ALGEBRAS OF HOLOMORPHIC FUNCTIONS

By

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Introduction

Let $D$ be an open subset of a Stein manifold $X$, and $\mathcal{O}(D)$ the algebra of all holomorphic functions on $D$. The spectrum of $\mathcal{O}(D)$, denoted by $\mathcal{F}_D$ is the set of continuous multiplicative linear functionals on $\mathcal{O}(D)$. We can represent $\mathcal{O}(D)$ as an algebra of functions on $\mathcal{F}_D$ by defining $\hat{f}(\zeta) = \zeta(f)$ for $f \in \mathcal{O}(D)$, $\zeta \in \mathcal{F}_D$. The set $\mathcal{F}_D$ is endowed with weak* topology. The mapping $\mathcal{O}(D) \to \mathcal{O}(D^*)$; $f \to \hat{f}$ will be referred to as the Gelfand transformation, where $\mathcal{O}(D^*)$ is the algebra of all continuous functions on $\mathcal{F}_D$. For any point $x \in D$, we define a continuous multiplicative functionals on $\mathcal{O}(D)$ by $\xi_x(f) = f(x)$, then $\xi_x \in \mathcal{F}_D$, and $x \to \xi_x$ is a continuous mapping of $D$ into $\mathcal{F}_D$. Now, since $X$ is a Stein manifold, $X$ can be imbedded to some complex number space $\mathbb{C}^N$, where $N$ is a sufficiently large positive integer. Let $F: = (f_1, \ldots, f_N): X \to \mathbb{C}^N$ the imbedding, where $f_j$'s are holomorphic functions on $X$ $(1 \leq j \leq N)$. Let $\hat{f}_j$ be the Gelfand transformation of $f_j$ $(1 \leq j \leq N)$. Then $F: = (\hat{f}_1, \ldots, \hat{f}_N)$ is a mapping of $\mathcal{F}_D$ into $\mathbb{C}^N$. By the theorem of Bishop [1] and Rossi [4], $\mathcal{F}_D$ is a Stein manifold and $F$ is a holomorphic mapping of $\mathcal{F}_D$ into $\mathbb{C}^N$. We put $\sigma_D(g_1, \ldots, g_k): = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k; \text{ the ideal generated by } (\lambda_1 - g_1), \ldots, (\lambda_k - g_k) \text{ is different from } \mathcal{O}(D)\}$, where $g_1, \ldots, g_k$ are elements of $\mathcal{O}(D)$. The set $\sigma_D(g_1, \ldots, g_k)$ is called the joint spectrum of $g_1, \ldots, g_k$.

In this note, we shall show the following theorem.

Theorem. Let $D$ be an open subset of a Stein manifold $X$, and $\mathcal{O}(D)$ the algebra of all holomorphic functions on $D$. Let $F: = (f_1, \ldots, f_N): X \to \mathbb{C}^N$ be an imbedding, where $N$ is a sufficiently large positive integer. Then the open set $D$ is $p_\tau$-convex in the sense of Docquier-Grauert [2], if and only if $\sigma_D(f_1, \ldots, f_N) = F(D)$.
Proof of theorem

Let $g_1, \ldots, g_n$ be holomorphic functions on $D$, and $\hat{g}_1, \ldots, \hat{g}_n$ the Gelfand transformations of $g_1, \ldots, g_n$. The mapping $\hat{G} = (\hat{g}_1, \ldots, \hat{g}_n)$ is a holomorphic mapping of $\mathcal{D}$ into $\mathbb{C}^n$. Then we have the following lemma.

**Lemma.** $\sigma_D(g_1, \ldots, g_n) = \hat{G}(\mathcal{D})$.

**Proof.** If $\lambda \in \sigma_D(g_1, \ldots, g_n)$ for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, then there exist holomorphic functions $h_1, \ldots, h_n$ on $D$ such that $1 = \sum h_j \cdot (\lambda_j - g_j)$. Since $1 = \zeta(1) = \zeta(\sum_{j=1}^n h_j \cdot (\lambda_j - g_j)) = \sum_{j=1}^n \zeta(h_j) \cdot (\lambda_j - \zeta(g_j))$ for any $\zeta \in \mathcal{D}$, $\lambda_j \neq \zeta(g_j)$ for some $j$ $(1 \leq j \leq n)$. Thus $\lambda \in \hat{G}(\mathcal{D})$. Therefore $\sigma_D(g_1, \ldots, g_n) \subset \hat{G}(\mathcal{D})$. Conversely, if $\lambda \in \hat{G}(\mathcal{D})$, then there exist $\hat{g}_j$ $(1 \leq j \leq n)$ such that $\lambda_j - \hat{g}_j(\zeta) \neq 0$ for any $\zeta \in \mathcal{D}$. Thus $\zeta \cdot (\lambda_j - g_j) \neq 0$ for any $\sigma \in \mathcal{D}$. Hence $(\lambda_j - g_j)^{-1} \in \mathcal{D}(D)$, namely $\lambda \in \sigma_D(g_1, \ldots, g_n)$. Therefore $\sigma_D(g_1, \ldots, g_n) = \hat{G}(\mathcal{D})$.

**Proof of Theorem.** We put $H_\varepsilon := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n; |z_1| < 1 + \varepsilon, |z_j| < 1 (j = 2, \ldots, n)\} \cup \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n; 1 - \varepsilon < |z_1| < 1 + \varepsilon, |z_j| < 1 + \varepsilon (j = 2, \ldots, n)\}$, and $P_\varepsilon := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n; |z_j| < 1 + \varepsilon (1 \leq j \leq n)\}$. If $D$ is not $p_\gamma$-convex, then there exists a continuous mapping $\varphi$ of $P_\varepsilon$ in $X$ such that the set $\varphi(H_\varepsilon)$ is contained in $D$ but the set $\varphi(P_\varepsilon)$ is not contained in $D$. For every holomorphic function $f \in \mathcal{O}(D)$, there exists a holomorphic function $h$ on $P_\varepsilon$ such that $f|_{H_\varepsilon} = h|_{H_\varepsilon}$. Now, there exist a point $x_0$ in $X$, and a point $z_0$ in $P_\varepsilon - H_\varepsilon$ such that $x_0 = \varphi(z_0)$, $x_0 \in \varphi(P_\varepsilon)$ but $x_0 \in D$. We put $\zeta_{x_0} = \zeta(x_0)$. Then $\zeta_{x_0} \in \mathcal{D}$, because such a function $h$ is uniquely determined. Since $\hat{F}(\mathcal{D}) = \sigma_D(f_1, \ldots, f_n)$, $F(D)$ by above lemma and the assumption, there exists a point $x \in D$ such that $f_\zeta(x_0) = \zeta_\zeta(x_0) = \zeta(f_j(x_0))$ $(1 \leq j \leq N)$. Since $f_j$'s are holomorphic on $X$, $\zeta_{x_0}(f_j) = \hat{f}_\zeta(f_j(x_0))$ $(1 \leq j \leq N)$. Since the mapping $F = (f_1, \ldots, f_n)$ $X \rightarrow \mathbb{C}^N$ is an imbedding, $x_0 = x$. Thus $x_0 \in D$. This is a contradiction.

Conversely, if $D$ is $p_\gamma$-convex, then $D$ is a Stein open subset of $X$ by the theorem of Docquier-Grauert [2]. We have only to show that $\hat{F}(\mathcal{D}) \subset \mathcal{F}(D)$. Now, the ideal generated by $(f_1 - \zeta(f_1)), \ldots, (f_N - \zeta(f_N))$, where $\zeta \in \mathcal{D}$, has common zero in $D$, because $D$ is a Stein open set. Thus there exists a point $x \in D$ such that $f_j(x) = \zeta(f_j)$ $(1 \leq j \leq N)$. Therefore $\hat{F}(\mathcal{D}) \subset \mathcal{F}(D)$.

**Remark.** By the theorem of Docquier-Grauert [2], $D$ is Stein if and only if $\mathcal{F}(D) = \sigma_D(f_1, \ldots, f_n)$. In particular, let $D$ be an open subset of $\mathbb{C}^n$, and $z_1, \ldots, z_n$
the coordinate functions on $\mathbb{C}^n$. Then $D$ is an open set of holomorphy if and only if $D = \sigma_D(z_1, \ldots, z_n)$, in fact $D$ is homeomorphic to $\mathscr{H}_D$.

References


