho_l(\text{Frob}(p)) = a_p, \quad \det \circ \rho_l(\text{Frob}(p)) = \chi(p)p^{k-1}$$

for every prime $p \nmid Nl$, where $\mathbf{Z}_l$ is the ring of l -adic integers and $\text{Frob}(p)$ denotes a Frobenius element of p in $\text{Gal}(K_{N,l}/\mathbf{Q})$ ([2]). Let $\tilde{\rho}_l: \text{Gal}(K_{N,l}/\mathbf{Q}) \rightarrow GL(2, \mathbf{F}_l)$ be the reduction of ρ_l modulo l , where $\mathbf{F}_l = \mathbf{Z}_l/l\mathbf{Z}_l$. Then one of the following must hold ([15]):

- i) The image of $\tilde{\rho}_l$ contains $SL(2, \mathbf{F}_l)$.
- ii) The image of $\tilde{\rho}_l$ is contained in a Borel subgroup of $GL(2, \mathbf{F}_l)$.
- iii) The image of $\tilde{\rho}_l$ is contained in the normalizer of a Cartan subgroup of $GL(2, \mathbf{F}_l)$, but not in the Cartan subgroup itself.
- iv) The image of $\tilde{\rho}_l$ in $PGL(2, \mathbf{F}_l) = GL(2, \mathbf{F}_l)/\mathbf{F}_l^*$ is isomorphic to the alternating groups $\mathfrak{A}_4$, $\mathfrak{A}_5$, or the symmetric group $\mathfrak{S}_4$.

Since L is infinite, at least one case among i)~iv) occurs for infinitely many $l \in L$ and such a case implies that P has density zero.

If the case i) occurs for infinitely many l , our assertion follows from the next lemma and the Čebotarev density theorem.

LEMMA 5. *Let G be a subgroup of $GL(2, \mathbf{F}_l)$ which contains $SL(2, \mathbf{F}_l)$ and G° the subset of G consisting of all elements whose characteristic polynomials have double roots. Then $\#G^\circ/\#G \leq 2l/(l^2 - 1)$.*

PROOF. The number of elements of G° with a given determinant is at most $2l^2$, so that our lemma holds.

Next suppose the case ii) occurs for infinitely many l . Without loss of generality we may assume the image of $\tilde{\rho}_l$ is contained in the group of non-singular upper triangular matrices for each such l . We define the homomorphism $\mu_l: \text{Gal}(K_{N,l}/\mathbf{Q}) \rightarrow \mathbf{F}_l^*$ as the composition of $\tilde{\rho}_l$ and the homomorphism of the group of non-singular upper triangular matrices into $\mathbf{F}_l^*$ given by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto ad^{-1}$. Then μ_l factors through $\text{Gal}(K_{N,l}^b/\mathbf{Q}) (\cong \prod_{p|Nl} \mathbf{Z}_p^*)$. Accordingly, for each l , there exists a positive integer $\mathbf{f}_l$ composed of prime factors of Nl , such that the image of $\text{Frob}(p)$ by μ_l is determined by $p \bmod \mathbf{f}_l$. If the orders of the images of μ_l are not bounded, our assertion holds by the Čebotarev density theorem. Therefore, suppose the orders of the images of μ_l are bounded. Then we may assume they are constant and equal to a positive integer m . If $l \nmid \mathbf{f}_l$ for infinitely many l , every prime $p \nmid Nl$ in P must satisfy certain congruences modulo l for each such l , so that P has density zero. If $l \nmid \mathbf{f}_l$ except for finitely many l , then $\mathbf{f}_l$ must be bounded. Moreover, for every prime $p \nmid Nl$, we have

$$a_p^2 \equiv \chi(p)p^{k-1} \{ \zeta^{\mathbf{e}_l(p)} + \zeta^{-\mathbf{e}_l(p)} + 2 \} \pmod{l\mathbf{Z}_l}$$

for some integer $\mathbf{e}_l(p) \in \{0, 1, \dots, m-1\}$, where ζ is an m -th root of unity, and each subset of primes $\{p \mid \mathbf{e}_l(p) = i\}$ ($0 \leq i \leq m-1$) has density $1/m$. This follows from (4.1). Since $\mathbf{f}_l$ are bounded, there exists an l such that

$$a_p^2 = \chi(p)p^{k-1} \{ \zeta^{\mathbf{e}_l(p)} + \zeta^{-\mathbf{e}_l(p)} + 2 \}$$

for every prime $p \nmid Nl$. This contradicts the estimate

$$(4.2) \quad \sum_{p \leq x} |a_p|^2 p^{-(k-1)} \sim \frac{x}{\log x} \quad \text{as } x \rightarrow \infty$$

([2]). Hence our assertion holds in this case.

Now suppose the case iii) occurs for infinitely many l . For each such l , we denote by C_l and N_l the Cartan subgroup and its normalizer respectively. Then the surjective homomorphism

$$\text{Gal}(K_{N,l}/\mathbf{Q}) \longrightarrow N_l \longrightarrow N_l/C_l \cong \{\pm 1\}$$

factors through $\text{Gal}(K_{N,l}^{ab}/\mathbf{Q})$, so that there exists a rational prime $\mathbf{q}_l$ dividing Nl such that the image of $\text{Frob}(p)$ by the above homomorphism is equal to the quadratic residue symbol $\left(\frac{p}{\mathbf{q}_l}\right)$. Let G_l be the image of $\tilde{\rho}_l$ in $\text{PGL}(2, \mathbf{F}_l)$. If the orders of G_l are not bounded for these l , our assertion follows from the Čebotarev density theorem. Note that the order of G_l is not divisible by l , so that the image of $\tilde{\rho}_l(\text{Frob}(p))$ in $\text{PGL}(2, \mathbf{F}_l)$ is the identity element for every prime $p \nmid Nl$ in P in this case. Therefore we may assume the orders of G_l are constant and equal to a positive even integer $2m$. If $\tilde{\rho}_l(\text{Frob}(p)) \notin C_l$, then $\text{Tr} \circ \tilde{\rho}_l(\text{Frob}(p)) = 0$, so that $a_p \equiv 0 \pmod{l\mathbf{Z}_l}$ for all primes $p \nmid Nl$ with $\left(\frac{p}{\mathbf{q}_l}\right) = -1$. If $\mathbf{q}_l = l$ for infinitely many l , P has density zero because $\left(\frac{p}{\mathbf{q}_l}\right) = +1$ for every prime $p \nmid Nl$ in P . Therefore we also assume $\mathbf{q}_l$ are constant and equal to a prime factor q of N . Then we have $a_p = 0$ for all primes $p \nmid N$ with $\left(\frac{p}{q}\right) = -1$. On the other hand, for every prime $p \nmid N$ with $\left(\frac{p}{q}\right) = +1$, there exists an integer $\mathbf{e}(p) \in \{0, 1, \dots, m-1\}$ such that

$$a_p^2 = \chi(p)p^{k-1} \{ \zeta^{\mathbf{e}(p)} + \zeta^{-\mathbf{e}(p)} + 2 \}$$

as before. If k is even, this is a contradiction because K_f is a finite extension of $\mathbf{Q}$. Accordingly, suppose k is odd. Let

$$\chi_l: \text{Gal}(K_{N,l}/\mathbf{Q}) \longrightarrow \text{Gal}(K_{1,l}^{ab}/\mathbf{Q}) \longrightarrow \mathbf{Z}_l^*$$

be the canonical l -adic cyclotomic character, where the first arrow is the restriction homomorphism and the latter the canonical isomorphism by class field theory. Then $\rho_l \otimes \chi_l^{-(k-1)/2}$ is an l -adic representation of $\text{Gal}(K_{N,l}/\mathbf{Q})$ into $\text{GL}(2, \mathbf{Q}_l)$ and the image of $\text{Tr} \circ (\rho_l \otimes \chi_l^{-(k-1)/2})$ is finite.

LEMMA 6. *Let H be a compact subgroup of $\text{GL}(2, \mathbf{Q}_l)$ such that the set of traces $\{\text{Tr}(\alpha) \mid \alpha \in H\}$ is finite. Then H is finite, or H is conjugate to a subgroup of $\text{GL}(2, \mathbf{Q}_l)$ consisting of upper triangular matrices, whose diagonal components are only finitely many roots of unity.*

PROOF. First note that the finiteness of the set of traces implies that the eigenvalues of the matrices in H are finitely many roots of unity. From this fact we easily obtain that H is finite or conjugate to a subgroup consisting of upper triangular matrices according as all the elements in H are semi-simple or not.

By this lemma, the l -adic representation $\rho_l \otimes \chi_l^{-(k-1)/2}$ has a finite image, or is isomorphic to an l -adic representation of the form $\begin{pmatrix} \varepsilon_1 & * \\ 0 & \varepsilon_2 \end{pmatrix}$ for some l -adic characters $\varepsilon_1, \varepsilon_2$ of finite order. If $\rho_l \otimes \chi_l^{-(k-1)/2}$ has a finite image, it defines an Artin L -function, which yields a contradiction from the point of view of the functional equations ([10]). If $\rho_l \otimes \chi_l^{-(k-1)/2}$ is isomorphic to $\begin{pmatrix} \varepsilon_1 & * \\ 0 & \varepsilon_2 \end{pmatrix}$, then

$$a_p = p^{(k-1)/2} \{ \varepsilon_1(\text{Frob}(p)) + \varepsilon_2(\text{Frob}(p)) \}$$

for every prime $p \nmid Nl$, which contradicts the estimate (4.2). Hence our assertion holds in this case.

Finally we note that the case iv) occurs for at most a finite number of $l \in L$. In fact, in this case, the image of $\tilde{\rho}_l(\text{Frob}(p))$ in $PGL(2, \mathbf{F}_l)$ is of order 1, 2, 3, 4, or 5, and a_p^2 is congruent to $4\chi(p)p^{k-1}$, 0, $\chi(p)p^{k-1}$, $2\chi(p)p^{k-1}$, or $\omega\chi(p)p^{k-1}$ modulo $l\mathbf{Z}_l$ for every prime $p \nmid Nl$, where ω is a root of $u^2 - 3u + 1 = 0$. Therefore, if the case iv) occurs for infinitely many l , then a_p^2 must be equal to one of the values above, which is a contradiction as in the preceding case. This completes the proof of Theorem 3.

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Added in proof.

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