ON A PROBLEM OF KUJALA

By

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1. Introduction

In [4], R. O. Kujala raised three problems. One of them is on the complete characterization of the zero sets of functions in the Nevanlinna class, by the Blaschke condition. This problem was solved affirmatively by G. M. Henkin [3] and H. Skoda [6], independently. In this paper we shall study one of the other two problems of Kujala.

Let $n \geq 2$ be an integer. Let $H^\infty(B_n)$ denote the space of bounded holomorphic functions in the open unit ball B_n in the complex n -dimensional Euclidean space $\mathbb{C}^n$. If $f \in H^\infty(B_n)$ and $f(0) \neq 0$, then the zero-divisor v_f of f satisfies the condition

$$\sup_{\zeta \in \partial B_n} N_{v_f}(1; \zeta) < \infty,$$

where ∂B_n is the boundary of B_n (See below §2, Lemma 2.). Kujala asked the converse problem. Let μ be a positive divisor on B_n . Suppose $\text{supp } \mu \not\equiv 0$, where $\text{supp } \mu$ is the support of μ . Then, is the condition

$$\sup_{\zeta \in \partial B_n} N_\mu(1; \zeta) < \infty$$

sufficient to be $\mu = v_f$ for some $f \in H^\infty(B_n)$? By giving a counter example, we shall answer negatively to this problem.

2. Divisors of bounded holomorphic functions in B_n

Let Ω be a connected open subset of $\mathbb{C}^n$. Let f be a holomorphic function in Ω . Take a point $a \in \Omega$. Assume $f \not\equiv 0$ in Ω . Then a series

$$f(z) = \sum_{k=m}^{\infty} P_k(z-a)$$

converges in some neighborhood of a and represents f in this neighborhood. Here P_k is a homogeneous polynomial of degree k and $P_m \not\equiv 0$. The polynomials P_k depend on f and a only. The integer

$$v_f(a) = m \geq 0$$

is called the zero multiplicity of f at a . The integer-valued function v_f defined in Ω is called the zero-divisor of f .

Let v be a non-negative integer-valued function defined in Ω . Then v is called a positive divisor on Ω if and only if it is locally the zero-divisor of some holomorphic function, that is, for each point $a \in \Omega$ there exist a connected neighborhood V of a and a holomorphic function f in V such that $f \not\equiv 0$ and $v = v_f$ in V .

Since the open ball B_n is a Cousin II domain, every positive divisor on B_n is the zero-divisor of some holomorphic function in B_n (See e.g. [2], Chap. V, §2.).

Let μ be a positive divisor on the open unit disc B_1 in the complex plane $\mathbf{C}$. Assume $\text{supp } \mu \not\equiv 0$.

Define

$$n_\mu(r) = \sum_{\lambda \in rB_1} \mu(\lambda),$$

$$N_\mu(r) = \int_0^r n_\mu(t) \frac{dt}{t}$$

for $0 < r \leq 1$, where $rB_1 = \{\lambda \in \mathbf{C}; |\lambda| < r\}$.

LEMMA 1. (See Kujala [4], p. 252.) Let μ be a positive divisor on B_1 . Suppose $\text{supp } \mu \not\equiv 0$. Then $N_\mu(1) < \infty$ if and only if μ satisfies the Blaschke condition i.e.

$$\sum_{\lambda \in B_1} \mu(\lambda) (1 - |\lambda|) < \infty.$$

PROOF. There exists a holomorphic function f in B_1 such that $\mu = v_f$. By the ordinary Jensen's formula,

$$\log |f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta - N_\mu(r)$$

for $0 < r < 1$. Hence,

$$(1) \quad N_\mu(1) < \infty \text{ if and only if } \sup_{0 < r < 1} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta < \infty.$$

On the other hand,

$$(2) \quad \sup_{0 < r < 1} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta < \infty \quad \text{if and only if} \quad \sum_{\lambda \in B_1} \mu(\lambda) (1 - |\lambda|) < \infty$$

(See e.g. [1], p. 18.). By (1) and (2), the lemma is proved.

Let $n \geq 2$ be an integer. Let v be a positive divisor on B_n . Assume $\text{supp } v \neq \emptyset$. Take a boundary point $\zeta \in \partial B_n$.

Define

$$v[\zeta](\lambda) = v(\lambda\zeta) \quad \text{for } \lambda \in B_1.$$

Then $v[\zeta]$ is a positive divisor on B_1 and $\text{supp } v[\zeta] \neq \emptyset$.

Define

$$n_v(r; \zeta) = n_{v[\zeta]}(r),$$

$$N_v(r; \zeta) = N_{v[\zeta]}(r)$$

for $\zeta \in \partial B_n$, $0 < r \leq 1$.

LEMMA 2. (See Kujala [4], p. 260.) Suppose $f \in H^\infty(B_n)$ and $f(0) \neq 0$. Then the zero-divisor v_f of f satisfies the condition

$$\sup_{\zeta \in \partial B_n} N_{v_f}(1; \zeta) < \infty.$$

PROOF. By the ordinary Jensen's formula,

$$\log |f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta}\zeta)| d\theta - N_{v_f}(r; \zeta)$$

for $\zeta \in \partial B_n$, $0 < r < 1$. Therefore,

$$\begin{aligned} \sup_{\zeta \in \partial B_n} N_{v_f}(1; \zeta) &= \sup_{\zeta \in \partial B_n, 0 < r < 1} N_{v_f}(r; \zeta) \\ &= \sup_{\zeta \in \partial B_n, 0 < r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta}\zeta)| d\theta - \log |f(0)| \\ &\leq \log \frac{\|f\|_\infty}{|f(0)|} < \infty, \end{aligned}$$

where $\|f\|_\infty = \sup_{z \in B_n} |f(z)|$.

3. The Main result

THEOREM. Let $n \geq 2$ be an integer. Then there exists a positive divisor

μ on B_n such that $\text{supp } \mu \ni 0$ and

$$\sup_{\zeta \in \partial B_n} N_\mu(1; \zeta) < \infty$$

but $\mu \neq v_f$ for any $f \in H^\infty(B_n)$.

PROOF. Put

$$(1) \quad r_k = 1 - 2^{-k}, \quad k = 1, 2, 3, \dots,$$

$$(2) \quad T_k = \{r_k e^{i\theta}; \theta = \frac{2\pi j}{2^k}, 1 \leq j \leq 2^k\},$$

$$(3) \quad E_k = \{(z_1, z_2, \dots, z_n) \in B_n; z_1 \in T_k\},$$

$$(4) \quad E = \bigcup_{k=1}^{\infty} E_k.$$

By (1), the infinite product

$$(5) \quad P(\lambda) = \prod_{k=1}^{\infty} \left\{ 1 - \left(\frac{\lambda}{r_k} \right)^{2^k} \right\}$$

converges uniformly on every compact subset of B_1 and defines a holomorphic function in B_1 . We note

$$(6) \quad Z(P) \equiv \{\lambda \in B_1; P(\lambda) = 0\} = \bigcup_{k=1}^{\infty} T_k$$

and

$$(7) \quad |P(\lambda)| \leq \frac{c}{(1 - |\lambda|^2)^3} \quad (\lambda \in B_1),$$

where c is a positive constant (See Rudin [5], p. 138.).

For $(z_1, z_2, \dots, z_n) \in B_n$, define

$$(8) \quad g(z_1, z_2, \dots, z_n) = P(z_1).$$

Then g is a holomorphic function in B_n and

$$(9) \quad Z(g) \equiv \{z \in B_n; g(z) = 0\} = E.$$

Put $e_1 = (1, 0, \dots, 0)$. Then, by (1), (2), (6) and (8),

$$(10) \quad \sum_{\lambda \in B_1} v_g[e_1](\lambda)(1 - |\lambda|) = \sum_{\lambda \in Z(P)} v_g[e_1](\lambda)(1 - |\lambda|) = \sum_{k=1}^{\infty} 2^k(1 - r_k) = \sum_{k=1}^{\infty} 1 = \infty.$$

By Lemma 1 and (10),

$$(11) \quad N_{v_g}(1; e_1) = N_{v_g[e_1]}(1) = \infty.$$

By Lemma 2 and (11),

$$(12) \quad v_g \neq v_f \quad \text{for any } f \in H^\infty(B_n).$$

Next, define a function h in $\bar{B}_n$ as follows: for $z = (z_1, z_2, \dots, z_n) \in \bar{B}_n$,

$$(13) \quad \begin{aligned} h(z) &= (z_2^2 + \dots + z_n^2)^4 P(z_1) & \text{if } |z_1| < 1, \\ h(z) &= 0 & \text{if } |z_1| = 1. \end{aligned}$$

Then, by (8),

$$(14) \quad h(z) = (z_2^2 + \dots + z_n^2)^4 P(z_1) = (z_2^2 + \dots + z_n^2)^4 g(z)$$

for $z = (z_1, z_2, \dots, z_n) \in B_n$, and so h is holomorphic in B_n .

By (7) and (13),

$$(15) \quad |h(z)| \leq c(|z_2|^2 + \dots + |z_n|^2)$$

for $z = (z_1, z_2, \dots, z_n) \in \bar{B}_n$. Hence

$$(16) \quad h \in H^\infty(B_n).$$

(In fact, it follows from (13) and (15) that h is continuous in $\bar{B}_n$).

By (9) and (14),

$$\begin{aligned} Z(h) &\equiv \{z \in B_n; h(z) = 0\} \\ &= E \cup \{z = (z_1, z_2, \dots, z_n) \in B_n; z_2^2 + \dots + z_n^2 = 0\}. \end{aligned}$$

Choose a point $w \in B_n \setminus Z(h)$. Since $\text{Aut}(B_n)$ (the group of biholomorphic mappings of B_n onto B_n) acts transitively on B_n (See e.g. [5], p. 27.), there exists a $\psi \in \text{Aut}(B_n)$ such that $\psi(0) = w$.

Put

$$(17) \quad \tilde{g} = g \circ \psi \quad \text{and} \quad \tilde{h} = h \circ \psi.$$

Then $\tilde{g}$ and $\tilde{h}$ are holomorphic functions in B_n .

Put $\mu = v_{\tilde{g}}$. Since the space $H^\infty(B_n)$ is invariant under $\text{Aut}(B_n)$, By (12) and (17),

$$\mu \neq v_f \quad \text{for any } f \in H^\infty(B_n).$$

By (16) and (17),

$$(18) \quad \tilde{h} \in H^\infty(B_n).$$

Since $\tilde{h}(0) = h(\psi(0)) = h(w) \neq 0$, $0 \notin \text{supp } \nu_{\tilde{h}}$.

Hence $0 \notin \text{supp } \mu$, by (14) and (17).

Finally, by (14), (17), (18) and Lemma 2,

$$\sup_{\zeta \in \partial B_n} N_\mu(1; \zeta) \leq \sup_{\zeta \in \partial B_n} N_{\nu_{\tilde{h}}}(1; \zeta) < \infty.$$

Q. E. D.

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