ON THE GROUP OF SELF-HOMOTOPY EQUIVALENCES OF H -SPACES OF LOW RANK, I

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(Received February 9, 1981)

Introduction

The set $\mathcal{E}(X)$ of homotopy classes of homotopy equivalences of a based space X forms a group under composition of maps. Several methods have been given (e.g. [1], [5], [8], [11]) for calculating the group $\mathcal{E}(X)$ at least up to extension. They are effective when X is equipped with pertinent structure, for example, X is a simply connected finite CW complex with few cells, has a low stage Postnikov system, is an H -space of low rank, etc. In case X is a simply connected, rank 2 H -space, Mimura and Sawashita [8] have recently completed the calculation of $\mathcal{E}(X)$, after the works of Oka-Sawashita-Sugawara [11], Sawashita [8] and Rutter [13].

In this paper, we shall give a method for calculating certain subgroup $\mathcal{E}_0(X)_k$ of $\mathcal{E}(X)$ for a simply connected, homotopy associative, finite H -space X under suitable condition, and study the group $\mathcal{E}(X)$ for several rank 3 H -spaces by combining the known method with ours.

For a simply connected, finite CW complex X , the group $\mathcal{E}(X)$ contains an important subgroup $\mathcal{E}_0(X)$ of all elements acting trivially on homology groups. $\mathcal{E}_0(X)$ is known to be nilpotent [17], [5], and the difficulty in calculating $\mathcal{E}(X)$ lies in doing $\mathcal{E}_0(X)$ rather than the factor group $\mathcal{E}(X)/\mathcal{E}_0(X)$. We shall filter $\mathcal{E}_0(X)$ by the subgroups $\mathcal{E}_0(X)_k$, $1 < k \leq \dim X$, of all elements whose restrictions to the k -skeleton of X are the identity (for the precise definition, see Definition 1.3). When X is moreover a (homotopy associative) H -space, we shall give in §1 a method for calculating $\mathcal{E}_0(X)_k$ for $\dim X - k$ small; in our case, $\mathcal{E}_0(X)_k$ to be consequently a nilpotent group of class at most 2 (Theorem 1.16).

In §2, we shall, in particular, study $\mathcal{E}_0(X)_k$ for the Lie groups $SU(n)$ and

This research was partially supported by Grant-in-Aid for Scientific Research (No. 434003), Ministry of Education.

$Sp(n)$, and conclude that $\mathcal{E}_0(SU(n))_k$, $k \geq \dim SU(n) - 8$, and $\mathcal{E}_0(Sp(n))_k$, $k \geq \dim Sp(n) - 10$, are nilpotent groups of class at most 2 (Theorems 2.4, 2.9).

If we know the group $\mathcal{E}(X^m)$ of the m -skeleton X^m of X , which may be calculated by the usual method when m is suitably small, for m up to the maximal dimension of generators of the cohomology ring $H^*(X)$ and up to $\mathcal{E}_0(X)_m$ to be known by the method in §1, then the group $\mathcal{E}(X)$ can be calculated from $\mathcal{E}(X^m)$ and $\mathcal{E}_0(X)_m$. We shall find such an m for several H -spaces of rank 3, namely, $SU(4)$, $SU(3) \times S^7$, the H -spaces G_2 , G_3 of type (3, 5, 7) discovered by Curtis and Mislin [3], [4], and $Sp(3)$. We shall determine in §3 and §5 the groups $\mathcal{E}(X)$, $\mathcal{E}_0(X)$ for $X = SU(4)$, $SU(3) \times S^7$, G_2 , G_3 , up to group extension (Theorem 3.9), and we shall describe in §4 the group structures of $\mathcal{E}(Sp(3))$ and of $\mathcal{E}_0(Sp(3))$, up to extension, by means of some homotopy groups concerning $Sp(3)$ (Theorem 4.3). In particular, we shall see that $\mathcal{E}_0(SU(4))$ and $\mathcal{E}_0(Sp(3))$ are nilpotent groups of class at most 2 (Corollary 3.11, Theorem 4.3, (iii)).

To know the precise group structure of $\mathcal{E}_0(X)_k$ by our method, we have to determine the homomorphism ξ defined in §1. In a forth-coming paper of the same title, we shall do it for $X = SU(4)$ to obtain complete result on $\mathcal{E}_0(SU(4))$.

All spaces we treat in this paper are simply connected CW complexes with base points. Maps and homotopies are also understood to preserve base points. We will often use the following notations. Let X, Y, X', Y' be spaces and $f: X \rightarrow Y, f': X' \rightarrow Y'$ be maps. Then, $[X, Y]$ denotes the set of homotopy classes of maps from X to Y . Sometimes the same symbol is used for a map and its homotopy class. $1 = 1_X$ denotes the identity map (and its homotopy class) of X . $X \vee Y$ denotes the wedge of X and Y , f and f' define a map $X \vee X' \rightarrow Y \vee Y'$ in an obvious way, which we denote by $f \vee f'$. $c: X \vee X \rightarrow X$ denotes the folding map, Σ^n the n -times iterated suspension functor, $l: \Sigma X \rightarrow \Sigma X \vee \Sigma X$ the comultiplication, CX the cone over X , $Y \cup_f CX$ or briefly $C(f)$ the mapping cone (cofiber) of f . Collapsing $X \times \left\{ \frac{1}{2} \right\} (\subset CX)$ in $C(f)$ to the base point defines a map $C(f) \rightarrow C(f) \vee \Sigma X$, called the coaction map, which we denote by l or l with subscript or superscript such as $l_1, l', \bar{l}$, etc. An additive, cyclic group generated by an element α of order n (resp. infinity) will be denoted by $\mathbf{Z}/n \cdot \alpha$ (resp. $\mathbf{Z} \cdot \alpha$).

§1. Subgroups of group of self-homotopy equivalences of an H -space

We shall begin by defining the groups $\mathcal{E}(X)$, $\mathcal{E}_0(X)$, $\mathcal{E}_0(X)_k$. Let X be a

simply connected finite CW complex.

Definition 1.1. $\mathcal{E}(X) = \{\alpha \in [X, X] \mid \alpha \text{ is a homotopy equivalence}\}.$

Definition 1.2. $\mathcal{E}_0(X) = \{\alpha \in \mathcal{E}(X) \mid \alpha_* = \text{id}: H_*(X) \rightarrow H_*(X)\}.$

$\mathcal{E}(X)$ becomes a group under the composition of maps, and $\mathcal{E}_0(X)$ is clearly a normal subgroup of $\mathcal{E}(X)$. The factor group $\mathcal{E}(X)/\mathcal{E}_0(X)$ is a subgroup of $\prod_{i>0} \text{Aut} H_i(X)$. It is known [17, §9], [5, Th. 3.3] that $\mathcal{E}_0(X)$ is a nilpotent group.

Definition 1.3. Let X^k be the k -skeleton of X , and $i: X^k \rightarrow X$ the inclusion. Then we put

$$\mathcal{E}_0(X)_k = \{\alpha \in \mathcal{E}_0(X) \mid \alpha i = i\}.$$

By the cellular approximation theorem, this is also a normal subgroup of $\mathcal{E}(X)$ (and of $\mathcal{E}_0(X)$). Clearly $\mathcal{E}_0(X)_{k+1} \subset \mathcal{E}_0(X)_k$, $\mathcal{E}_0(X)_{\dim X} = \{1\}$ and $\mathcal{E}_0(X)_l = \mathcal{E}_0(X)$ if X is l -connected.

Henceforth, let us consider the case that the space X is 1-connected finite H -complex of dimension n and of rank ≥ 2 . Then X has no cells of dimension 1, 2, $n-2$ or $n-1$, and we have

$$X = Y \cup_f e^n \text{ with } Y = X^{n-3}, \text{ the } (n-3)\text{-skeleton of } X.$$

There is a cofibration

$$(1.1) \quad Y \xrightarrow{i} X \xrightarrow{\pi} S^n.$$

Since $\text{rank } X \geq 2$, $\pi_n(X)$ is a finite group, and $\pi_*: \pi_n(X) \rightarrow \pi_n(X, Y) \cong \pi_n(S^n)$ is zero. Therefore

$$(1.2) \quad i_*: \pi_n(Y) \longrightarrow \pi_n(X) \text{ is onto.}$$

The set $[A, X]$ has an operation

$$+ : [A, X] \times [A, X] \longrightarrow [A, X]$$

defined by $\alpha + \beta = \mu(\alpha \times \beta)d$, $d: A \rightarrow A \times A$ the diagonal, $\mu: X \times X \rightarrow X$ the multiplication on X . Notice that, for any map $f: B \rightarrow A$, $f^*: [A, X] \rightarrow [B, X]$ is a homomorphism:

$$(1.3) \quad f^*(\alpha + \beta) = f^*\alpha + f^*\beta, \quad \alpha, \beta \in [A, X].$$

Define the map

$$l: X \longrightarrow X \vee S^n$$

by collapsing $S^{n-1} \times \left\{ \frac{1}{2} \right\}$ in $X = Y \cup CS^{n-1}$ to the base point. We call this map the *coaction map*.

Definition 1.4. We define

$$\lambda: \pi_n(X) \longrightarrow \mathcal{E}_0(X) (= [X, X])$$

in the following two ways:

$$\lambda(\alpha) = 1 + \alpha\pi, \quad \lambda(\alpha) = c(1 \vee \alpha)l,$$

where $c: X \vee X \rightarrow X$ is the folding map.

These definitions coincide, because $jl = (1 \times \pi)d$ and $\mu j = c$, where $j: X \vee X \rightarrow X \times X$ is the inclusion. Moreover λ is well-defined, i.e., $\text{Im } \lambda \subset \mathcal{E}_0(X)$. For, any element α in $\pi_n(X)$ induces the zero homomorphism on homology groups, by (1.2).

Let Δ be the subgroup of $\pi_n(X)$:

$$(1.4) \quad \Delta = \text{Im } (\Sigma f)^*: [\Sigma Y, X] \longrightarrow \pi_n(X),$$

where Σ denotes the suspension functor.

Lemma 1.5. *The map λ is a homomorphism, i. e.,*

$$\lambda(\alpha)\lambda(\alpha') = \lambda(\alpha + \alpha'), \quad \alpha, \alpha' \in \pi_n(X),$$

and $\text{Ker } \lambda = \Delta$, $\text{Im } \lambda = \mathcal{E}_0(X)_{n-3}$.

Proof. By the standard method such as [8, Cor. 2.2], [11, Lemmas 1.4, 1.7, Th. 2.12], [13, Th. 2], (1.2) implies that λ is a homomorphism and $\text{Ker } \lambda = \Delta$. $\text{Im } \lambda = \mathcal{E}_0(X)_{n-3}$ is clear.

Corollary 1.6. $\lambda: \pi_n(X)/\Delta \rightarrow \mathcal{E}_0(X)_{n-3}$ is an isomorphism. In particular, $\mathcal{E}_0(X)_{n-3}$ is abelian.

For $\alpha \in [X, X]$, there is a unique element $\beta \in [Y, Y]$ with $\alpha i = i\beta$. Letting $R(\alpha) = \beta$, we obtain homomorphisms

$$\begin{aligned}
 (1.5) \quad & R: \mathcal{E}(X) \longrightarrow \mathcal{E}(Y), \\
 & R: \mathcal{E}_0(X) \longrightarrow \mathcal{E}_0(Y), \\
 & R: \mathcal{E}_0(X)_k \longrightarrow \mathcal{E}_0(Y)_k,
 \end{aligned}$$

which we call the *restriction homomorphisms*. Then a similar discussion as in Lemma 1.5 shows the following

Theorem 1.7. *There are exact sequences*

$$\begin{aligned}
 0 &\longrightarrow \pi_n(X)/\Delta \xrightarrow{\lambda} \mathcal{E}_0(X) \xrightarrow{R} \mathcal{E}_0(Y), \\
 0 &\longrightarrow \pi_n(X)/\Delta \xrightarrow{\lambda} \mathcal{E}_0(X)_k \xrightarrow{R} \mathcal{E}_0(Y)_k.
 \end{aligned}$$

Now, we shall restrict ourself to the X satisfy the condition that there is an integer m with $3 \leq m \leq n-4$ ^(*) which satisfies the following three assumptions:

Assumption (A). *There is a complex K together with a map $g: K \rightarrow Z$, where $Z = X^m$, such that Y is a mapping cone of g (hence $\dim K = n-4$ and the connectivity l of K is greater than or equal to $m-1$), and that $g_* = 0: H_m(K) \rightarrow H_m(Z)$ if $l = m-1$.*

Assumption (B). $\mathcal{E}_0(\Sigma K) = \{1\}$.

Assumption (C). $\pi'_* f = 0$ in $\pi_{n-1}(\Sigma K)$.

Here we denote the cofibration for $Y = Z \cup_g CK$ by

$$(1.1)' \quad Z \xrightarrow{i'} Y \xrightarrow{\pi'} \Sigma K.$$

Since the attaching map f of the top cell is stably nullhomotopic [2], we have

Proposition 1.8. *If $n \leq 2l+3$ (in particular, $n \leq 2m+1$), then (C) holds.*

Let Γ and Δ' be the following subgroups of $[\Sigma K, Y]$:

$$\begin{aligned}
 (1.4)' \quad & \Gamma = (\pi'_*)^{-1}(0), \pi'_*: [\Sigma K, Y] \longrightarrow [\Sigma K, \Sigma K], \\
 & \Delta' = \text{Im}(\Sigma g)^*: [\Sigma Z, Y] \longrightarrow [\Sigma K, Y].
 \end{aligned}$$

Definition 1.9. We define a map

^(*) Hence $n = \dim X \geq 7$. Since X is 1-connected and $\text{rank } X \geq 2$, such an inequality holds except only $X = S^3 \times S^3$, for which, however, the group $\mathcal{E}(X)$ is known by Sawashita [14].

$$\lambda' : \Gamma \longrightarrow \mathcal{E}_0(Y)$$

by $\lambda'(\beta) = c(1 \vee \beta)l' \in [Y, Y]$, where $l' : Y \rightarrow Y \vee \Sigma K$ is the coaction map similarly defined as l .

For any $\beta \in \Gamma$, $\beta_* = 0$ on homology, by (A). Hence $\text{Im } \lambda' \subset \mathcal{E}_0(Y)$. Moreover λ' is a restriction of a map

$$\lambda' : [\Sigma K, Y] \longrightarrow [Y, Y]$$

similarly defined, which is equivalent to a map

$$(1.6) \quad \lambda'_0 : [\Sigma K, X] \longrightarrow [Y, X], \quad h \longmapsto i + h\pi' = c(i \vee h)l',$$

through the bijections $i_* : [\Sigma K, Y] \rightarrow [\Sigma K, X]$ and $i_* : [Y, Y] \rightarrow [Y, X]$. The former i_* is an isomorphism of groups, and hence $[\Sigma K, Y]$ is abelian. Since $(\Sigma g)_* = 0$ on homology by (A), we have $\Delta' \subset \Gamma$ by (C), so we can form a factor group Γ/Δ' .

We consider the kernel of λ'_0 in (1.6), following Rutter [12]. For a given space A , let X^A be the space of based maps $A \rightarrow X$ with compact-open topology. The operation $+$ defined after (1.3) gives rise to an H -structure on X^A and the map $f^* : X^B \rightarrow X^A$; $h \rightarrow hf$, for a map $f : A \rightarrow B$, is an H -map by the same reason as (1.3). Since X is a finite CW -complex, X has a left-inversion $\sigma : X \rightarrow X$, i.e., $\sigma + 1$ is connected with the constant map 0 by a path in X^X . For $h \in X^A$, put $-h = \sigma h$ so that $-h + h$ is connected with 0 by a path in X^A . Let $G : X^Z \rightarrow X^K$ be a map given by $G(h) = -ii'g + (ii'g + hg)$, where Z, K and g are as above. Put $0' = G(0)$, which is connected with 0. Then, by Rutter [12, Cor. 3.3.2], $(\lambda'_0)^{-1}(i)$ is the image of the composite

$$[\Sigma Z, X] \cong \pi_1(X^Z, 0) \xrightarrow{G_*} \pi_1(X^K, 0') \cong \pi_1(X^K, 0) \cong [\Sigma K, X].$$

In our case, the above image coincides with $\text{Im } (\Sigma g)^* = i_* \Delta'$, because $G(h)$ is connected with $hg = g^*(h)$ by a path in X^K , for $h \in X^Z$. Therefore

$$(1.7) \quad (\lambda')^{-1}(0) = \Delta', \quad (\lambda'_0)^{-1}(i) = i_* \Delta'.$$

Proposition 1.10. *If either X is homotopy associative or $(i')_* : [\Sigma K, Z] \rightarrow [\Sigma K, Y]$ is onto (in this case, $\Gamma = [\Sigma K, Y]$), then λ' is a homomorphism, i.e.,*

$$\lambda'(\beta)\lambda'(\beta') = \lambda'(\beta + \beta'), \quad \beta, \beta' \in \Gamma,$$

$\text{Ker } \lambda' = \Delta'$ and $\text{Im } \lambda' = \mathcal{E}_0(Y)_m$. Hence λ' induces an isomorphism

$$\lambda' : \Gamma / \Delta' \longrightarrow \mathcal{E}_0(Y)_m.$$

Proof. By the definition of λ' , we have

$$(1.8) \quad \pi' \lambda'(\beta) = \pi' \quad \text{for } \beta \in \Gamma.$$

If X is homotopy associative, we have from (1.3) and (1.8)

$$\begin{aligned} i_* \lambda'(\beta) \lambda'(\beta') &= \lambda'_0(i\beta) \lambda'(\beta') \\ &= (i + i\beta\pi') \lambda'(\beta') \\ &= i \lambda'(\beta') + i\beta\pi' \\ &= i + i\beta'\pi' + i\beta\pi' \\ &= i + i(\beta' + \beta)\pi' \\ &= i_* \lambda'(\beta' + \beta) = i_* \lambda'(\beta + \beta'). \end{aligned}$$

Therefore λ' is a homomorphism. If $(i')_*$ is onto, λ' is also a homomorphism by [11, Lemma 1.4]. $\text{Ker } \lambda' = \Delta'$ follows immediately from (1.7). The map l' in Definition 1.9 satisfies $l'i' = i_1 i'$, $i_1 : Y \rightarrow Y \vee \Sigma K$ the first injection. Then $\text{Im } \lambda' \subset \mathcal{E}_0(Y)_m$. The converse $\mathcal{E}_0(Y)_m \subset \text{Im } \lambda'$ follows from [12, Cor. 3.3.2] or the following fact:

(1.9) Let $v : [Y, Y] \times [\Sigma K, Y] \rightarrow [Y, Y]$ be the action of the group $[\Sigma K, Y]$ on $[Y, Y]$ which is defined by $v(h, \beta) = c(h \vee \beta)l'$. Then, for $h, h' \in [Y, Y]$, $hi' = h'i'$ if and only if there is $\beta \in [\Sigma K, Y]$ with $h' = v(h, \beta)$.

Remark 1.11. Without the assumption in the proposition, λ' is a homomorphism if we change the group structure on $[\Sigma K, Y]$, [11, § 1], $\text{Ker } \lambda'$ is more complicated, but $\text{Im } \lambda' = \mathcal{E}_0(Y)_m$ still holds.

Next we shall study the group $\mathcal{E}_0(X)_m$ under the assumptions (A), (B) and (C). By (C), X/Z is a wedge $\Sigma K \vee S^n$. Let

$$\begin{aligned} \bar{\pi} : X &\longrightarrow \Sigma K \vee S^n, \\ \pi_1 = p_1 \bar{\pi} : X &\longrightarrow \Sigma K \end{aligned}$$

be the collapsing maps, then

$$(1.10) \quad \begin{aligned} \pi &= p_2 \bar{\pi} : X \longrightarrow S^n \\ \pi' &= \pi_1 i : Y \longrightarrow \Sigma K, \end{aligned}$$

where p_i is the i -th projection. By (C), the coextension construction [16, p. 13] gives us a map

$$\tilde{f}: S^n \longrightarrow \Sigma Z \text{ with } (\Sigma i')\tilde{f} = \Sigma f,$$

and there is a sequence of cofibrations:

$$(1.11) \quad Z \xrightarrow{ii'} X \xrightarrow{\bar{\pi}} \Sigma K \vee S^n \xrightarrow{c(\Sigma g \vee \tilde{f})} \Sigma Z \xrightarrow{\Sigma(ii')} \Sigma X.$$

Definition 1.12. Define a map

$$\mu: \Gamma \longrightarrow \mathcal{E}_0(X)$$

by $\mu(\beta) = 1 + i\beta\pi_1 \in [X, X]$.

By (1.10) $\mu(\beta)i = i + i\beta\pi' = \lambda'_0(i\beta) = i\lambda'(\beta)$. This implies

Lemma 1.13. $R\mu = \lambda'$, $\text{Im } \mu = \mathcal{E}_0(X)_m$ and $R: \mathcal{E}_0(X)_m \rightarrow \mathcal{E}_0(Y)_m$ is epic. If either X is homotopy associative or $(i')_*: [\Sigma K, Z] \rightarrow [\Sigma K, Y]$ is onto, then μ has a factorization

$$\Gamma \longrightarrow \Gamma/\Delta' \xrightarrow{\mu} \mathcal{E}_0(X)_m \subset \mathcal{E}_0(X),$$

in which μ is one-to-one.

For any $\beta \in \Gamma/\Delta'$, which is represented by $\beta \in \Gamma$, there is a homotopy commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\mu(\beta)} & X \\ \downarrow \bar{\pi} & & \downarrow \bar{\pi} \\ \Sigma K \vee S^n & \xrightarrow{\beta'} & \Sigma K \vee S^n, \end{array}$$

for some $\beta' \in \mathcal{E}_0(\Sigma K \vee S^n)$, by the cellular approximation theorem. By (B), the map

$$\rho: \pi_n(\Sigma K) \longrightarrow \mathcal{E}_0(\Sigma K \vee S^n); \gamma \longmapsto 1 + i_1\gamma p_2,$$

is an isomorphism, where $1 = 1_{\Sigma K \vee S^n}$ and $+$ is defined by the co- H -structure on $\Sigma K \vee S^n = \Sigma(K \vee S^{n-1})$. Hence $\beta' = \rho(\gamma)$ for some $\gamma \in \pi_n(\Sigma K)$. Letting $\xi(\beta) = \gamma$, we obtain maps

$$(1.12) \quad \begin{aligned} \xi: \Gamma &\longrightarrow \pi_n(\Sigma K), \\ \xi: \Gamma/\Delta' &\longrightarrow \pi_n(\Sigma K). \end{aligned}$$

The latter ξ is defined if either X is homotopy associative or $(i')_*$ is onto. Generally, ξ is defined on $\Gamma/\text{Ker } \lambda'$ without this assumption.

Lemma 1.14. *Let K be l -connected, $l \geq m-1$.*

(i) *If $l > m-1$, then ξ is well-defined (single valued).*

(ii) *If $l = m-1$ and $n \leq 2m+1$, then*

$$(1.12)' \quad \xi: \Gamma/\Delta' \longrightarrow \pi_n(\Sigma K)/(\Sigma f)^*[\Sigma Y, \Sigma K]$$

is well-defined.

Proof. It suffices to show that $\beta' \bar{\pi} = \bar{\pi}$, $\beta' \in \mathcal{E}_0(\Sigma K \vee S^n)$, implies $\beta' = 1$ in case (i) or $\beta' \in \rho(\text{Im } (\Sigma f)^*)$ in case (ii). Consider the exact sequence of based sets:

$$[\Sigma Z, \Sigma K \vee S^n] \xrightarrow{h^*} [\Sigma K \vee S^n, \Sigma K \vee S^n] \xrightarrow{\bar{\pi}^*} [X, \Sigma K \vee S^n]$$

defined from (1.11), where $h = c(\Sigma g \vee \tilde{f})$ and the group $[\Sigma K \vee S^n, \Sigma K \vee S^n]$ acts on the set $[X, \Sigma K \vee S^n]$ in a manner similar to (1.9). If $\beta' \bar{\pi} = \bar{\pi}$, i.e., $\bar{\pi}^* \beta' = \bar{\pi}^* 1$, there is an element $\beta'' \in [\Sigma Z, \Sigma K \vee S^n]$ with $\beta' = v(1, \beta'')$ by (1.9). In case (i), we have $[\Sigma Z, \Sigma K \vee S^n] = 0$ and $\beta' = v(1, 0) = 1$, proving the lemma. In case (ii), we have

$$\beta'' = i_1 \beta_1, \quad \beta_1 \in [\Sigma Z, \Sigma K].$$

We shall calculate the element $\beta' = v(1, i_1 \beta_1)$. Let

$$\tilde{l}: \Sigma K \vee S^n \longrightarrow \Sigma K \vee S^n \vee \Sigma Z$$

be the coaction map similarly defined as l and l' . Then

$$(*) \quad j\tilde{l} = (1 \times h)d$$

where $j: (\Sigma K \vee S^n) \vee \Sigma Z \rightarrow (\Sigma K \vee S^n) \times \Sigma Z$ is the inclusion, and we have $\beta' = c(1 \vee i_1 \beta_1)\tilde{l}$. Then

$$(**) \quad \begin{aligned} p_2 \beta' &= c(p_2 \vee p_2 i_1 \beta_1)\tilde{l} = p_2, \\ p_1 \beta' &= c(1 \vee \beta_1)p_{1,3}\tilde{l}, \end{aligned}$$

where $1 = 1_{\Sigma K}$ and $p_{1,3}: \Sigma K \vee S^n \vee \Sigma Z \rightarrow \Sigma K \vee \Sigma Z$ collapses S^n to the base point. The element $\beta_2 = (1 \vee \beta_1)p_{1,3}\tilde{l}$ lies in

$$[\Sigma K \vee S^n, \Sigma K \vee \Sigma K] = (p_1)^*[\Sigma K, \Sigma K \vee \Sigma K] \oplus (p_2)^* \pi_n(\Sigma K \times \Sigma K) \oplus (p_2)^* F,$$

$$F \cong \pi_{n+1}(\Sigma K \times \Sigma K, \Sigma K \vee \Sigma K),$$

and $F=0$ in case $n < 2m+1$. In case $n=2m+1$, F is generated by the universal Whitehead products $[u_1, u_2] \in \pi_n(\Sigma K \vee \Sigma K)$, $u_i \in \pi_{m+1}(\Sigma K)$, which are detected by the cup products. Since $\bar{l}$ induces an isomorphism of $H^n(-, R)$ for any coefficient ring R , $H^{n+1}(C(\bar{l}), R)=0$, where $C(-)$ denotes the mapping cone. Therefore the cup products $H^{m+1}(C(\beta_2), R) \otimes H^{m+1}(C(\beta_2), R) \rightarrow H^{n+1}(C(\beta_2), R)$ is zero. Hence the component of β_2 in $(p_2)^*F$ is trivial, even if $n=2m+1$. By (*), we have $p_1\beta' = c\beta_2 = p_1 + \beta_1 h$. Since $\beta_1 h i_1 = \beta_1(\Sigma g) \in [\Sigma K, \Sigma K]$ induces the zero homomorphism on homology by (A), it is in fact trivial, by (B). Consider the exact sequence:

$$[\Sigma Y, \Sigma K] \xrightarrow{(\Sigma i')^*} [\Sigma Z, \Sigma K] \xrightarrow{(\Sigma g)^*} [\Sigma K, \Sigma K].$$

Again by (A) and (B), $(\Sigma g)^*=0$ and $(\Sigma i')^*$ is onto. Hence $\beta_1 \tilde{f} = \beta_3(\Sigma i') \tilde{f} = \beta_3(\Sigma f)$ for some $\beta_3 \in [\Sigma Y, \Sigma K]$. Therefore $\beta_1 h i_2 = \beta_1 \tilde{f} = \beta_3(\Sigma f)$, $\beta_1 h = \beta_3(\Sigma f) p_2$ and $p_1 \beta' = p_1 + \beta_3(\Sigma f) p_2$. Together with (**), we obtain $\beta' = 1 + i_1 \beta_3(\Sigma f) p_2$, proving the lemma in case (ii).

Lemma 1.15. *Let $\alpha \in \pi_n(X)/\Delta$, $\beta \in \Gamma/\Delta'$ ($\beta \in \Gamma$ if neither X is homotopy associative nor $(i')_*: [\Sigma K, Z] \rightarrow [\Sigma K, Y]$ is onto). In case $l=m-1$, we assume $n \leq 2m+1$. Then*

- (i) $\pi\lambda(\alpha) = \pi$, $\pi\mu(\beta) = \pi$.
- (ii) $\pi_1\lambda(\alpha) = (p_1 + \pi_1\alpha p_2)\bar{\pi}$, $\pi_1\mu(\beta) = (p_1 + \xi(\beta)p_2)\bar{\pi}$, where $+$ is understood to be defined by the co- H -structure on $\Sigma K \vee S^n$.
- (iii) ξ is a homomorphism.

Proof. (i) is easy. The map $\bar{\pi}$ is homotopic to $(\pi_1 \vee 1)l: X \rightarrow X \vee S^n \rightarrow \Sigma K \vee S^n$. This implies the first half of (ii). The second half of (ii) follows from the definition of ξ . (iii) follows from (ii).

Theorem 1.16. *Let X be a simply connected homotopy associative H -complex of dimension n and of rank ≥ 2 . Let m be an integer with $3 \leq m \leq n-4$, and assume that the three conditions (A), (B) and (C) hold. Then,*

- (i) *Any element in $\mathcal{E}_0(X)_m$ is written uniquely as*

$$\lambda(\alpha)\mu(\beta), \quad \text{for } \alpha \in \pi_n(X)/\Delta, \quad \beta \in \Gamma/\Delta'.$$

- (ii) *The group structure of $\mathcal{E}_0(X)_m$ is given by the following relations:*

$$\lambda(\alpha_1)\lambda(\alpha_2) = \lambda(\alpha_1 + \alpha_2),$$

$$\mu(\beta)\lambda(\alpha) = \lambda(\alpha + i\beta\pi_1\alpha)\mu(\beta),$$

$$\mu(\beta_1)\mu(\beta_2) = \lambda(i\beta_1\xi(\beta_2))\mu(\beta_1 + \beta_2).$$

(iii) $\mathcal{E}_0(X)_m$ is a nilpotent group of class at most 2.

Proof. (i) follows from the second exact sequence in Theorem 1.7 and Lemma 1.13. The first relation in (ii) is proved in Lemma 1.5, and the others follow from Lemma 1.15. From the relations in (ii), we have

$$\begin{aligned} [\lambda(\alpha_1), \lambda(\alpha_2)] &= 1, \quad [\mu(\beta), \lambda(\alpha)] = \lambda(i\beta\pi_1\alpha), \\ [\mu(\beta_1), \mu(\beta_2)] &= \lambda(i\beta_1\xi(\beta_2) - i\beta_2\xi(\beta_1)), \end{aligned}$$

where $[x, y] = xyx^{-1}y^{-1}$, the commutator in $\mathcal{E}_0(X)_m$. Since $\pi_1 i\beta = \pi' \beta = 0$, all the triple commutators $[x, [y, z]]$ are 1, proving (iii).

Corollary 1.17. *With the same notations and assumptions as above, there holds*

$$\mu(\beta)^n = (n\beta)\lambda\left(\frac{n(n-1)}{2}\beta\xi(\beta)\right).$$

Let β be of order m , then the order of $\mu(\beta)$ is

$$\begin{array}{ll} 2m & \text{if } m \text{ is even and } (m/2)\beta\xi(\beta) \neq 0, \\ m & \text{otherwise.} \end{array}$$

Remark 1.18. If we replace the homotopy associativity of X by the condition that $(i')_*: [\Sigma K, Z] \rightarrow [\Sigma K, Y]$ is onto, the part (i) and the first relation in (ii) still hold, in the above theorem. For the second and the third relations in (ii), we need the homotopy associativity of X .

§2. Applications to $SU(n)$ and $Sp(n)$

In this section, we shall apply Theorem 1.16 to the special unitary group $SU(n)$ and the symplectic group $Sp(n)$.

We shall use the following notations of the typical elements in homotopy groups of spheres:

$$\begin{aligned} \eta &= \eta_{\mathbf{C}} \in \pi_3(S^2), \text{ the complex Hopf map,} \\ \omega &\in \pi_6(S^3), \text{ the Blackers-Massey element,} \\ \eta_{\mathbf{H}} &\in \pi_7(S^4), \text{ the quaternionic Hopf map.} \end{aligned}$$

The iterated suspensions of these elements will be denoted by the same letters.

η , ω , η_H . The 2-primary parts of ω and η_H are v' and v in Toda's book [16]. The following results are well known:

$$\begin{aligned} \pi_{n+1}(S^n) &= \begin{cases} \mathbf{Z} \cdot \eta & \text{for } n=2, \\ \mathbf{Z}/2 \cdot \eta & \text{for } n \geq 3; \end{cases} \\ \pi_{n+2}(S^n) &= \mathbf{Z}/2 \cdot \eta^2, \eta^2 = \eta\eta, & \text{for } n \geq 2; \\ \pi_{n+3}(S^n) &= \begin{cases} \mathbf{Z}/2 \cdot \eta^3 & \text{for } n=2, \\ \mathbf{Z}/12 \cdot \omega & \text{for } n=3, \\ \mathbf{Z} \cdot \eta_H \oplus \mathbf{Z}/12 \cdot \omega & \text{for } n=4, \\ \mathbf{Z}/24 \cdot \eta_H & \text{for } n \geq 5; \end{cases} \end{aligned}$$

where $6\omega = \eta^3$ for $n \geq 3$ and $2\eta_H = \omega$ for $n \geq 5$.

Here and henceforth, we shall use the notation $\mathbf{Z} \cdot \alpha$ (resp. $\mathbf{Z}/k \cdot \beta$) for a cyclic group of order infinity (resp. k) with generator α (resp. β).

We first consider $X = SU(n)$ with $n \geq 4$. Then $\dim X = n^2 - 1$ and $\text{rank } X = n - 1$. The 7-skeleton of X has a cell decomposition:

$$(2.1) \quad X^7 = (S^3 \cup_{\eta} e^5) \cup_{i\omega} e^7,$$

where $i: S^3 \rightarrow S^3 \cup e^5 = X^5$ is the inclusion. By a theorem of Browder-Spanier [2], X^{n^2-4}/X^{n^2-9} is the S -dual to X^7 , and hence

$$(2.2) \quad X/X^{n^2-9} = \{(S^{n^2-8} \vee S^{n^2-6}) \cup_{\alpha} e^{n^2-4}\} \vee S^{n^2-1},$$

$$\text{with } \alpha = (\omega \vee \eta)l = (2\eta_H \vee \eta)l,$$

where $l: \Sigma A \rightarrow \Sigma A \vee \Sigma A$ denotes the co- H -structure. We put

$$(2.3) \quad K = (S^{n^2-9} \vee S^{n^2-7}) \cup_{\alpha} e^{n^2-5},$$

so that $X/X^{n^2-9} = \Sigma K \vee S^{n^2-1}$.

Lemma 2.1. *Let $X = SU(n)$, $n \geq 4$, and K as above. Then there is a map $g: K \rightarrow X^{n^2-9}$ such that the cofiber (mapping cone) of g is homotopy equivalent to X^{n^2-4} .*

Proof. Put $n' = n^2 - 9$ for the simplicity ($\dim X = n' + 8$). Let $\gamma \in \pi_{n'+1}(X^{n'+1}, X^{n'})$ and $\delta \in \pi_{n'+3}(X^{n'+3}, X^{n'+1})$ be the elements such that $\partial\gamma \in \pi_{n'}(X^{n'})$ and $\partial\delta \in \pi_{n'+2}(X^{n'+1})$ are the attaching maps of the top cells in $X^{n'+1} =$

$X^{n'} \cup e^{n'+1}$ and $X^{n'+3} = X^{n'+1} \cup e^{n'+3}$ respectively. By (2.2) and by the exact sequence

$$\pi_{n'+2}(X^{n'}) \longrightarrow \pi_{n'+2}(X^{n'+1}) \longrightarrow \pi_{n'+2}(X^{n'+1}/X^{n'}),$$

the element $\partial\delta$ is in the image from $\pi_{n'+2}(X^{n'})$, i.e., there is $\delta_1 \in \pi_{n'+3}(X^{n'+3}, X^{n'})$ with $j^*\delta_1 = \delta$, j an inclusion. Using the homotopy exact sequence for the triple $(X^{n'+3}, X^{n'+1}, X^{n'})$ and Blakers-Massey theorem, we see that

$$\begin{aligned}\pi_{n'+1}(X^{n'+3}, X^{n'}) &= \mathbf{Z} \cdot j_*\gamma, \\ \pi_{n'+2}(X^{n'+3}, X^{n'}) &= \mathbf{Z}/2 \cdot j_*\gamma\eta, \\ \pi_{n'+3}(X^{n'+3}, X^{n'}) &= \mathbf{Z}/2 \cdot j_*\gamma\eta^2 \oplus \mathbf{Z} \cdot \delta_1 \oplus \text{Ker } p_*,\end{aligned}$$

where $j: (X^{n'+1}, X^{n'}) \rightarrow (X^{n'+3}, X^{n'})$ is the inclusion and $p: (X^{n'+3}, X^{n'}) \rightarrow (X^{n'+3}/X^{n'}, *) = (S^{n'+1} \vee S^{n'+3}, *)$ is the collapsing map.

Let F be the fiber of the inclusion $X^{n'} \rightarrow X^{n'+3}$. Since X has no torsion, $H^i(X) \rightarrow H^i(X^k)$ is isomorphic for $i \leq k$. Then the spectral sequence argument leads us that

$$H^i(F) = \begin{cases} \mathbf{Z} & \text{for } i = n', \\ \mathbf{Z} \oplus \mathbf{Z} & \text{for } i = n' + 2, \\ \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} & \text{for } i = n' + 4, \\ 0 & \text{for } 0 < i < n', i = n' + 1 \text{ and for } i = n' + 3. \end{cases}$$

Take a complex $L = S^{n'} \cup e^{n'+2} \cup e^{n'+3}$ and a map $f: L \rightarrow F$ such that $f_*: \pi_i(L) \rightarrow \pi_i(F)$ is isomorphic for $i \leq n' + 2$ and epic for $i = n' + 3$. We have $\pi_{n'+1}(L) \cong \pi_{n'+1}(F) \cong \pi_{n'+2}(X^{n'+3}, X^{n'}) \cong \mathbf{Z}/2$. Hence L must be of the form

$$L = S^{n'} \vee S^{n'+2} \vee S^{n'+3}.$$

Let $i_1: S^{n'} \rightarrow L$, $i_k: S^{n'+2} \rightarrow L$ ($k=2, 3$) be the inclusion to the k -th factor. Through the isomorphism $\tau: \pi_i(L) \cong \pi_{i+1}(X^{n'+3}, X^{n'})$, $i \leq n' + 2$, $j_*\gamma$ and δ_1 correspond with i_1 and i_2 , respectively, and $\text{Ker } p_*$ ($\subset \pi_{n'+3}(X^{n'+3}, X^{n'})$) is $\mathbf{Z}$, generated by an element δ_2 which corresponds with i_3 . Then we have

$$\begin{aligned}\pi_{n'+3}(X^{n'+3}, X^{n'}) &= \mathbf{Z}/2 \cdot j_*\gamma\eta^2 \oplus \mathbf{Z} \cdot \delta_1 \oplus \mathbf{Z} \cdot \delta_2, \\ \pi_{n'+4}(X^{n'+3}, X^{n'}) &= \mathbf{Z}/24 \cdot j_*\gamma\eta\eta_H \oplus \mathbf{Z}/2 \cdot \delta_1\eta \oplus (\mathbf{Z}/2 \text{ or } 0) \cdot \delta_2\eta,\end{aligned}$$

because the map $\tau: \pi_{n'+3}(L) = \mathbf{Z}/24 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \rightarrow \pi_{n'+4}(X^{n'+3}, X^{n'})$ is epic and $p_*\tau|_{\mathbf{Z}/24 \oplus \mathbf{Z}/2 \oplus 0}$ is isomorphic.

Let $\varepsilon \in \pi_{n'+4}(X^{n'+3})$ be the attaching map of the top cell in $X^{n'+5} = X^{n'+3} \cup e^{n'+5}$, and consider the element $k_*\varepsilon \in \pi_{n'+4}(X^{n'+3}, X^{n'})$, where $k: (X^{n'+3}, *) \rightarrow (X^{n'+3}, X^{n'})$ is the inclusion. By (2.2), $p_*k_*\varepsilon = \alpha = (2\eta_H \vee \eta)l$. Therefore

$$k_*\varepsilon \equiv 2j_*\gamma\eta_H + \delta_1\eta \pmod{\delta_2\eta}.$$

Let $\pi_K: K \rightarrow S^{n'+4}$ be the map collapsing $K^{n'+2}$ to the base point. Consider the following exact sequences of $P = (X^{n'+3}, X^{n'})$:

$$\begin{aligned} \pi_{n'+1}(P) \oplus \pi_{n'+3}(P) &\xrightarrow{\omega^* + \eta^*} \pi_{n'+4}(P) \xrightarrow{(\pi_K)^*} [CK', K'; P], \\ [K, X^{n'}] &\xrightarrow{i_*} [K, X^{n'+3}] \xrightarrow{k_*} [CK', K'; P], \end{aligned}$$

where $K' = \Sigma^{-1}K$ and $i: X^{n'} \rightarrow X^{n'+3}$ is the inclusion. The image of $\omega^* + \eta^*$ is generated by $j_*\gamma\omega = 2j_*\gamma\eta_H$, $\delta_1\eta$ and $\delta_2\eta$, and hence, contains $k_*\varepsilon$. Therefore $(\pi_K)_*\varepsilon \in \text{Ker } k_*$, and

$$\varepsilon\pi_K = ig \quad \text{for some } g \in [K, X^{n'}].$$

Then, there is a map $G: C(g) \rightarrow X^{n'+5} = X^{n^2-4}$ with commutative diagram of cofibrations:

$$\begin{array}{ccccc} K & \xrightarrow{g} & X^{n'} & \longrightarrow & C(g) \\ \downarrow \pi_K & & \downarrow i & & \downarrow G \\ S^{n'+4} & \xrightarrow{\varepsilon} & X^{n'+3} & \longrightarrow & X^{n'+5} \\ \downarrow \text{collapse} & & \downarrow \text{collapse} & & \\ S^{n'+1} \vee S^{n'+3} & \xrightarrow{\text{id}} & X^{n'+3}/X^{n'} & & \end{array}$$

Consider the diagram of homology groups induced by the above. Then, by the diagram chasing method, the map G induces isomorphisms of homology groups. Thus G is a homotopy equivalence.

Lemma 2.2. $\mathcal{E}_0(\Sigma K) = \{1\}$.

Proof. $[\Sigma K, \Sigma K]$ is in the stable range, and easily computed. In consequence, $[\Sigma K, \Sigma K] = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ and the map $[\Sigma K, \Sigma K] \rightarrow \Sigma_{i>0} \text{Hom}(H_i(\Sigma K), H_i(\Sigma K))$ is monic.

Lemma 2.3. Let Γ be the subgroup of $[\Sigma K, X^{n^2-4}]$ in (1.4)'. Then $\Gamma = [\Sigma K, X^{n^2-4}] \cong [\Sigma K, X]$.

Proof. $(\pi')_* = 0: [\Sigma K, X^{n^2-4}] \rightarrow [\Sigma K, \Sigma K]$, because $[\Sigma K, X]$ is finite and $[\Sigma K, \Sigma K]$ is free.

For $X = SU(n)$, $m = n^2 - 9$ and K as above, the condition (A) holds because $H_*(X)$ is free, (B) does by Lemma 2.2, and (C) also does by Proposition 1.8. Therefore all the conclusions of Theorem 1.16 hold. Let $f: S^{n^2-2} \rightarrow X^{n^2-4}$ be the attaching map of the top cell in X . Then we have obtained

Theorem 2.4. *Let $n \geq 4$, K be the complex (2.3). Then any element in $\mathcal{E}_0(SU(n))_{n^2-9}$ has a unique expression*

$$\lambda(\alpha)\mu(\beta) \quad \text{with } \alpha \in \pi_{n^2-1}(SU(n))/\Delta, \beta \in [\Sigma K, SU(n)]/\Delta',$$

where

$$\Delta = \text{Im}(\Sigma f)^*: [\Sigma(SU(n)^{n^2-4}), SU(n)] \longrightarrow \pi_{n^2-1}(SU(n)),$$

$$\Delta' = \text{Im}(\Sigma g)^*: [\Sigma(SU(n)^{n^2-9}), SU(n)] \longrightarrow [\Sigma K, SU(n)].$$

The group structure of $\mathcal{E}_0(SU(n))_{n^2-9}$ is given by the relations in Theorem 1.16, (ii), and $\text{nil } \mathcal{E}_0(SU(n))_{n^2-9} \leq 2$.

Here $\text{nil } G \leq l$ means the $(l+1)$ -th term $\Gamma^{l+1}G$ of the lower central series of G is the trivial group.

Remark 2.5. For $n=4$, $SU(4)^7$ has a cell structure (2.1) and $\mathcal{E}_0(SU(4)^7)$ is easily computed by the standard method. Together with the result on $\mathcal{E}_0(SU(4))_7$ above, we shall conclude in §3 that $\mathcal{E}_0(SU(4)) = \mathcal{E}_0(SU(4))_7$. Moreover we shall determine the homomorphism ξ in (1.12)' and Lemma 1.15, (iii), to obtain the precise group structure of $\mathcal{E}_0(SU(4))$. Details will be seen in the second part of the paper. For $n=5$, if we know the structure of $\mathcal{E}_0(SU(5)^{16})$, then the structure of $\mathcal{E}_0(SU(5))$ can be describe.

Next we shall consider $\bar{X} = Sp(n)$ with $n \geq 3$. Put $N = 2n^2 + n$. Then $\dim \bar{X} = N$, $\text{rank } \bar{X} = n$, and $\bar{X} = \bar{X}^{N-10} \cup e^{N-7} \cup e^{N-3} \cup e^N$. We put

$$\bar{K} = S^{N-8} \cup_{2\eta_H} e^{N-4}.$$

Then the following lemmas and theorem are proved in a similar way.

Lemma 2.6. *There is a map $\bar{g}: \bar{K} \rightarrow \bar{X}^{N-10}$ such that $C(\bar{g}) = \bar{X}^{N-3}$.*

Lemma 2.7. $\mathcal{E}_0(\Sigma \bar{K}) = 1$.

Lemma 2.8. $\Gamma = [\Sigma \bar{K}, \bar{X}^{N-3}] \cong [\Sigma \bar{K}, X]$.

Theorem 2.9. Let $n \geq 3$. Any element in $\mathcal{E}_0(Sp(n))_{2n^2+n-10}$ is written uniquely as

$$\lambda(\alpha)\mu(\beta), \alpha \in \pi_{2n^2+n}(Sp(n))/\Delta, \beta \in [\Sigma \bar{K}, Sp(n)]/\Delta',$$

where

$$\Delta = \text{Im}(\Sigma \bar{f})^*: [\Sigma(Sp(n)^{2n^2+n-3}), Sp(n)] \longrightarrow \pi_{2n^2+n}(Sp(n)),$$

$$\Delta' = \text{Im}(\Sigma \bar{g})^*: [\Sigma(Sp(n)^{2n^2+n-10}), Sp(n)] \longrightarrow [\Sigma \bar{K}, Sp(n)],$$

and $\bar{f}$ is the attaching map of the top cell in $Sp(n)$. The group structure of $\mathcal{E}_0(Sp(n))_{2n^2+n-10}$ is given by the relations in Theorem 1.16, (ii), and $\text{nil } \mathcal{E}_0(Sp(n))_{2n^2+n-10} \leq 2$.

Remark 2.10. The calculation of $\mathcal{E}_0(Sp(3)^{11})$ yields $\mathcal{E}_0(Sp(3))_{11} = \mathcal{E}_0(Sp(3))$, see § 4.

§ 3. The group of self-equivalences of the principal $SU(3)$ -bundles over S^7

The principal $SU(3)$ -bundles over S^7 are classified by elements in $\pi_7(BSU(3))$, which is isomorphic to $\pi_6(SU(3)) = \mathbb{Z}/6 \cdot i\omega$, $i: S^3 \rightarrow SU(3)$ the inclusion. The generator $i\omega$ corresponds with the usual bundle

$$(3.1) \quad SU(3) \xrightarrow{i} G_1 = SU(4) \xrightarrow{p} S^7.$$

Pulling back the bundle (3.1) by the map $S^7 \rightarrow S^7$ of degree k , we obtain the bundle

$$(3.1)' \quad SU(3) \xrightarrow{i} G_k \xrightarrow{p} S^7$$

with classifying element $ki\omega$. Obviously,

$$G_0 = SU(3) \times S^7.$$

By the mixing homotopy method, Curtis and Mislin proved the following

Theorem 3.1 (Curtis-Mislin [3], [4, Th. II and § 4]). *There are just four distinct homotopy types of total spaces G_k , namely, $G_0 = SU(3) \times S^7$, $G_1 = SU(4)$, G_2 and G_3 , and they are all H -spaces.*

The cohomology with coefficient $\mathbb{Z}$ of G_k is an exterior algebra:

$$(3.2) \quad H^*(G_k) = E(x_3, x_5, x_7), \quad \deg x_i = i, \text{ with } x_i \text{ primitive,}$$

and G_k has a cell decomposition:

$$(3.3) \quad G_k = S^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}.$$

Let G_k^n be the n -skeleton of G_k with respect to (3.3). From a similar discussion as in Lemmas 2.1 and 2.2, the following two lemmas follow easily.

Lemma 3.2. (i) $G_k^5 = S^3 \cup_{\eta} e^5$, $G_k^7 = G_k^3 \cup_{k\iota\omega} e^7 \simeq S^3 \cup_{c(\eta \vee k\omega)} C(S^4 \vee S^6)$, where $i: S^3 \rightarrow G_k^5$ is the inclusion.

(ii) Let $K_k = (S^7 \vee S^9) \cup_{\kappa} e^{11}$, $\kappa = (2k\eta_H \vee \eta)l$. Then,

$$G_k^{12} = G_k^7 \cup_{g_k} CK_k$$

for some map $g_k: K_k \rightarrow G_k^7$.

Lemma 3.3. (i) $[\Sigma K_k, \Sigma K_k] = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ and the usual ring homomorphism

$$[\Sigma K_k, \Sigma K] \longrightarrow \Sigma_{i>0} \text{Hom}(H_i(\Sigma K_k), H_i(\Sigma K_k))$$

is monic.

(ii) $\mathcal{E}_0(\Sigma K_k) = \{1\}$.

The homotopy groups of G_k are computed from those of $SU(3)$, $SU(4)$ [9], by making use of the exact sequences induced by (3.1)'. By [9], $\pi_5(SU(3)) = \mathbf{Z} \cdot [2\epsilon_5]$, where the generator $[2\epsilon_5]$ satisfies $p_*[2\epsilon_5] = 2\epsilon_5$, $p: SU(3) \rightarrow S^5$ the S^3 -bundle projection. Then,

Lemma 3.4 (cf. [3, Lemma 4.2]).

$$\pi_3(G_k) = \mathbf{Z}, \pi_5(G_k) = \mathbf{Z} \cdot i[2\epsilon_5], \pi_6(G_k) = \mathbf{Z}/(6/k') \cdot i_*\omega,$$

$$\pi_n(G_k) = 0 \quad \text{for } n = 1, 2, 4.$$

Here the integer k' is the order of the classifying element $k\iota\omega$, which is given as follows:

$$(3.4) \quad k' = 1, 6, 3 \text{ or } 2 \text{ according as } k = 0, 1, 2 \text{ or } 3.$$

Since $H_*(G_k)$ has no torsion by (3.2), the condition (A) in §1 is satisfied for $X = G_k$, $Y = G_k^{12}$, $Z = G_k^7$, $K = K_k$, $g = g_k$, $m = 7$, $n = 15$. The condition (B) is also satisfied by the above lemma. Let

$$f_k: S^{14} \longrightarrow G_k^{12}$$

be the attaching map for the top cell in G_k . By Proposition 1.8, the condition

(C) is satisfied too.

As in § 1, let

$$\begin{aligned} S^{14} &\xrightarrow{f_k} G_k^{12} \xrightarrow{i} G_k \xrightarrow{\pi} S^{15}, \\ K_k &\xrightarrow{g_k} G_k^7 \xrightarrow{i'} G_k^{12} \xrightarrow{\pi'} \Sigma K_k \end{aligned}$$

be the cofibrations (see (1.1), (1.1)'), and

$$\begin{aligned} \bar{\pi}: G_k &\longrightarrow G_k/G_k^7 = \Sigma K_k \vee S^{15}, \\ (\pi = p_2 \bar{\pi}: G_k &\longrightarrow G_k/G_k^{12} = S^{15}), \\ \pi_1 = p_1 \bar{\pi}: G_k &\longrightarrow \Sigma K_k, \\ (\pi' = \pi_1 i: G_k^{12} &\longrightarrow \Sigma K_k) \end{aligned}$$

the collapsing maps (see (1.10)). Then the subgroups Δ , Γ , Δ' in (1.4), (1.4)' for our G_k are the following

$$\begin{aligned} \Delta &= \text{Im } (\Sigma f_k)^*: [\Sigma G_k^{12}, G_k] \longrightarrow \pi_{15}(G_k), \\ \Gamma &= (\pi'_*)^{-1}(0), \quad \text{where } \pi'_*: [\Sigma K_k, G_k^{12}] \longrightarrow [\Sigma K_k, \Sigma K_k], \\ \Delta' &= \text{Im } (\Sigma g_k)^*: [\Sigma G_k^{12}, G_k^{12}] \longrightarrow [\Sigma K_k, G_k^{12}]. \end{aligned}$$

Lemma 3.5. (i) $\Delta = 0$.

(ii) $\Gamma = [\Sigma K_k, G_k^{12}] \cong [\Sigma K_k, G_k]$.

(iii) $\Delta' = 0$.

Proof. (i) Let $\pi^5: G_k^{12} \rightarrow G_k^{12}/G_k^5$ be the collapsing map. Then $G_k^{12}/G_k^5 = (S^7 \vee S^8 \vee S^{10}) \cup_a e^{12}$, $a = (0 \vee 2k\eta_H \vee \eta)l$, and $\pi^5 f_k$ has a factorization

$$S^{14} \xrightarrow{u} S^7 \vee S^8 \subset G_k^{12}/G_k^5,$$

where u is the universal Whitehead product map. Since it is trivial after suspension, we have

$$(*) \quad \Sigma(\pi^5 f_k) = 0.$$

By Lemma 3.2 (i) and Lemma 3.4, we have an exact sequence:

$$\begin{array}{ccccccc} \pi_5(G_k) & \xrightarrow{\eta^*} & \pi_6(G_k) & \longrightarrow & [\Sigma G_k^5, G_k] & \longrightarrow & \pi_4(G_k) \\ \mathbf{Z} \cdot i[2\epsilon_5] & & \mathbf{Z}/(6/k') \cdot i_* \omega & & & & 0 \end{array}$$

Then $\eta_*(i[2\epsilon_5]) = i_*\{\eta, 2\epsilon_4, \eta\} = \pm 3i_*\omega$ by [9, Th. 2.1] and [16, (5.4)], where $\{-, -, -\}$ denotes the Toda bracket. Therefore

$$(**) \quad [\Sigma G_k^5, G_k] = \begin{cases} 0 & \text{for } k=1, 2, \\ \mathbf{Z}/3 & \text{for } k=0, 3. \end{cases}$$

In the diagram with exact row

$$\begin{array}{ccccc} [\Sigma(G_k^{12}/G_k^5), G_k] & \xrightarrow{(\Sigma\pi^5)^*} & [\Sigma G_k^{12}, G_k] & \longrightarrow & [\Sigma G_k^5, G_k] \\ & \searrow \Sigma(\pi^5 f_k)^* = 0 & \downarrow (\Sigma f_k)^* & & \\ & & \pi_{15}(G_k), & & \end{array}$$

the oblique arrow and the right top group are trivial for $k=1, 2$, by (*) and (**), and hence $(\Sigma f_k)^* = 0$, proving (i). For $k=0, 3$, $\text{Im}(\Sigma f_k)^*$ is a 3-group by (*) and (**); on the other hand G_k is 3-equivalent to the product $S^3 \times S^5 \times S^7$ in this case, so Σf_k is 3-equivalent to the constant map. Hence $\text{Im}(\Sigma f_k)^*$ vanishes too.

(ii) Similar to Lemma 2.3.

(iii) Let $\pi^3: G_k^7 \rightarrow G_k^7/S^3 = S^5 \vee S^7$ be the collapsing map. We shall prove $\Sigma(\pi^3 g_k) = (\Sigma\pi^3)(\Sigma g_k) = 0$. Then (iii) follows easily from $\pi_4(G_k) = 0$. The element $\pi^3 g_k$ lies in $[K_k, S^5 \vee S^7]$, which is generated by four elements:

$$\gamma_1 \quad \text{with} \quad \gamma_1 i_1 = j_1 \eta^2, \quad \gamma_1 i_2 = 0;$$

$$\gamma_2 \quad \text{with} \quad \gamma_2 i_1 = k' i_2, \quad \gamma_2 i_2 = \begin{cases} 0 & \text{for } k=0, \\ j_2 \eta^2 & \text{for } k=1, 2, 3; \end{cases}$$

$$\gamma_3 = j_1 v^2 \pi_K; \text{ and } \gamma_4 = [j_1, j_2] \pi_K,$$

where $i_1: S^7 \rightarrow K_k$, $i_2: S^9 \rightarrow K_k$, $j_1: S^5 \rightarrow S^5 \vee S^7$, $j_2: S^7 \rightarrow S^5 \vee S^7$ are the inclusions, $\pi_K: K_k \rightarrow S^{11}$ the projection, v^2 is the generator of $\pi_{n+6}(S^n) = \mathbf{Z}/2$, $n \geq 5$, [16], and $[j_1, j_2]$ is the universal Whitehead product.

The mapping cone of $\pi^3 g_k$ is homotopy equivalent to G_k^{12}/S^3 , which contains $SU(3)/S^3$, the space obtained from $SU(3)$ by the identification $x \sim_*$ for all $x \in S^3$, as a subcomplex. By [7, (2.5)], $SU(3)/S^3 = S^5 \vee S^8$, which means that the γ_1 -part of $\pi^3 g_k$ vanishes. Also the γ_2 -part vanishes because g_k (and hence $\pi^3 g_k$) induces the zero homomorphism of homology groups. Let E_k be the S^3 -bundle over S^7 classified by the element $k\omega \in \pi_6(S^3) \cong \pi_7(BS^3)$. Then there is a fibration

$$E_k \xrightarrow{i} G_k \xrightarrow{p} S^5$$

with classifying element [15] $\tilde{k}: S^4 \times E_k \rightarrow E_k$, whose prototype is the principal bundle

$$Sp(2) \longrightarrow SU(4) \longrightarrow S^5.$$

Then G_k is homotopy equivalent to $\tilde{G}_k = E_k \cup (D^5 \times E_k) / \sim$, where $(x, y) \in S^4 \times E_k \subset D^5 \times E_k$ is identified with $\tilde{k}(x, y) \in E_k$, i is equivalent to the inclusion $E_k \subset E_k \cup (D^5 \times E_k)$ to the first factor, p is equivalent to the map $\tilde{p}: \tilde{G}_k \rightarrow S^5$ such that $\tilde{p}(E_k) = *$, $\tilde{p}(x, y) = h(x)$, $h: (D^5, S^4) \rightarrow (S^5, *)$ the relative homeomorphism preserving orientation, and the homotopy equivalence $G_k \simeq \tilde{G}_k$ is fiberwise. [15, Prop. (1)]. Identifying G_k with $\tilde{G}_k$, we see that $G_k/E_k = D^5 \times E_k/S^4 \times E_k$, because $\tilde{k}|_{* \times E_k}$ is the identity of E_k . Since $D^5 \times E_k/(S^4 \times E_k \cup D^5 \times *) = \Sigma^5 E_k$, the following is a cofibration

$$S^5 = D^5/S^4 \longrightarrow G_k/E_k \longrightarrow \Sigma^5 E_k,$$

and p gives a retraction $G_k/E_k \rightarrow S^5$. Therefore $\Sigma(G_k/E_k)$ is homotopy equivalent to $\Sigma(S^5 \vee \Sigma^5 E_k)$. Since $E_k = S^3 \cup e^7 \cup e^{10}$ is a subcomplex of G_k , we see that $\Sigma(G_k^{12}/(S^3 \cup e^7 \cup e^{10})) = S^6 \vee (S^9 \cup_{k\omega} e^{13})$, from which the γ_3 -part of $\pi^3 g_k$ is in the kernel of Σ . Since $\Sigma\gamma_4 = 0$, the γ_4 -part also vanishes after suspension. Thus $\Sigma(\pi^3 g_k) = 0$.

Remark 3.6. In the second part of this paper, we shall develop the above discussion about the triviality of the γ_3 -part, to determine the precise group structure of $\mathcal{E}_0(SU(4))$.

Lemma 3.7. *The homomorphism $i'_*: [\Sigma K_k, G_k^7] \rightarrow [\Sigma K_k, G_k^{12}]$ is epic.*

Proof. Let F be the fiber of the inclusion $i': G_k^7 \rightarrow G_k^{12}$. Then, by the spectral sequence argument, F has a cell structure, up to homotopy, with $F^{12} = S^7 \cup e^9 \cup e^9 \cup e^{11} \cup e^{11} \cup e^{11}$. An arbitrary complex L consisting of cells such as in F^{12} is stable and has a homotopy type of one of the following

$$\begin{aligned} & [(S^7 \cup_{x\eta} e^9) \cup_{y\eta\mathbf{H}} e^{11}] \vee [S^9 \cup_{x'\eta} e^{11}] \vee S^{11}, \\ & [((S^7 \cup_{x\eta} e^9) \vee S^9) \cup_{(y\eta\mathbf{H} \vee \eta)l} e^{11}] \vee S^{11} \vee S^{11}, \end{aligned}$$

where $x, x' \in \mathbf{Z}/2$, $y \in \mathbf{Z}/24$. Then $[K_k, L]$ is in the stable range and easily computed. In consequence, $[K_k, L]$ is a free group of rank 5, in particular, $[K_k, F] \cong [K_k, F^{12}]$ is free. Since

$$[\Sigma K_k, G_k^7] \xrightarrow{(i')_*} [\Sigma K_k, G_k^{12}] \longrightarrow [K_k, F]$$

is exact and $[\Sigma K_k, G_k^{12}] \cong [\Sigma K_k, G_k]$ is finite, the lemma is proved.

Lemma 3.8.

$$(i) \quad \pi_{15}(G_k) \cong \begin{cases} \mathbf{Z}/36 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 & \text{for } k=0, 2, \\ \mathbf{Z}/72 \oplus \mathbf{Z}/2 & \text{for } k=1, 3. \end{cases}$$

$$(ii) \quad [\Sigma K_k, G_k] \cong \begin{cases} \mathbf{Z}/120 \oplus \mathbf{Z}/120 \oplus \mathbf{Z}/12 \oplus \mathbf{Z}/6 & \text{for } k=0, \\ \mathbf{Z}/360 \oplus \mathbf{Z}/120 \oplus \mathbf{Z}/4 & \text{for } k=1, \\ \mathbf{Z}/360 \oplus \mathbf{Z}/120 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2 & \text{for } k=2, \\ \mathbf{Z}/120 \oplus \mathbf{Z}/120 \oplus \mathbf{Z}/12 \oplus \mathbf{Z}/2 & \text{for } k=3. \end{cases}$$

The proof of Lemma 3.8 will be given in § 5.

We now state our result on the group $\mathcal{E}(G_k)$.

Theorem 3.9. (i) *The factor group $\mathcal{E}(G_k)/\mathcal{E}_0(G_k)$ is isomorphic to $Z_2 \times Z_2^{(*)}$ for $k=1, 2$, and to $Z_2 \times Z_2 \times Z_2$ for $k=0, 3$; and the exact sequence*

$$1 \longrightarrow \mathcal{E}_0(G_k) \longrightarrow \mathcal{E}(G_k) \longrightarrow \mathcal{E}(G_k)/\mathcal{E}_0(G_k) \longrightarrow 1$$

is split at least for $k=0, 1, 2$.

(ii) *There is an exact sequence of groups*

$$0 \longrightarrow \pi_{15}(G_k) \xrightarrow{\lambda} \mathcal{E}_0(G_k) \xrightarrow{(\lambda')^{-1}R} [\Sigma K_k, G_k] \longrightarrow 0,$$

where the group structures of $\pi_{15}(G_k)$ and of $[\Sigma K_k, G_k]$ are given in Lemma 3.8 (for the homomorphisms λ, λ', R , see Definitions 1.4, 1.9, and (1.5)).

(iii) *The order of $\mathcal{E}(G_k)$ is $2^{17}3^65^2$, $2^{14}3^55^2$, $2^{16}3^55^2$ or $2^{15}3^65^2$ according as $k=0, 1, 2$ or 3 .*

Proof. *First Step.* We construct elements in $\mathcal{E}(G_k)$ which generate $\mathcal{E}(G_k)/\mathcal{E}_0(G_k)$. For $G_1 = SU(4)$, the maps

$$\sigma: G_1 \longrightarrow G_1; \sigma(x) = x^{-1},$$

$$\tau: G_1 \longrightarrow G_1; \tau(x) = \bar{x} \text{ (the conjugate to } x),$$

are involutions such that $\sigma\tau = \tau\sigma$. For $G_0 = SU(3) \times S^7$, the maps

(*) Z_2 stands for the multiplicative cyclic group of order 2; the multiplication given by composition of maps.

$$\begin{aligned}\sigma: G_0 &\longrightarrow G_0; \sigma(x, y) = (x^{-1}, y^{-1}), \\ \tau: G_0 &\longrightarrow G_0; \tau(x, y) = (\bar{x}, y),\end{aligned}\quad (x \in SU(3), y \in S^7)$$

are also involutions with $\sigma\tau = \tau\sigma$. G_k is rationally equivalent to the product of spheres and these σ 's and τ 's are, respectively, rationally equivalent to each other. For any prime p , the p -localization of G_k is that of G_0 or G_1 , so the localization technique [6, Th. 5.1, Cor. 5.12] gives us elements

$$\sigma: G_k \longrightarrow G_k, \quad \tau: G_k \longrightarrow G_k$$

with $\sigma^2 = 1$, $\tau^2 = 1$, $\sigma\tau = \tau\sigma$, for all k . They act on the cohomology ring (3.2) as follows:

$$\begin{aligned}\sigma^*(x_i) &= -x_i, \quad i = 3, 5, 7, \\ \tau^*(x_3) &= x_3, \quad \tau^*(x_5) = -x_5, \quad \tau^*(x_7) = x_7.\end{aligned}$$

Therefore they generate a subgroup H_k of $\mathcal{E}(G_k)$ isomorphic to $Z_2 \times Z_2$, and $\mathcal{E}_0(G_k) \cap H_k = \{1\}$.

For $k=0, 3$, we define the third element ρ . For $k=0$, we put

$$\rho = 1 \times \sigma': G_0 \longrightarrow G_0,$$

where σ' is the inversion on S^7 . Then σ, τ and ρ generate a subgroup $\bar{H}_0$ isomorphic to $Z_2 \times Z_2 \times Z_2$, and $\bar{H}_0 \cap \mathcal{E}_0(G_0) = \{1\}$. For $k=3$, the classifying map θ for G_3 is of order 2, hence we have a homotopy commutative diagram

$$\begin{array}{ccccccc} SU(3) & \xrightarrow{i} & G_3 & \xrightarrow{p} & S^7 & \xrightarrow{\theta} & BSU(3) \\ \downarrow 1 & & \downarrow \rho & & \downarrow \sigma' & & \downarrow 1 \\ SU(3) & \xrightarrow{i} & G_3 & \xrightarrow{p} & S^7 & \xrightarrow{\theta} & BSU(3) \end{array}$$

giving rise to an element

$$\rho: G_3 \longrightarrow G_3,$$

which acts on the cohomology as follows:

$$\rho^*(x_3) = x_3, \quad \rho^*(x_5) = x_5, \quad \rho^*(x_7) = -x_7.$$

The ring structure (3.2) tells us that the factor group $\mathcal{E}(G_k)/\mathcal{E}_0(G_k)$ is a subgroup of $\text{Aut } H_3(G_k) \times \text{Aut } H_5(G_k) \times \text{Aut } H_7(G_k) = Z_2 \times Z_2 \times Z_2$; on the other hand, the images of the above three or two elements generate $Z_2 \times Z_2 \times Z_2$ or $Z_2 \times Z_2$. Therefore $\mathcal{E}(G_k)/\mathcal{E}_0(G_k) = Z_2 \times Z_2 \times Z_2$ for $k=0, 3$, and $= Z_2 \times Z_2$ or

$Z_2 \times Z_2 \times Z_2$ for $k=1, 2$. To show the last case can not happen, we shall compute $\mathcal{E}(G_k^7)$ in the next step. The splitness of the sequence in (i) follows immediately from the above results on H_k and $\bar{H}_0$.

Second Step. We determine $\mathcal{E}(G_k^7)$. Let CP^n be the complex projective space of (complex) dimension n . There is an embedding [18, § 5]

$$f_n: \Sigma CP^n \longrightarrow SU(n+1)$$

and we can identify G_1^7 with $f_3(\Sigma CP^3)$. The S^1 -bundle projection $q: S^5 \rightarrow CP^2$ is the attaching map of the top cell in $CP^3 = CP^2 \cup e^6$, and so, we have, by (2.1) or Lemma 3.1 (i), $\Sigma q = i\omega$. Then,

$$G_k^7 = (S^3 \cup_{\eta} e^5) \cup_{k\Sigma q} e^7 = \Sigma(CP^2 \cup_{kq} e^6).$$

It is well known that $\mathcal{E}(CP^n) = Z_2$ with generator a_n which is of degree $(-1)^i$ on the $2i$ -cell of $CP^n = \bigcup_{i=1}^n e^{2i}$. a_2 satisfies $a_2 q = -q$, so there is a map

$$b: CP^2 \cup_{kq} e^6 \longrightarrow CP^2 \cup_{kq} e^6$$

such that $b|_{CP^2} = a_2$ and b is of degree $(-1)^i$ on the $2i$ -cell. Let u be the reflexion with respect to the suspension coordinate of $G_k^7 = \Sigma(CP^2 \cup e^6)$, and put $v = \Sigma b$. Then u and v generate a subgroup H'_k of $\mathcal{E}(G_k^7)$, and $H'_k = Z_2 \times Z_2$. By the cellular approximation theorem, for $k=0, 3$, there is a map (not unique)

$$w = \rho|G_k^7: G_k^7 \longrightarrow G_k^7,$$

which is a homotopy equivalence, because w induces isomorphisms of homology groups.

Consider the following exact sequence of Barcus-Barratt [1, Th. (6.1)], cf. [11, Th. 2.11]:

$$(3.5) \quad \begin{array}{ll} \pi_7(G_k^5) \xrightarrow{\lambda_1 \circ (j_1)_*} \mathcal{E}(G_k^7) \xrightarrow{R} \mathcal{E}(G_k^5) & \text{for } k=1, 2, \\ \pi_7(G_k^5) \xrightarrow{\lambda_1 \circ (j_1)_*} \mathcal{E}(G_k^7) \xrightarrow{(R, R')} \mathcal{E}(G_k^5) \times \mathcal{E}(S^7) & \text{for } k=0, 3, \end{array}$$

where $\lambda_1(\xi) = c(1 \vee \xi)l$ for $\xi \in \pi_7(G_k^7)$, $l: G_k^7 \rightarrow G_k^7 \vee S^7$ the coaction map, $j_1: G_k^5 \rightarrow G_k^7$ the inclusion, R the restriction homomorphism, and R' is the map giving the mapping degree on the top cell. Since $R(H'_k) = \mathcal{E}(G_k^5)$ and $R'(w)$ generates $\mathcal{E}(S^7)$, R and (R, R') in (3.5) are epic.

Let $\phi \in \pi_7(G_k^5) = \pi_7(S^3 \cup_{\eta} e^5)$ be the attaching map of the top cell in $SU(3) = G_k^5 \cup e^8$, and $\psi \in \pi_7(G_k^7)$ be the map which is of degree k' on the 7-cells, where

k' is the integer in (3.4). Then, the homotopy exact sequences of the pairs $(SU(3), G_k^5), (G_k^7, G_k^5)$ yield

$$\pi_7(G_k^5) = \mathbf{Z} \cdot \phi, \quad \pi_7(G_k^7) = \mathbf{Z} \cdot j_1 \phi \oplus \mathbf{Z} \cdot \psi.$$

Since $(G_k^7 \times S^7, G_k^7 \vee S^7)$ is 9-connected, $\pi_7(G_k^7 \vee S^7)$ is isomorphic to $\pi_7(G_k^7) \oplus \pi_7(S^7)$, and, through the isomorphism, $l_*: \pi_7(G_k^7) \rightarrow \pi_7(G_k^7 \vee S^7)$ is equivalent to $\text{id} + (p_1)_*$, where $p_1: G_k^7 \rightarrow S^7$ is the collapsing map. Then $\lambda_1 \circ (j_1)_*(n\phi)$, $n \in \mathbf{Z}$, induces an automorphism of $\pi_7(G_k^7) = \mathbf{Z} \oplus \mathbf{Z}$, which is given by the matrix

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

This means $\lambda_1 \circ (j_1)_*$ in (3.5) is monic, and the sequence (3.5) is short exact. In particular, $\mathcal{E}_0(G_k^7) = \mathbf{Z}$, generated by $\lambda_1 \circ (j_1)_*(\phi)$, and $\mathcal{E}(G_k^7)/\mathcal{E}_0(G_k^7) = \mathbf{Z}_2 \times \mathbf{Z}_2$ for $k=1, 2$, $= \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ for $k=0, 3$. The group $\mathcal{E}(G_k)/\mathcal{E}_0(G_k)$ is isomorphic to $\mathcal{E}(G_k^7)/\mathcal{E}_0(G_k^7)$, proving (i) completely.

Remark 3.10. Put $x = \lambda_1(j_1\phi)$. Further discussion about the action on $\pi_7(G_k^7)$ of the elements u, v, w leads us that the group $\mathcal{E}(G_k^7)$ has the following presentation:

$$\begin{aligned} & \{u, v, w, x | u^2, v^2, w^2, [u, v], [u, w], [u, x], [v, w], [v, x]x^2, [w, x]\} && \text{for } k=0, \\ & \{u, v, x | u^2, v^2, [u, v], [u, x], [v, x]x^2\} && \text{for } k=1, 2, \\ & \{u, v, w, x | u^2, v^2, w^2, [u, v], [u, w]x^\varepsilon, [u, x], [v, w], [v, x]x^2, [w, x]\} && \text{for } k=3, \end{aligned}$$

where $\varepsilon=0$ or 2 .

Final Step. We prove (ii). The part (iii) is an easy consequence of (i), (ii) and Lemma 3.8. By Lemma 3.7, Lemma 1.13 is applicable for G_k even if G_k is not homotopy associative. By Theorem 1.7, Remark 1.18, Lemma 1.13 and Lemma 3.5,

$$0 \longrightarrow \pi_{15}(G_k) \xrightarrow{\lambda} \mathcal{E}_0(G_k)_7 \xrightarrow{R} \mathcal{E}_0(G_k^{12})_7 \longrightarrow 1$$

is exact and

$$\lambda': [\Sigma K_k, G_k^{12}] \longrightarrow \mathcal{E}_0(G_k^{12})_7$$

is an isomorphism. Let $i': G_k^7 \rightarrow G_k^{12}$ be the inclusion. For any $\alpha \in \mathcal{E}_0(G_k^{12})$, there is $\alpha': G_k^7 \rightarrow G_k^7$ with $i'\alpha' = \alpha i'$. Then α' is in $\mathcal{E}_0(G_k^7)$, so $\alpha' = \lambda_1(nj_1\phi)$, $n \in \mathbf{Z}$. Since $SU(3) \subset G_k^{12}$, $i'j_1\phi = 0$ in $\pi_7(G_k^{12})$. Then $i'\alpha' = i'$ and $\alpha i' = i'$, which implies $\mathcal{E}_0(G_k^{12})_7 = \mathcal{E}_0(G_k^{12})$. This completes the proof of (ii).

Corollary 3.11. $\text{nil } \mathcal{E}_0(SU(4)) \leq 2$.

Proof. Immediate from Theorems 2.4 and 3.9.

§ 4. Some results on self-equivalences of $Sp(3)$.

Throughout this section, we put

$$\bar{X} = Sp(3) \quad \text{and} \quad \bar{K} = S^{13} \cup_{\omega} e^{17}.$$

The cohomology with coefficient $\mathbf{Z}$ of $\bar{X}$ is the exterior algebra:

$$H^*(\bar{X}) = E(x_3, x_7, x_{11}),$$

and $\bar{X}$ has a cell decomposition

$$\bar{X} = S^3 \cup e^7 \cup e^{10} \cup e^{11} \cup e^{14} \cup e^{18} \cup e^{21}.$$

Concerning the cell structure, we have

- Lemma 4.1.** (i) $\bar{X}^7 = S^3 \cup_{\omega} e^7$ and $\bar{X}^{10} = Sp(2)$.
(ii) (cf. [10, § 2]) $\bar{X}^{11} = \bar{X}^{10} \cup_{\theta} e^{11}$ and the attaching map θ is a generator of $\pi_{10}(Sp(2)) = \mathbf{Z}/120$.
(iii) (Lemma 2.6) $\bar{X}^{18} = \bar{X}^{11} \cup_{\bar{g}} C\bar{K}$ for some map $\bar{g}: \bar{K} \rightarrow \bar{X}^{11}$.

Let $\lambda_1: \pi_{10}(Sp(2)) \rightarrow \mathcal{E}(Sp(2))$ be a homomorphism similarly defined as the previous λ 's, and $\sigma' \in \mathcal{E}(Sp(2))$ the inversion. Then,

Lemma 4.2. ([11, Example 4.5]). $\mathcal{E}_0(Sp(2))$ is the cyclic group of order 120, and generated by $\lambda_1(\theta)$. $\mathcal{E}(Sp(2)) = D_{120} = \{\lambda_1(\theta), \sigma' | \lambda_1(\theta)^{120}, (\sigma')^2, [\lambda_1(\theta), \sigma']\lambda_1(\theta)^2\}$.

Let $\bar{f}: S^{20} \rightarrow \bar{X}^{18}$ be the attaching map of the top cell in $\bar{X}$. Then the subgroups Δ of $\pi_{21}(\bar{X})$ and Δ' of $[\Sigma\bar{K}, \bar{X}^{18}] \cong [\Sigma\bar{K}, \bar{X}]$ in Theorem 2.9 are the following

$$\begin{aligned} \Delta &= \text{Im } (\Sigma\bar{f})^*: [\Sigma\bar{X}^{18}, \bar{X}] \longrightarrow \pi_{21}(\bar{X}), \\ \Delta' &= \text{Im } (\Sigma\bar{g})^*: [\Sigma\bar{X}^{11}, \bar{X}] \longrightarrow [\Sigma\bar{K}, \bar{X}]. \end{aligned}$$

Theorem 4.3. (i) *There is a split exact sequence*

$$1 \longrightarrow \mathcal{E}_0(Sp(3)) \longrightarrow \mathcal{E}(Sp(3)) \longrightarrow Z_2 \longrightarrow 1.$$

(ii) *There is an exact sequence*

$$0 \longrightarrow \pi_{21}(Sp(3))/\Delta \xrightarrow{\lambda} \mathcal{E}_0(Sp(3)) \xrightarrow{(\lambda')^{-1}R} [\Sigma\bar{K}, Sp(3)]/\Delta' \longrightarrow 0.$$

(iii) $\text{nil } \mathcal{E}_0(Sp(3)) \leq 2$.

Proof. The inversion $\sigma \in \mathcal{E}(\bar{X})$ generates Z_2 and its restriction to $\bar{X}^{10}$ is σ' in Lemma 4.2. Let $j': \bar{X}^{10} \rightarrow \bar{X}^{11}$, $j: \bar{X}^{11} \rightarrow \bar{X}$ be the inclusions. Then, $j'\theta=0$ by Lemma 4.1 (ii), so $j'\lambda_1(\theta)=j'$. Therefore $(j')_*\mathcal{E}(\bar{X}^{10})=\{j', j'\sigma'=\sigma j\}$ by Lemma 4.2. Let $p': \bar{X}^{11} \rightarrow S^{11}$ be the map collapsing $\bar{X}^{10}$ to the base point. p' is the restriction of the $Sp(2)$ -bundle projection $\bar{X} \rightarrow S^{11}$, and we have $\pi_{11}(\bar{X}^{11}) \cong \pi_{11}(\bar{X}) = \mathbf{Z} \cdot \gamma$, $p'\gamma = 120\epsilon_{11}$, by [10]. For any $\alpha \in \mathcal{E}(\bar{X}^{11})$, $\alpha j' \in (j')_*\mathcal{E}(\bar{X}^{10})$, and hence $\alpha j' = \sigma^\epsilon j'$, where $\sigma \in \mathcal{E}(\bar{X}^{11})$ is the restriction of σ and $\epsilon=0$ or 1 . Put $\alpha' = \sigma^\epsilon \alpha$ so that $\alpha' j' = j'$. Then $\alpha' = \lambda_2(n\gamma)$, $n \in \mathbf{Z}$, by (1.9), where $\lambda_2 = v(1, -): \pi_{11}(\bar{X}^{11}) \rightarrow [\bar{X}^{11}, \bar{X}^{11}]$. The action of $\lambda_2(n\gamma)$ on the cohomology group $H^{11}(\bar{X}^{11}) = \mathbf{Z}$ is the multiplication by $120n+1$. Since $\alpha' = \lambda_2(n\gamma)$ is a homotopy equivalence, n must be zero and $\alpha' = 1$. Therefore, we have obtained $\mathcal{E}(\bar{X}^{11}) = Z_2$ with generator σ , $\mathcal{E}_0(\bar{X}^{11}) = \{1\}$. The ring $H^*(\bar{X})$ is generated by elements of dimension up to 11, hence $\mathcal{E}(\bar{X})/\mathcal{E}_0(\bar{X})$ is isomorphic to $\mathcal{E}(\bar{X}^{11})/\mathcal{E}_0(\bar{X}^{11})$, proving (i).

It follows from $\mathcal{E}_0(\bar{X}^{11}) = \{1\}$ that $\mathcal{E}_0(\bar{X})_{11} = \mathcal{E}_0(\bar{X})$. Then (ii) and (iii) are consequences of Theorem 2.9.

Remark 4.4. To determine $\mathcal{E}_0(Sp(3))$, we have to calculate the factor groups in both sides in the sequence (ii). This seems to be attended with difficulties. Here we shall state merely several partial results: (1) $\pi_{21}(\bar{X}) = \mathbf{Z}/12 \oplus \mathbf{Z}/2$ by Mimura-Toda [10]; (2) $\Delta = 0$ or $\mathbf{Z}/2$ because $S^{21} \xrightarrow{f} \bar{X}^{18} \rightarrow \bar{X}^{18}/\bar{X}^7$ is null-homotopic and $[\Sigma\bar{X}^7, \bar{X}] = \mathbf{Z}/2$; (3) $[\Sigma\bar{K}, \bar{X}]$ is of order $2^9 3^5 5^2 7^2$ because

$$0 \longrightarrow \pi_{18}(\bar{X}) \longrightarrow [\Sigma\bar{K}, \bar{X}] \longrightarrow \pi_{14}(\bar{X}) \longrightarrow 0$$

is short exact, and $\pi_{18}(\bar{X}) = \mathbf{Z}/(3 \cdot 7!)$, $\pi_{14}(\bar{X}) = \mathbf{Z}/(2 \cdot 7!)$ [10]; and (4) (unfortunately!) $\Delta' \neq 0$. Thus the order of $\mathcal{E}_0(Sp(3))$ is a divisor of $2^{12} 3^6 5^2 7^2$. Moreover the group $\mathcal{E}(Sp(3))$ contains the subgroup generated by $\lambda(\pi_{21}(Sp(3))/\Delta)$ and σ , which is isomorphic to $D(\pi_{21}(Sp(3))/\Delta)$ with notation in [11, (3.10)], especially $\mathcal{E}(Sp(3))$ is not abelian.

§5. Computations; proof of Lemma 3.8.

In this section, we shall make computations of homotopy groups $\pi_n(G_k)$ and related groups, and prove Lemma 3.8. First we recall the principal bundles

$$(5.1) \quad \begin{aligned} S^3 &\xrightarrow{i_1} SU(3) \xrightarrow{p_1} S^5, \\ SU(3) &\xrightarrow{i} G_k \xrightarrow{p} S^7. \end{aligned}$$

The exact sequences of homotopy groups derived from (5.1) lead us that there exist elements

$$(5.2) \quad \iota \in \pi_3(G_k), \quad \iota' \in \pi_5(G_k), \quad \iota'' \in \pi_7(G_k),$$

which are, respectively, of degree 1 on the 3-cells, degree 2 on the 5-cells, degree k' on the 7-cells, where k' is the integer defined in (3.4). They are unique and generate the homotopy groups modulo torsion.

We next introduce notations of elements in $\pi_n(SU(3))$, $\pi_n(G_k)$. For $\beta \in \pi_n(S^5)$, $[\beta]$ denotes an element (if it exists) in $\pi_n(SU(3))$ such that $(p_1)_*[\beta] = \beta$. The image of $[\beta]$ in $\pi_n(G_k)$ under i_* is denoted by $i[\beta]$ instead of $i_*[\beta]$ (for the simplicity, we omit the subscript $*$). For $\gamma \in \pi_n(S^7)$, $[\gamma]'$ denotes an element (if it exists) in $\pi_n(G_k)$ such that $p_*[\gamma]' = \gamma$. We use freely the names of elements in homotopy groups of spheres in Toda's book [16]. In particular, the generators (5.2) of free part are written as

$$(5.3) \quad \iota = ii_1, \quad \iota' = i[2\epsilon_5], \quad \iota'' = [k'\epsilon_7]'$$

Thus every element in $\pi_n(G_k)$ is a sum of elements $\iota\alpha$, $i[\beta]$ and $[\gamma]'$ ($\alpha \in \pi_n(S^3)$, $\beta \in \pi_n(S^5)$, $\gamma \in \pi_n(S^7)$); sometimes $i[\beta]$ and $[\gamma]'$ are, respectively, replaced by the forms $\iota'\beta$ and $\iota''\gamma$.

By $X \underset{p}{\simeq} Y$, we mean X is p -equivalent to Y , where p is a prime. Then we have

$$(5.4) \quad \begin{aligned} G_0 &= SU(3) \times S^7 \underset{p}{\simeq} S^3 \times S^5 \times S^7 && \text{for } p \geq 3; \\ G_1 &= SU(4) \underset{p}{\simeq} \begin{cases} Sp(2) \times S^5 & \text{for } p = 3, \\ S^3 \times S^5 \times S^7 & \text{for } p \geq 5; \end{cases} \\ G_2 &\underset{p}{\simeq} \begin{cases} G_0 & \text{for } p \neq 3, \\ G_1 & \text{for } p = 3; \end{cases} \end{aligned}$$

$$G_3 \underset{p}{\simeq} \begin{cases} G_0 & \text{for } p \neq 2, \\ G_1 & \text{for } p = 2. \end{cases}$$

Mimura and Toda [9] computed homotopy groups of $SU(3)$, $Sp(2)$, $SU(4)$ from the results on homotopy groups of spheres in [16]. We then easily obtain homotopy groups of G_k from their results and (5.4). The following are the results dimension up to 15, which contain Lemma 3.8 (i).

Proposition 5.1. (i) (= Lemma 3.4) $\pi_1(G_k)=0$, $\pi_2(G_k)=0$, $\pi_3(G_k)=\mathbf{Z} \cdot \iota$, $\pi_4(G_k)=0$, $\pi_5(G_k)=\mathbf{Z} \cdot \iota'$, $\pi_6(G_0)=\mathbf{Z}/2 \cdot \iota v' \oplus \mathbf{Z}/3 \cdot \iota \alpha_1$, $\pi_6(G_1)=0$, $\pi_6(G_2)=\mathbf{Z}/2 \cdot \iota v'$, $\pi_6(G_3)=\mathbf{Z}/3 \cdot \iota \alpha_1$.

$$(ii) \quad \pi_7(G_k)=\mathbf{Z} \cdot \iota''.$$

$$(iii) \quad \pi_8(G_k) = \begin{cases} \mathbf{Z}/4 \cdot \iota' v_5 \oplus \mathbf{Z}/2 \cdot \iota'' \eta_7 \oplus \mathbf{Z}/3 \cdot \iota' \alpha_1 & \text{for } k=0, 2, \\ \mathbf{Z}/8 \cdot [\eta_7]' \oplus \mathbf{Z}/3 \cdot \iota' \alpha_1 & \text{for } k=1, 3, \end{cases}$$

and there holds the relation $2[\eta_7]' = \iota' v_5$ for $k=1, 3$.

$$(iv) \quad \pi_9(G_k) = \begin{cases} \mathbf{Z}/2 \cdot \iota'' \eta_7^2 \oplus \mathbf{Z}/3 \cdot \iota \alpha_1^2 & \text{for } k=0, \\ \mathbf{Z}/2 \cdot [\eta_7]' \eta_8 & \text{for } k=1, \\ \mathbf{Z}/2 \cdot \iota'' \eta_7^2 & \text{for } k=2, \\ \mathbf{Z}/2 \cdot [\eta_7]' \eta_8 \oplus \mathbf{Z}/3 \cdot \iota \alpha_1^2 & \text{for } k=3. \end{cases}$$

$$(v) \quad \pi_{10}(G_k) = A \oplus \begin{cases} \mathbf{Z}/8 \cdot \iota'' v_7 \oplus \mathbf{Z}/3 \cdot \iota'' \alpha_1 & \text{for } k=0, \\ \mathbf{Z}/8 \cdot [v_7]' & \text{for } k=1, \\ \mathbf{Z}/8 \cdot \iota'' v_7 & \text{for } k=2, \\ \mathbf{Z}/8 \cdot [v_7]' \oplus \mathbf{Z}/3 \cdot \iota'' \alpha_1 & \text{for } k=3, \end{cases}$$

$$A = \mathbf{Z}/2 \cdot i[v_5 \eta_2^8] \oplus \mathbf{Z}/3 \cdot \iota \alpha_2 \oplus \mathbf{Z}/5 \cdot \iota \alpha_1.$$

$$(vi) \quad \pi_{11}(G_k) = \mathbf{Z}/4 \cdot i[v_3^2], \text{ and there holds } 2i[v_3^2] = \iota \varepsilon_3.$$

$$(vii) \quad \pi_{12}(G_k) = \mathbf{Z}/4 \cdot i[\sigma'''] \oplus \mathbf{Z}/3 \cdot \iota' \alpha_2 \oplus \mathbf{Z}/5 \cdot \iota' \alpha_1,$$

and there holds $2i[\sigma'''] = \iota \mu_3$.

$$(viii) \quad \pi_{13}(G_k) = \begin{cases} \mathbf{Z}/2 \cdot \epsilon \epsilon' \oplus \mathbf{Z}/2 \cdot \epsilon'' v_7^2 \oplus \mathbf{Z}/3 \cdot \epsilon \alpha_1 \alpha_2 & \text{for } k=0, \\ \mathbf{Z}/4 \cdot [v_7]' v_{10} & \text{for } k=1, \\ \mathbf{Z}/2 \cdot \epsilon \epsilon' \oplus \mathbf{Z}/2 \cdot \epsilon'' v_7^2 & \text{for } k=2, \\ \mathbf{Z}/4 \cdot [v_7]' v_{10} \oplus \mathbf{Z}/3 \cdot \epsilon \alpha_1 \alpha_2 & \text{for } k=3, \end{cases}$$

and there holds $2[v_7]' v_{10} = \epsilon \epsilon'$ for $k=1, 3$.

$$(ix) \quad \pi_{14}(G_k) = \begin{cases} \mathbf{Z}/2 \cdot \epsilon \mu' \oplus \mathbf{Z}/4 \cdot i[v_5^2] v_{11} \oplus \mathbf{Z}/8 \cdot \epsilon'' \sigma' \oplus \mathbf{Z}/3 \cdot \epsilon'' \alpha_2 \oplus B & \text{for } k=0, \\ \mathbf{Z}/2 \cdot i[v_5^2] v_{11} \oplus \mathbf{Z}/16 \cdot [\sigma']' \oplus B & \text{for } k=1, \\ \mathbf{Z}/2 \cdot \epsilon \mu' \oplus \mathbf{Z}/4 \cdot i[v_5^2] v_{11} \oplus \mathbf{Z}/8 \cdot \epsilon'' \sigma' \oplus B & \text{for } k=2, \\ \mathbf{Z}/2 \cdot i[v_5^2] v_{11} \oplus \mathbf{Z}/16 \cdot [\sigma']' \oplus \mathbf{Z}/3 \cdot \epsilon'' \alpha_2 \oplus B & \text{for } k=3, \end{cases}$$

$$B = \mathbf{Z}/3 \cdot \epsilon \alpha_3 \oplus \mathbf{Z}/5 \cdot \epsilon'' \alpha_1 \oplus \mathbf{Z}/7 \cdot \epsilon \alpha_1,$$

and there hold $2i[v_5^2] v_{11} = \epsilon \epsilon_3 v_{11}$ for $k=0, 2$, and $8[\sigma']' = \epsilon \mu'$ for $k=1, 3$.

$$(x) \quad \pi_{15}(G_k) = \begin{cases} \mathbf{Z}/4 \cdot \epsilon' v_5 \sigma_8 \oplus \mathbf{Z}/2 \cdot \epsilon'' \sigma' \eta_{14} \oplus \mathbf{Z}/2 \cdot \epsilon'' \bar{v}_7 \oplus \mathbf{Z}/2 \cdot \epsilon'' \epsilon_7 \\ \oplus \mathbf{Z}/9 \cdot \epsilon' \beta_1 & \text{for } k=0, 2, \\ \mathbf{Z}/8 \cdot [\eta_7]' \sigma_8 \oplus \mathbf{Z}/2 \cdot [\sigma']' \eta_{14} \oplus \mathbf{Z}/9 \cdot \epsilon' \beta_1 & \text{for } k=1, 3. \end{cases}$$

We next compute the group $[\Sigma K_k, G_k]$ together with generators and prove Lemma 3.8 (ii). Recall that

$$\Sigma K_k = (S^8 \vee S^{10}) \cup_{\kappa} e^{12}, \quad \kappa = ((2k(v_8 + \alpha_1)) \vee \eta_{10})l.$$

We put

$$Q = S^{10} \cup_{\eta_{10}} e^{12}, \quad R = S^8 \cup_{2v_8} e^{12}, \quad R' = S^8 \cup_{\alpha_1} e^{12}.$$

Then we have the following p -equivalences

$$(5.5) \quad \begin{aligned} \Sigma K_0 &= S^8 \vee Q \underset{p}{\simeq} S^8 \vee S^{10} \vee S^{12} & \text{for } p \geq 3; \\ \Sigma K_1 &\underset{p}{\simeq} \begin{cases} R' \vee S^{10} & \text{for } p=3, \\ S^8 \vee S^{10} \vee S^{12} & \text{for } p \geq 5, \end{cases} \\ (\Sigma K_1)/S^{10} &\underset{2}{\simeq} R; \end{aligned}$$

$$\Sigma K_2 \underset{p}{\simeq} \begin{cases} S^8 \vee Q & \text{for } p=2, \\ R' \vee S^{10} & \text{for } p=3, \\ S^8 \vee S^{10} \vee S^{12} & \text{for } p \geq 5; \end{cases}$$

$$\Sigma K_3 \underset{p}{\simeq} \begin{cases} \Sigma K_1 & \text{for } p=2, \\ S^8 \vee S^{10} \vee S^{12} & \text{for } p \geq 3. \end{cases}$$

Let $i_k: S^8 \vee S^{10} \rightarrow \Sigma K_k$ be the inclusion, $\pi_k: \Sigma K_k \rightarrow S^{12}$ the map collapsing $S^8 \vee S^{10}$ to the base point, $p_1: S^8 \vee S^{10} \rightarrow S^8$ the first projection, and $p_2: S^8 \vee S^{10} \rightarrow S^{10}$ the second projection. For given elements $\alpha \in \pi_8(X)$ and $\beta \in \pi_{10}(X)$, $\langle \alpha \rangle_1$ and $\langle \beta \rangle_2$ denote elements in $[\Sigma K_k, X]$ (if they exist) such that $\langle \alpha \rangle_1 i_k = \alpha p_1$ and $\langle \beta \rangle_2 i_k = \beta p_2$. Then

Proposition 5.2.

$$\begin{aligned} [\Sigma K_0, G_0] &= \mathbf{Z}/8 \cdot \epsilon'' \langle v_7 \rangle_2 \oplus \mathbf{Z}/2 \cdot \epsilon'' \langle \eta_7 \rangle_1 \oplus C \\ &\quad \oplus \mathbf{Z}/3 \cdot \epsilon' \alpha_2 \pi_K \oplus \mathbf{Z}/3 \cdot \epsilon' \langle \alpha_1 \rangle_1 \oplus \mathbf{Z}/3 \cdot \epsilon'' \langle \alpha_1 \rangle_2 \oplus D, \\ [\Sigma K_1, G_1] &= \mathbf{Z}/8 \cdot \chi \oplus C \oplus \mathbf{Z}/9 \cdot \epsilon' \langle \alpha_1 \rangle_1 \oplus D, \\ [\Sigma K_2, G_2] &= \mathbf{Z}/8 \cdot \epsilon'' \langle v_7 \rangle_2 \oplus \mathbf{Z}/2 \cdot \epsilon'' \langle \eta_7 \rangle_1 \oplus C \oplus \mathbf{Z}/9 \cdot \epsilon' \langle \alpha_1 \rangle_1 \oplus D, \\ [\Sigma K_3, G_3] &= \mathbf{Z}/8 \cdot \chi \oplus C \oplus \mathbf{Z}/3 \cdot \epsilon' \alpha_2 \pi_K \oplus \mathbf{Z}/3 \cdot \epsilon' \langle \alpha_1 \rangle_1 \oplus \mathbf{Z}/3 \cdot \epsilon'' \langle \alpha_1 \rangle_2 \oplus D, \end{aligned}$$

where

$$\begin{aligned} C &= \mathbf{Z}/8 \cdot i \langle [v_5 \eta_8^2] \rangle_2 \oplus \mathbf{Z}/4 \cdot \epsilon' \langle v_5 \rangle_1, \\ D &= \mathbf{Z}/3 \cdot \epsilon \langle \alpha_2 \rangle_2 \oplus \mathbf{Z}/5 \cdot \epsilon \langle \alpha_1 \rangle_2 \oplus \mathbf{Z}/5 \cdot \epsilon' \alpha_1 \pi_K, \end{aligned}$$

and χ is an element such that $\chi i_k = [\eta_7]' p_1 + [v_7]' p_2$.

The following relations hold:

$$\begin{aligned} 2i \langle [v_5 \eta_8^2] \rangle_2 &= i[\sigma'''] \pi_K, \quad 4i \langle [v_5 \eta_8^2] \rangle_2 = \epsilon \mu_3 \pi_K \quad \text{for all } k; \\ 2\chi &= \epsilon' \langle v_5 \rangle_1 + \langle 2[v_7]' \rangle_2 \quad \text{for } k=1, 3; \\ 3\epsilon' \langle \alpha_1 \rangle_1 &= \pm k \epsilon' \alpha_2 \pi_K \quad \text{for } k=1, 2. \end{aligned}$$

Proof. By (5.4) and (5.5), for the 2-primary part $[\Sigma K_k, G_k]_{(2)}$, the following isomorphisms result:

$$\begin{aligned}
[\Sigma K_2, G_2]_{(2)} &\cong [\Sigma K_0, G_0]_{(2)} \\
&\cong \pi_8(SU(3))_{(2)} \oplus [Q, SU(3)]_{(2)} \oplus \pi_8(S^7)_{(2)} \oplus [Q, S^7]_{(2)}, \\
[\Sigma K_3, G_3]_{(2)} &\cong [\Sigma K_1, G_1]_{(2)},
\end{aligned}$$

where the subscript (2) indicates the localization at 2. So, it suffices to compute $[Q, S^7]_{(2)}$, $[Q, SU(3)]_{(2)}$, $[\Sigma K_1, G_1]_{(2)}$. The following is a commutative diagram of cofibrations

$$(5.6) \quad \begin{array}{ccccccc}
S^{11} & \xrightarrow{\kappa} & S^8 \vee S^{10} & \xrightarrow{i_K} & \Sigma K_1 & \xrightarrow{\pi_K} & S^{12} \\
\parallel & & \downarrow p_2 & & \downarrow q & & \parallel \\
S^{11} & \xrightarrow{\eta} & S^{10} & \xrightarrow{i_Q} & Q & \xrightarrow{\pi_Q} & S^{12},
\end{array}$$

where $\eta = \eta_{10}$. For $\beta \in \pi_{10}(X)$, we denote by $\langle \beta \rangle_Q$ (if it exists) an element in $[Q, X]$ with $\langle \beta \rangle_Q i_Q = \beta$.

STATEMENT (1). $[Q, S^7]_{(2)} = \mathbf{Z}/8 \cdot \langle v_7 \rangle_Q$.

Because $(i_Q)^*: [Q, S^7] \rightarrow \pi_{10}(S^7)$ is isomorphic.

STATEMENT (2). *There exist elements $a_Q \in [Q, S^{10}]$, $b_Q \in \pi_{12}(Q)$ such that $a_Q i_Q = 2\epsilon_{10}$, $\pi_Q b_Q = 2\epsilon_{12}$ and $2 \cdot 1_Q = i_Q a_Q + b_Q \pi_Q$.*

For, $[Q, Q] = \mathbf{Z} \cdot 1_Q \oplus \mathbf{Z} \cdot i_Q a_Q$, and compute the $(i_Q)^*$ - and $(\pi_Q)_*$ -images of the equality $b_Q \pi_Q = x \cdot 1_Q + y i_Q a_Q$, to get $x=2$, $y=-1$.

We put $\tau = v_5 \eta_8^2 \in \pi_{10}(S^5)$ for the simplicity.

STATEMENT (3). $[Q, SU(3)]_{(2)} = \mathbf{Z}/8 \cdot \langle [\tau] \rangle_Q$ and $2\langle [\tau] \rangle_Q = [\sigma''] \pi_Q$.

PROOF. From (5.6), we have the exact sequence

$$\pi_{11} \xrightarrow{\eta^*} \pi_{12} \longrightarrow [Q, SU(3)]_{(2)} \longrightarrow \pi_{10} \xrightarrow{\eta^*} \pi_{11},$$

where $\pi_n = \pi_n(SU(3))_{(2)}$, $\pi_{11} = \mathbf{Z}/4 \cdot [v_5^2]$, $\pi_{12} = \mathbf{Z}/4 \cdot [\sigma'']$, $\pi_{10} = \mathbf{Z}/2 \cdot [\tau]$. By [9] Theorem 2.1 with $\Delta\alpha = \eta_3$, $\beta = v_4^2$, $\gamma = \eta_{10}$,

$$\eta^*[v_5^2] = (i_1)_* \{ \eta_3, v_4^2, \eta_{10} \}$$

where i_1 is the inclusion in (5.1) and $\{-, -, -\}$ denotes the Toda bracket, and by [9] Theorem 2.6 with $\alpha = \epsilon_5$, $\beta = \tau$, $\gamma = \eta_{10}$,

$$\eta^*[\tau] = (i_1)_* E^* \{ \eta_4, \tau, \eta_{10} \},$$

where $E^*: \pi_{12}(S^4) \rightarrow \pi_{11}(S^3)$ is the left inverse of Σ defined in [9]. In our case, E^* is an isomorphism. From the relations

$$\{\eta_3, v_4^2, \eta_{10}\} \equiv 0 \pmod{\eta_3 e_4}, \{\eta_4, \tau, \eta_{10}\} \equiv 0 \pmod{0},$$

it follows that η^* 's are both trivial. By Statement (2),

$$\begin{aligned} 2\langle[\tau]\rangle_Q &= [\tau]a_Q + \langle[\tau]\rangle_Q b_Q \pi_Q \\ &\in \{[\tau], 2\epsilon_{10}, \eta_{10}\}\pi_Q + \{[\tau], \eta_{10}, 2\epsilon_{11}\}\pi_Q. \end{aligned}$$

We have $(p_1)_*\{[\tau], 2\epsilon_{10}, \eta_{10}\} = \{\tau, 2\epsilon_{10}, \eta_{10}\} = v_5 \eta_8 \{\eta_9, 2\epsilon_{10}, \eta_{10}\} = 0$, which implies $\{[\tau], 2\epsilon_{10}, \eta_{10}\} \equiv 0 \pmod{2[\sigma'']}$. From $H\{\tau, \eta_{10}, 2\epsilon_{11}\} = 2\Delta^{-1}(4v_4^2) = 4v_9 = H\sigma'''$ (H and Δ are in [16]), $\{\tau, \eta_{10}, 2\epsilon_{11}\} = \sigma'''$ and we have $\{[\tau], \eta_{10}, 2\epsilon_{11}\} \equiv [\sigma'''] \pmod{2[\sigma''']}$. Hence

$$2\langle[\tau]\rangle_Q \equiv [\sigma''']\pi_Q \pmod{2[\sigma''']\pi_Q}.$$

We may replace $\langle[\tau]\rangle_Q$ by $\langle[\tau]\rangle_Q + [\sigma''']\pi_Q$, proving the statement.

Replacing Q by R in the diagram (5.6) gives

$$(5.7) \quad \begin{array}{ccccccc} S^{11} & \xrightarrow{\kappa} & S^8 \vee S^{10} & \xrightarrow{i_K} & \Sigma K_1 & \xrightarrow{\pi_K} & S^{12} \\ \parallel & & \downarrow x_{p_1} & & \downarrow r & & \parallel \\ S^{11} & \xrightarrow{2v} & S^8 & \xrightarrow{i_R} & R & \xrightarrow{\pi_R} & S^{12}, \end{array}$$

with $v = v_8$, x a suitable odd integer. Then Statement (2) with Q replaced by R is the following

STATEMENT (4). *There exist elements $a_R \in [R, S^8]$, $b_R \in \pi_{12}(R)$ such that $a_R i_R = 4\epsilon_8$, $\pi_R b_R = 4\epsilon_{12}$ and $4 \cdot 1_R = i_R a_R + b_R \pi_R$.*

STATEMENT (5). $[\Sigma K_1, SU(3)]_{(2)} = \mathbf{Z}/8 \cdot \epsilon \langle[\tau]\rangle_2 \oplus \mathbf{Z}/4 \cdot [2\epsilon_5] \langle v_5 \rangle_1$ and $2\langle[\tau]\rangle_2 = [\sigma''']\pi_K$.

PROOF. Applying $[-, SU(3)]_{(2)}$ to (5.6) and (5.7) gives the following commutative diagram with exact rows

$$(5.8) \quad \begin{array}{ccccccc} & & & & \mathbf{Z}/2 \cdot [\tau] & & \\ & & & & \parallel & & \\ 0 & \longrightarrow & \pi_{12} & \longrightarrow & [Q, SU(3)]_{(2)} & \longrightarrow & \pi_{10} \longrightarrow 0 \\ & & \parallel & & \downarrow q^* & & \downarrow \\ 0 & \longrightarrow & \pi_{12} & \longrightarrow & [\Sigma K_1, SU(3)]_{(2)} & \longrightarrow & \pi_8 \oplus \pi_{10} \longrightarrow 0 \\ & & \parallel & & \uparrow r^* & & \uparrow \\ 0 & \longrightarrow & \pi_{12} & \longrightarrow & [R, SU(3)]_{(2)} & \longrightarrow & \pi_8 \longrightarrow 0, \\ & & \parallel & & \downarrow & & \parallel \\ & & \mathbf{Z}/4 \cdot [\sigma'''] & & & & \mathbf{Z}/4 \cdot [2\epsilon_5] v_5 \end{array}$$

where $\pi_n = \pi_n(SU(3))_{(2)}$. Then q^* and r^* are monic. We can take $\langle [\tau] \rangle_2$ as $q^* \langle [\tau] \rangle_Q$, and $2 \langle [\tau] \rangle_2 = [\sigma''] \pi_K$ by Statement (3). From the relation $v_5(2v_8) = 0$, there is an element $\langle v_5 \rangle_R \in [R, S^5]$ with $\langle v_5 \rangle_R i_R = v_5$. Then we can take $\langle v_5 \rangle_1$ as $r^* \langle v_5 \rangle_R$, and $\langle [2\epsilon_5] v_5 \rangle_1$ as $[2\epsilon_5] \langle v_5 \rangle_1 (= [2\epsilon_5] \langle v_5 \rangle_R r)$. By Statement (4),

$$4[2\epsilon_5] \langle v_5 \rangle_R = \gamma \pi_R + \gamma' \pi_R, \text{ where}$$

$$\gamma' \in \{[2\epsilon_5] v_5, 2v_8, 4\epsilon_{11}\} = [2\epsilon_5] \{v_5, 2\epsilon_8, 4\epsilon_{11}\},$$

$$\gamma' \in \{[2\epsilon_5] v_5, 4\epsilon_8, 2v_8\} = [2\epsilon_5] \{v_5, 8\epsilon_8, v_8\}.$$

From the relations, $\{v_5, 2v_8, 4\epsilon_{11}\} = \sigma'''$, $\{v_5, 8\epsilon_8, v_8\} = \sigma'''$ and $2\sigma''' = 0$, we get $4[2\epsilon_5] \langle v_5 \rangle_R = 0$, and $[2\epsilon_5] \langle v_5 \rangle_1$ is of order 4. Thus the group extension at the middle row in (5.8) is solved.

STATEMENT (6). Applying $[-, G_1]_{(2)}$ to the top row in (5.6) gives a short exact sequence

$$0 \longrightarrow \mathbf{Z}/4 \cdot i[\sigma'''] \xrightarrow{(\pi_K)^*} [\Sigma K_1, G_1]_{(2)} \xrightarrow{(i_K)^*} \mathbf{Z}/8 \cdot \chi_0 \oplus \mathbf{Z}/4 \cdot 2[v_7]' p_2 \oplus \mathbf{Z}/2 \cdot i[\tau] p_2 \longrightarrow 0,$$

where $\chi_0 = [\eta_7]' p_1 + [v_7]' p_2$.

PROOF. The top row in (5.8) lead us that $(\pi_K)^*$ is monic and $i[\tau] p_2$ is in $\text{Im}(i_K)^*$. Also $\text{Im}(i_K)^*$ is the kernel of

$$\kappa^* = 2v^* + \eta^* : \pi_8(G_1)_{(2)} \oplus \pi_{10}(G_1)_{(2)} \longrightarrow \pi_{11}(G_1)_{(2)}.$$

By the relations in Proposition 5.1,

$$2v^*[\eta_7]' = v^* \epsilon' v_5 = 2i[v_5]^2 = \epsilon \epsilon_3,$$

$$\eta^*[\eta_7]' = \epsilon \{v', v_6, \eta_9\} = \epsilon \epsilon_3.$$

Hence $\text{Ker } \kappa^* = \mathbf{Z}/8 \cdot \chi_0 \oplus \mathbf{Z}/4 \cdot 2[v_7]' p_2 \oplus \mathbf{Z}/2 \cdot i[\tau] p_2$.

STATEMENT (7). $[\Sigma K_1, G_1]_{(2)} = \mathbf{Z}/8 \cdot \chi \oplus \mathbf{Z}/4 \cdot \epsilon' \langle v_5 \rangle_1 \oplus \mathbf{Z}/8 \cdot i \langle [\tau] \rangle_2$.

PROOF. There is an element $\xi \in [Q, G_1]_{(2)}$ with $\xi i_Q = 2[v_7]'$ by Statement (6) together with its proof. From the relation $4v_7 = \eta_7^3$, we have easily $4[v_7]' = [\eta_7]' \eta_8^2$. Then

$$2\xi i_Q a_Q = 4[v_7]' a_Q = [\eta_7]' \{\eta_8^2, 2\epsilon_{10}, \eta_{10}\} \pi_Q = 0,$$

$$2\xi b_Q = 2\{2[v_7]', \eta_{10}, 2\epsilon_{11}\} = 2[v_7]' \{2\epsilon_{10}, \eta_{10}, 2\epsilon_{11}\} = 0.$$

Therefore $4\xi = 2\xi i_Q a_Q + 2\xi b_Q \pi_Q = 0$, and hence $\langle 2[v_7]' \rangle_2 = q^* \xi$ is of order 4. Also $\epsilon' \langle v_5 \rangle_1 = i[2\epsilon_5] \langle v_5 \rangle_1$ is of order 4 by Statement (5). From the relation $2[\eta_7]' = \epsilon' v_5$ in Proposition 5.1,

$$2\chi_0 = (i_K)^*(\epsilon' \langle v_5 \rangle_1 + \langle 2[v_7]' \rangle_2),$$

which implies

$$2\chi \equiv \epsilon' \langle v_5 \rangle_1 + \langle 2[v_7]' \rangle_2 \pmod{i[\sigma''']\pi_K}.$$

Then $8\chi = 0$, proving Statement (7). We notice that, by the middle row in (5.8) and the exact sequence in Statement (6),

$$(5.9) \quad i_*: [\Sigma K_1, SU(3)]_{(2)} \longrightarrow [\Sigma K_1, G_1]_{(2)} \text{ is monic.}$$

By Statements (1), (3) and (7), the result on the 2-primary part of $[\Sigma K_k, G_k]$ is obtained. The odd primary parts are computed from the following isomorphisms

$$\begin{aligned} [\Sigma K_k, G_k]_{(p)} &\cong [S^8 \vee S^{10} \vee S^{12}, S^3 \times S^5 \times S^7]_{(p)} \\ &\text{for } k=1, 2, p \geq 5 \text{ and for } k=0, 3, p \geq 3; \\ [\Sigma K_k, G_k]_{(3)} &\cong [R' \vee S^{10}, Sp(2) \times S^5]_{(3)} \quad \text{for } k=1, 2. \end{aligned}$$

The calculations of $[R', S^5]$ and $[R', Sp(2)]$ at the prime 3 is similar to and easier than those in Statements (1) and (3), and we omit them. Complete the proof of Proposition 5.2.

Now, $SU(4) = G_1$ is a total space of another bundle

$$Sp(2) \xrightarrow{i'} SU(4) \xrightarrow{p'} S^5,$$

and $Sp(2)$ is a total space of the bundle

$$S^3 \xrightarrow{i_2} Sp(2) \xrightarrow{p_2} S^7.$$

The coset space $SU(4)/S^3$ is a complex Stiefel manifold, which is homotopy equivalent to $S^5 \times S^7$, that is, there is an S^3 -bundle

$$S^3 \longrightarrow SU(4) \xrightarrow{(p \times p')d} S^5 \times S^7.$$

In Propositions 5.1 and 5.2, the element χ and generators of the forms

$[-]$, $[-]'$, $\langle - \rangle_1$, $\langle - \rangle_2$ may, of course, be defined up to some subgroups. For the case of the element of the form $[\alpha]'$ in $[A, SU(4)]$ ($\alpha \in [A, S^7]$), it was defined up to the image of $i_*: [A, SU(3)] \rightarrow [A, SU(4)]$. If we could give the value $(p')_*[\alpha]'$ as well as $p_*[\alpha]'$, which is α , then the element $[\alpha]'$ could be defined up to the image of $[A, S^3] \rightarrow [A, SU(4)]$, which is smaller. The computation of $\pi_n(SU(4))$ of Mimura and Toda [9] was made by using exact sequence derived from the above S^3 -bundle over $S^5 \times S^7$. So the elements of them are chosen more precisely than ours. For example, the element $[\eta_7]'$ in $\pi_8(SU(4))$ in Proposition 5.1 is uniquely determined by two relations $p_*[\eta_7]' = \eta_7$ and $(p')_*[\eta_7]' = v_5$. For the generators of $[\Sigma K_1, SU(4)]$, we notice the following

Remark 5.3. (i) The element χ is determined, up to the subgroup $\bar{I} = \mathbf{Z}/4 \cdot i[\sigma'']\pi_K$, by the relations

$$p_*\chi = \langle \eta_7 \rangle_1 + \langle v_7 \rangle_2, \quad p'\chi i_1 = v_5 \text{ (or } (p')_*\chi = \langle v_5 \rangle_1),$$

where $\langle \eta_7 \rangle_1$, $\langle v_7 \rangle_2$ are uniquely defined, but $\langle v_5 \rangle_1$ is determined up to $\mathbf{Z}/2 \cdot \sigma''\pi_K$. The original definition of χ depends on the choice of $[\eta_7]'$, and we take here $[\eta_7]'$ as above.

- (ii) The element $i[\langle v_5 \eta_8^2 \rangle_2]$ is determined up to $\bar{I}$.
- (iii) The element $\iota'\langle v_5 \rangle_1$ is determined up to $I = \mathbf{Z}/2 \cdot \iota\mu_3\pi_K = 2\bar{I}$.
- (iv) The element $\iota'\langle \alpha_1 \rangle_1$ is determined up to a subgroup generated by $3\iota'\langle \alpha_1 \rangle_1 = \pm \iota'\alpha_2\pi_K$.
- (v) Generators of the subgroup D are all uniquely defined.

Let $[v_7]_{Sp}$ be an element in $\pi_{10}(Sp(2))$ of order 8 such that $(p_2)_*[v_7]_{Sp} = v_7$. Then it is unique and generates the 2-primary part of $\pi_{10}(Sp(2))$ [9]. Then the following lemma is proved in a similar way as (5.9) and Statement (5), so we omit the proof.

Lemma 5.4.

$$\begin{aligned} \text{(i)} \quad [\Sigma K_1, SU(3)] &= \mathbf{Z}/8 \cdot \langle [v_5 \eta_8^2] \rangle_2 \oplus \mathbf{Z}/4 \cdot [2\iota_5] \langle v_5 \rangle_1 \\ &\quad \oplus \mathbf{Z}/9 \cdot [2\iota_5] \langle \alpha_1 \rangle_1 \oplus \mathbf{Z}/3 \cdot i_1 \langle \alpha_2 \rangle_2 \\ &\quad \oplus \mathbf{Z}/5 \cdot [2\iota_5] \alpha_1 \pi_K \oplus \mathbf{Z}/5 \cdot i_1 \langle \alpha_1 \rangle_2, \end{aligned}$$

$i_*: [\Sigma K_1, SU(3)] \rightarrow [\Sigma K_1, SU(4)]$ is monic and its image is generated by $i[\langle v_5 \eta_8^2 \rangle_2]$, $\iota'\langle v_5 \rangle_1$, $\iota'\langle \alpha_1 \rangle_1$, $\iota\langle \alpha_2 \rangle_2$ (order 3), $\iota'\alpha_1\pi_K$ (order 5), $\iota\langle \alpha_1 \rangle_2$ (order 5).

$$\text{(ii)} \quad [\Sigma K_1, Sp(2)] = \mathbf{Z}/4 \cdot \langle 2[v_7]_{Sp} \rangle_2 \oplus \mathbf{Z}/2 \cdot i_2 \mu_3 \pi_K$$

$$\oplus \mathbb{Z}/3 \cdot i_2 \langle \alpha_2 \rangle_2 \oplus \mathbb{Z}/5 \cdot i_2 \langle \alpha_1 \rangle_2,$$

$(i')_*: [\Sigma K_1, Sp(2)] \rightarrow [\Sigma K_1, SU(4)]$ is monic and its image is generated by $2\chi - \iota' \langle v_5 \rangle_1$, $4i \langle [v_5 \eta_8^2] \rangle_2$, $\iota \langle \alpha_2 \rangle_2$ (order 3), $\iota \langle \alpha_1 \rangle_2$ (order 5).

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