A NONPARAMETRIC TEST OF TREND BASED ON AMALGAMATION

By

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§1. Summary

In this paper we deal with a nonparametric test of trend proposed by Brunk [3]. This test is identical with the test based on the number of distinct values obtained by applying the Pool Adjacent Violators Algorithm or amalgamation process to the data. (c.f. Barlow et al. [2]) We provide rather extensive table, and an example is taken from Barlow et al. [2]. We conducted simulations and tables of estimated power is also provided.

§2. Preliminaries

We cite necessary definitions and results. For a sequence of k numbers $(a_1, a_2, \dots, a_k)$, let $a_0 = 0$ and consider $k+1$ points on (X, Y) -plane (l, a_l) ($l=0, \dots, k$), and a function corresponding to the diagram which connects adjacent points (l, a_l) and $(l+1, a_{l+1})$ by segment, or a function

$$f(x) = a_l + (x-l)(a_{l+1} - a_l) \quad \text{for } l \leq x \leq l+1, l=0, \dots, k-1 \quad (1)$$

The greatest convex minorant function (GCMF) for $(a_1, a_2, \dots, a_k)$ is the maximum of the convex functions among those less than or equal to $f(x)$ of (1). The GCMF apparently exists uniquely and its diagram connects $(0, 0)$ and (k, a_k) and, if any, some of (l, a_l) $l=0, \dots, k$ by one or more segments. Thus the diagram of the GCMF of $(a_1, a_2, \dots, a_k)$ consists of at least one and at most k segments. This number, L , of segments is the test statistic in this paper.

For the maximum likelihood estimates in isotonic regression, the pool adjacent violators algorithm (PAVA) or amalgamation process is very important and the relation between this algorithm and the GCMF is excellently reviewed in R. E. Barlow et al. [2]. The relation is essentially the following. The number

of distinct values of estimates obtained by applying the pool adjacent violators algorithm to $(x_1, \dots, x_k)$ is equal to the number of segments of the GCMF of the numbers $(S_1, S_2, \dots, S_k)$ where $S_l = \sum_{i=1}^l x_i$. Let $\underline{X} = (X_1, \dots, X_k)'$ be k -dimensional real valued vector random variable with the distribution function $F(x_1, \dots, x_k) = F(\underline{x})$ and let $S_l = X_1 + \dots + X_l$ ($l = 1, 2, \dots, k$). $\underline{X}$ is said to be symmetrically dependent if $F(\underline{x})$ is invariant under any permutation of the components, i.e.

$$F(x_1, \dots, x_k) = F(x_{i_1}, x_{i_2}, \dots, x_{i_k})$$

for any permutation $(i_1, i_2, \dots, i_k)$ of $(1, 2, \dots, k)$.

We now state

THEOREM 1. *Let $\underline{X} = (X_1, \dots, X_k)'$ be symmetrically dependent random variable and satisfies*

$$\Pr \left[\frac{1}{l} S_l = \frac{1}{l'} S_{l'} \right] = 0, \quad 1 \leq l < l' \leq k, \quad (2)$$

and let the probability that the random variable L , the number of segments of the GCMF of $(S_0, \dots, S_k)$ or the number of distinct values obtained by applying PAVA to $(X_1, \dots, X_k)$, takes the value l be denoted by $p(l, k)$. Then we have

$$\begin{aligned} p(1, k) &= 1/k \\ p(l, k) &= \frac{1}{k} \{ (k-1)p(l, k-1) + p(l-1, k-1) \}, \quad l = 2, \dots, k-1 \\ p(k, k) &= 1/k! \end{aligned} \quad (3)$$

PROOF. The proof can be found by noting that Theorem 3.3 of R. E. Barlow et al. [2] is proved by E. S. Andersen [1]. The condition in this Theorem is identical with that of Theorem A in Chacko [4], and the result is from corollary B on page 145 of R. E. Barlow et al. [2].

COROLLARY. *For $Q(l, k) = \Pr(L \geq l) = \sum_{m=l}^k p(m, k)$, we have the similar recurrence relations.*

$$\begin{aligned} Q(1, k) &= 1 \\ Q(l, k) &= \frac{1}{k} \{ (k-1)Q(l, k-1) + Q(l-1, k-1) \}, \quad l = 2, \dots, k-1. \\ Q(k, k) &= \frac{1}{k!} = \frac{1}{k} Q(k-1, k-1) \end{aligned} \quad (4)$$

PROOF. This can be proved by straight forward calculations.

§3. Test based on the number of sides in GCMF

In [3], H. D. Brunk discussed the result of E. S. Andersen in more general condition and suggested that the number of sides in GCMF could be used to test the upward trend. His suggestion is made on an intuitive basis, and the present authors share the same intuition, and call this test as GCMF test.

About the first four moments, the skewness and the kurtosis, we have, after simple calculations, the following relations.

$$\text{Mean} = \mu = \sum_{n=1}^k 1/n$$

$$\text{Variance} = \sigma^2 = - \sum_{n=1}^k (1/n)^2 + \sum_{n=1}^k (1/n)$$

$$E(L - \mu)^3 = 2 \sum_{n=1}^k (1/n)^3 - 3 \sum_{n=1}^k (1/n)^2 + \mu$$

$$E(L - \mu)^4 = -6 \sum_{n=1}^k (1/n)^4 + 12 \sum_{n=1}^k (1/n)^3 + 3 \left(\sum_{n=1}^k (1/n)^2 \right)^2 \\ - (6\mu + 7) \sum_{n=1}^k (1/n)^2 + 3\mu^2 + \mu$$

For large values of k , they can be approximated by

$$\mu = \log k + \gamma = \log k + 0.57721$$

where γ is the Euler constant.

$$\sigma^2 = \log k - 1.06771.$$

$$E(L - \mu)^3 = \log k - 1.95347.$$

$$E(L - \mu)^4 = 3(\log k)^2 - 5.40631 \log k + 0.41348.$$

Finally we have

$$\text{Skewness of } L = 0(\sqrt{\mu}),$$

$$\text{Kurtosis of } L = 3 + 0(\mu).$$

Although L is distributed asymptotically normal with mean and variance

both equal to $\log k$, the convergence is too slow to be used in computing the upper tail probability for manageable size of sample. Fortunately we have as given in Corollary the recurrence relations (4) among upper tail probabilities, which enables us to compute the probabilities for a fairly large value of k .

Starting from the initial values $Q(1, 2)=1$, $Q(2, 2)=1/2$, and $Q(l, 2)=0$ ($l=3, \dots, 30$), we performed the recurrent calculations using Seiko Desk Top computer S500, which can handle 16 digits floating numbers between 10^{-99} and 10^{99} . In Table 1, for $\alpha=0.05$ and 0.01 the minimum of l such that $\Pr(L>l)<\alpha$ and the values of $\Pr(L>l)$ and $\Pr(L=l)$ are tabulated for $k=3, \dots, 150$. Calculations are made through $k=10,000$ and the summary figures are tabulated in Table 2.

TABLE 1. Table for the GCMF test.

k	l	$\Pr(L>l k)$	$\Pr(L=l k)$	l	$\Pr(L<l k)$	$\Pr(L=l k)$
3	3	0.	.166667	3	0.	.166667
4	3	.041667	.250000	4	0.	.041667
5	4	.008333	.083333	4	.008333	.083333
6	4	.022222	.118056	5	.001389	.020833
7	4	.039087	.145833	5	.004365	.034722
8	5	.008705	.048611	5	.008705	.048611
9	5	.014107	0.61863	6	.001607	.012500
10	5	.020293	.074219	6	.002857	.017436
11	5	.027040	.085601	6	.004442	.022598
12	5	.034174	.096024	6	.006325	.027849
13	5	.041560	.105541	6	.008467	.033093
14	5	.049099	.114222	7	.001758	.009073
15	6	.013382	.043331	7	.002363	.011019
16	6	.016090	.048257	7	.003052	.013039
17	6	.018929	.053028	7	.003819	.015110
18	6	.021875	.057637	7	.004658	.017217
19	6	.024908	.062078	7	.005564	.019344
20	6	.028012	.066353	7	.006532	.021481
21	6	.031172	.070462	7	.007554	.023618
22	6	.034375	.074410	7	.008628	.025747
23	6	.037610	.078202	7	.009747	.027863
24	6	.040869	.081843	8	.002359	.008550
25	6	.044142	.085339	8	.002701	.009406
26	6	.047425	.088696	8	.003062	.010277

TABLE 1. (continued.)

27	7	.014601	.036108	8	.003443	.011158
28	7	.015891	.038102	8	.003841	.012049
29	7	.017205	.040064	8	.004257	.012948
30	7	.018540	.041995	8	.004689	.013852
31	7	.019895	.043894	8	.005135	.014760
32	7	.021267	.045760	8	.005597	.015670
33	7	.022653	.047593	8	.006071	.016582
34	7	.024053	.049392	8	.006559	.017494
35	7	.025464	.051160	8	.007059	.018405
36	7	.026885	.052894	8	.007570	.019315
37	7	.028315	.054596	8	.008092	.020223
38	7	.029752	.056266	8	.008624	.021127
39	7	.031194	.057905	8	.009166	.022028
40	7	.032642	.059512	8	.009717	.022925
41	7	.034093	.061089	9	.002633	.007643
42	7	.035548	.062636	9	.002815	.008028
43	7	.037005	.064154	9	.003001	.008416
44	7	.038463	.065643	9	.003193	.008806
45	7	.039921	.067103	9	.003388	.009199
46	7	.041380	.068536	9	.003588	.009593
47	7	.042838	.069942	9	.003793	.009989
48	7	.044296	.071322	9	.004001	.010386
49	7	.045751	.072675	9	.004213	.010785
50	7	.047205	.074003	9	.004428	.011184
51	7	.048656	.075307	9	.004648	.011584
52	8	.016855	.033249	9	.004870	.011985
53	8	.017483	.034066	9	.005096	.012386
54	8	.018113	.034877	9	.005326	.012788
55	8	.018748	.035681	9	.005558	.013189
56	8	.019385	.036477	9	.005794	.013591
57	8	.020025	.037267	9	.006032	.013992
58	8	.020667	.038049	9	.006274	.014394
59	8	.021312	.038824	9	.006518	.014799
60	8	.021959	.039592	9	.006764	.015195
61	8	.022608	.040353	9	.007013	.015595
62	8	.023259	.041108	9	.007265	.015994
63	8	.023912	.041855	9	.007519	.016393
64	8	.024566	.042595	9	.007775	.016791
65	8	.025221	.043328	9	.008033	.017188

TABLE 1. (continued).

66	8	.025877	.044054	9	.008293	.017584
67	8	.026535	.044774	9	.008556	.017979
68	8	.027193	.045487	9	.008820	.018373
69	8	.027852	.046193	9	.009087	.018766
70	8	.028512	.046892	9	.009355	.019158
71	8	.029173	.047585	9	.009625	.019548
72	8	.029834	.048271	9	.009896	.019938
73	8	.030495	.048951	10	.002971	.007198
74	8	.031157	.049625	10	.003068	.007375
75	8	.031818	.050292	10	.003167	.007553
76	8	.032480	.050952	10	.003266	.007731
77	8	.033142	.051607	10	.003367	.007910
78	8	.033803	.052255	10	.003468	.008089
79	8	.034465	.052898	10	.003570	.008268
80	8	.035126	.053534	10	.003674	.008448
81	8	.035787	.054165	10	.003778	.008627
82	8	.036447	.054789	10	.003883	.008807
83	8	.037107	.055408	10	.003989	.008987
84	8	.037767	.056021	10	.004096	.009168
85	8	.038426	.056628	10	.004204	.009348
86	8	.039085	.057230	10	.004313	.009529
87	8	.039742	.057826	10	.004422	.009709
88	8	.040400	.058417	10	.004533	.009890
89	8	.041056	.059002	10	.004644	.010071
90	8	.041712	.059582	10	.004756	.010251
91	8	.042366	.060156	10	.004868	.010432
92	8	.043020	.060726	10	.004982	.010613
93	8	.043673	.061290	10	.005096	.010794
94	8	.044325	.061849	10	.005211	.010975
95	8	.044976	.062403	10	.005326	.011155
96	8	.045626	.062953	10	.005442	.011336
97	8	.046275	.063497	10	.005559	.011516
98	8	.046923	.064036	10	.005677	.011697
99	8	.047570	.064571	10	.005795	.011877
100	8	.048216	.065101	10	.005914	.012057
101	8	.048860	.065626	10	.006033	.012237
102	8	.049504	.066147	10	.006153	.012417
103	9	.018871	.031275	10	.006274	.012597
104	9	.019172	.031615	10	.006395	.012777

TABLE 1. (continued).

105	9	.019473	.031954	10	.006516	.012956
106	9	.019774	.032291	10	.006639	.013135
107	9	.020076	.032626	10	.006761	.013314
108	9	.020378	.032960	10	.006885	.013493
109	9	.020680	.033293	10	.007009	.013672
110	9	.020983	.033623	10	.007133	.013850
111	9	.021286	.033952	10	.007258	.014028
112	9	.021589	.034280	10	.007383	.014206
113	9	.021892	.034606	10	.007509	.014384
114	9	.022196	.034930	10	.007635	.014561
115	9	.022500	.035253	10	.007761	.014738
116	9	.022804	.035574	10	.007888	.014915
117	9	.023108	.035894	10	.008016	.015092
118	9	.023412	.036212	10	.008144	.015268
119	9	.023716	.036529	10	.008272	.015444
120	9	.024021	.036844	10	.008401	.015620
121	9	.024325	.037158	10	.008530	.015795
122	9	.024630	.037470	10	.008659	.015970
123	9	.024934	.037780	10	.008789	.016145
124	9	.025239	.038089	10	.008919	.016320
125	9	.025544	.038397	10	.009050	.016494
126	9	.025848	.038703	10	.009181	.016668
127	9	.026153	.039008	10	.009312	.016841
128	9	.026458	.039311	10	.009444	.017014
129	9	.026763	.039613	10	.009576	.017187
130	9	.027067	.039913	10	.009708	.017360
131	9	.027372	.040212	10	.009840	.017532
132	9	.027677	.040509	10	.009973	.017704
133	9	.027981	.040805	11	.003263	.006843
134	9	.028286	.041100	11	.003315	.006925
135	9	.028590	.041393	11	.003366	.007007
136	9	.028895	.041685	11	.003417	.007090
137	9	.029199	.041975	11	.003469	.007172
138	9	.029503	.042264	11	.003521	.007255
139	9	.029807	.042552	11	.003573	.007337
140	9	.030111	.042838	11	.003626	.007420
141	9	.030415	.043123	11	.003678	.007503
142	9	.030719	.043406	11	.003731	.007585
143	9	.031022	.043688	11	.003784	.007668

TABLE 1. (continued).

144	9	.031325	.043969	11	.003837	.007750
145	9	.031629	.044249	11	.003891	.007833
146	9	.031932	.044527	11	.003945	.007916
147	9	.032235	.044804	11	.003998	.007998
148	9	.032537	.045080	11	.004052	.008081
149	9	.032840	.045354	11	.004107	.008164
150	9	.033142	.045627	11	.004161	.008247

TABLE 2. Summary table of the distribution of GCMF statistic, providing values of k ($\leq 10,000$) with $\Pr(L \geq l|k) \leq \alpha < \Pr(L \geq l-1|k)$ for different values of l and α .

$\alpha \backslash l$	4	5	6	7	8	9
10%	4-5	6-10	11-20	21-42	43-88	89-188
5%	4	5-7	8-14	15-26	27-51	52-102
2.5%		5-6	7-10	11-19	20-34	35-64
1%		5	6-8	9-13	14-23	24-39
0.5%			6-7	8-11	12-18	19-30
0.1%				7-8	9-12	13-19

$\alpha \backslash l$	10	11	12	13	14	15
10%	189-406	407-888	889-1961	1962-4374	4375-9842	9843-
5%	103-207	208-424	425-884	885-1863	1864-3971	3972-8549
2.5%	65-123	124-239	240-473	474-949	950-1930	1931-3974
1%	40-72	73-132	133-246	247-467	468-899	900-1758
0.5%	31-52	53-92	93-165	166-301	302-559	560-1055
0.1%	20-30	31-48	49-80	81-135	136-234	235-411

$\alpha \backslash l$	16	17	18	19	20	21
10%						
5%	8550-					
2.5%	3975-8267	8268-				
1%	1759-3478	3479-6961	6962-			
0.5%	1056-2018	2019-3908	3909-7656	7657-		
0.1%	412-734	735-1330	1331-2441	2442-4537	4538-8527	8528-

EXAMPLE. An example is taken from the one hundred eleven numbers in Table 1.5 of Barlow et al. [2], which are the freezing dates of Lake Mendota. These are used for demonstrating the Pool Adjacent Violators Algorithm (PAVA) tacitly assuming that there is an increasing trend. After applying the PAVA to the data, there are found only seven different values. The question is whether this supports the assumption of increasing trend or not. The upper tail probability is $\Pr(L \geq 7) = 0.249984$, as shown in Table 3. Therefore the assumption of increasing trend cannot be supported.

TABLE 3. Distribution of L for $k=111$: $\Pr(L \geq l)$.

l	$\Pr(L \geq l)$	l	$\Pr(L \geq l)$
1	1.	11	.007258
2	.990991	12	.002205
3	.943403	13	.000601
4	.825087	14	.000148
5	.639103	15	.000033
6	.430025	16	.000007
7	.249984	17	.000001
8	.125872	18	.000000
9	.055238	19	.000000
10	.021286	20	.000000

§ 4. A study on power by simulation

The power of GCMF test is studied for the following two types of alternative hypotheses. Let $\{X_{ij}\}$ ($i=1, \dots, k$) be independently normally distributed with means $\{\theta_i\}$ ($i=1, \dots, k$) and variances all equal to one. The null hypothesis is $H_0: \theta_1 = \dots = \theta_k$ and the first type the alternative hypothesis is that of linear trend, i.e. $K: \theta_i - \theta_{i-1} = \Delta (>0)$ $i=2, \dots, k$, and the second type is that the first l means are same and the remaining $k-l$ means are slipped to the right by the same amount $\Delta (>0)$, i.e. $K_l: \theta_1 = \dots = \theta_l = \theta_{l+1} - \Delta = \dots = \theta_k - \Delta$. We studied two cases: $k=20$ and $k=100$, and the second type of alternatives are for $l=2(2)18$ in case $k=20$ and $l=10(10)90$ in case $k=100$. The value of Δ is determined in such a way that the test based on $\tilde{\chi}^2$ statistic: $\sum_{i=1}^k (x_i - \bar{x})^2$ would have the power equal to 0.5. The 20 or 100 normal random numbers were generated 10,000 times in each one of the null and alternative hypothesis. For instance, in case $k=20$, Table 1

indicates that $\Pr(L \geq 7) = 0.028012$ and $\Pr(L = 6) = 0.066353$, and a randomized test which reject H with probability 1 when $L \geq 7$ and with probability $(0.05 - 0.028010)/0.066353 = 0.33138$ when $L = 6$ will have the exact size 0.05. And the outcome of our simulation under H is that the number of cases when $L \geq 7$ is 269 and $L = 6$ is 643. We computed the expected number of H being rejected by the randomized test, $269 + 643 \times 0.33138 = 482.08$, and this is the estimate of the probability of rejecting H . In this way we estimated the power of GCMF test.

For the sake of comparison, we performed simulation with another non-parametric test: the rank test of Wilcoxon type for the problem of linear trend. The test statistic is $\sum iR_i$, where R_i is the rank of the i -th observation [6].

The result of simulations are tabulated in Table 4. The power of the test seems not unsatisfactory, especially for testing linear trend. For applications the algorithm stated in [5] may be utilized.

TABLE 4.

Estimate of power of GCMF test and Wilcoxon type test for $k=100$

hypothesis	H	K	K_{10}	K_{20}	K_{30}	K_{40}
GCMF test	.0490	.807	.332	.382	.399	.410
Wilcoxon type test	.0497	.999	.683	.938	.984	.993
hypothesis	K_{50}	K_{60}	K_{70}	K_{80}	K_{90}	
GCMF test	.412	.406	.401	.386	.341	
Wilcoxon type test	.995	.993	.985	.940	.680	

Estimate of power of GCMF test and Wilcoxon type test for $k=20$

hypothesis	H	K	K_2	K_4	K_6	K_8
GCMF test	.0482	.612	.202	.272	.305	.313
Wilcoxon type test	.0528	.941	.254	.568	.769	.841
hypothesis	K_{10}	K_{12}	K_{14}	K_{16}	K_{18}	
GCMF test	.321	.319	.303	.270	.203	
Wilcoxon type test	.859	.841	.767	.567	.260	

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