A CHARACTERIZATION OF BOUNDED HOLOMORPHIC FUNCTIONS

By

Yasuo MATSUGU

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1. Introduction

Let f be a holomorphic function in the open unit ball B_n of the complex n -dimensional Euclidean space $\mathbf{C}^n$. Associate with each boundary point $\zeta \in \partial B_n$ the function f_ζ defined by

$$f_\zeta(\lambda) = f(\lambda\zeta) \quad (\lambda \in B_1).$$

We call the functions f_ζ the slice functions of f .

Let $H^p(B_n)$ be the Hardy spaces on B_n ($0 < p < \infty$). If $f \in H^p(B_n)$, $0 < p < \infty$, then $f_\zeta \in H^p(B_1)$ for almost all $\zeta \in \partial B_n$ (See [3], §4.1.2.). However, $f \in H^p(B_n)$, $0 < p < \infty$, does not imply $f_\zeta \in H^p(B_1)$ for all $\zeta \in \partial B_n$. On the contrary, the situation is different in the case of the space $H^\infty(B_n)$ of bounded holomorphic functions in B_n , and $f \in H^\infty(B_n)$ implies $f_\zeta \in H^\infty(B_1)$ for all $\zeta \in \partial B_n$. In the present paper we study the converse problem. The first result is that the condition $f_\zeta \in H^\infty(B_1)$ for all $\zeta \in \partial B_n$ is not sufficient for $f \in H^\infty(B_n)$. We shall give a counter example.

THEOREM 1. Let $n \geq 2$ be a natural number. Then there exists a holomorphic function f in B_n such that $f_\zeta \in H^\infty(B_1)$ for all $\zeta \in \partial B_n$ but $f \notin H^\infty(B_n)$.

However, if we use a kind of generalized slice function instead of slice function, the converse turns out valid:

THEOREM 2. Let M be a separable, connected, complex manifold. Then a holomorphic function f in M is bounded if and only if $f \circ \psi \in H^\infty(B_1)$ for any holomorphic mapping ψ of B_1 into M .

2. Lemma 1. There exists a $g \in H^2(B_2)$ such that $g_\zeta \in H^\infty(B_1)$ for all $\zeta \in \partial B_2$ but $g \notin H^\infty(B_2)$.

PROOF. We use an example of Rudin ([5], §7.4.4.).

Put $c_m=0$ unless m is a power of 16. If $m=16^k$, put $c_m=2^{-k}$ ($k=1, 2, 3, \dots$). Then

$$\sum_{m=1}^{\infty} c_m = 1, \quad \sum_{m=1}^{\infty} m^{1/2} c_m^2 = \infty.$$

Define

$$(1) \quad h(z, w) = \sum_{m=1}^{\infty} c_m \binom{2m}{m}^{-1/2} \frac{w^{2m+1}}{(1-z)^{m+1}},$$

where $\binom{2m}{m} = \frac{(2m)!}{(m!)^2}$ means a binomial coefficient. Then the following conditions (2), (3) and (4) hold.

$$(2) \quad h \in H^2(B_2),$$

$$(3) \quad h \text{ is continuous on } \bar{B}_2 \text{ except one boundary point } e_1 = (1, 0) \in \partial B_2,$$

$$(4) \quad h \notin H^\infty(B_2).$$

(See [5], §7.4.4. pp. 147–149.)

Next, choose a positive number ε with $0 < \varepsilon < 1$.

Put

$$(5) \quad \Phi(z, w) = \left(\frac{\sqrt{1-\varepsilon^2}z}{1-\varepsilon w}, \frac{w-\varepsilon}{1-\varepsilon w} \right),$$

$$\Psi(z, w) = \left(\frac{\sqrt{1-\varepsilon^2}z}{1+\varepsilon w}, \frac{w+\varepsilon}{1+\varepsilon w} \right).$$

Then Φ is an automorphism of B_2 i.e. Φ is a biholomorphic mapping of B_2 onto B_2 and a homeomorphism of $\bar{B}_2$ onto $\bar{B}_2$. Ψ is the inverse mapping of Φ : $\Psi = \Phi^{-1}$ (See [2].). Put $\zeta_0 = \Psi(e_1) = (\sqrt{1-\varepsilon^2}, \varepsilon) \in \partial B_2$. Define $g = h \circ \Phi$. Since the spaces $H^2(B_2)$ and $H^\infty(B_2)$ are invariant under the automorphisms of B_2 ([5], p. 85), by (2), (3) and (4), the following conditions (6), (7) and (8) hold.

$$(6) \quad g \in H^2(B_2),$$

$$(7) \quad g \text{ is continuous on } \bar{B}_2 \text{ except one boundary point } \zeta_0,$$

$$(8) \quad g \notin H^\infty(B_2).$$

From the definition of slice function and (6), (7), it follows that $g_\zeta \in H^\infty(B_1)$

for $\zeta \in \partial B_2 \setminus \{e^{i\theta}\zeta_0; \theta \in \mathbf{R}\}$. We shall prove that $g_\zeta \in H^\infty(B_1)$ for $\zeta = e^{i\theta}\zeta_0$, $\theta \in \mathbf{R}$.

By (1) and (5), for any $\lambda \in B_1$,

$$\begin{aligned}
 (9) \quad g_\zeta(\lambda) &= g(\lambda\zeta) = h(\Phi(\lambda\zeta)) \\
 &= h\left(\frac{(1-\varepsilon^2)\lambda e^{i\theta}}{1-\varepsilon^2\lambda e^{i\theta}}, \frac{\varepsilon(\lambda e^{i\theta}-1)}{1-\varepsilon^2\lambda e^{i\theta}}\right) \\
 &= \sum_{m=1}^{\infty} c_m \binom{2m}{m}^{-1/2} \left\{ \frac{\varepsilon(\lambda e^{i\theta}-1)}{1-\varepsilon^2\lambda e^{i\theta}} \right\}^{2m+1} / \left\{ 1 - \frac{(1-\varepsilon^2)\lambda e^{i\theta}}{1-\varepsilon^2\lambda e^{i\theta}} \right\}^{m+1} \\
 &= - \sum_{m=1}^{\infty} c_m \binom{2m}{m}^{-1/2} \varepsilon^{2m+1} \left(\frac{1-\lambda e^{i\theta}}{1-\varepsilon^2\lambda e^{i\theta}} \right)^m.
 \end{aligned}$$

Put $T(\lambda) = \frac{1-\lambda e^{i\theta}}{1-\varepsilon^2\lambda e^{i\theta}}$. Then

$$T(B_1) = \left\{ z \in \mathbf{C}; \left| z - \frac{1}{1+\varepsilon^2} \right| < \frac{1}{1+\varepsilon^2} \right\}.$$

Thus

$$(10) \quad \left| \frac{1-\lambda e^{i\theta}}{1-\varepsilon^2\lambda e^{i\theta}} \right| < \frac{2}{1+\varepsilon^2} \quad \text{for } \lambda \in B_1.$$

By Stirling's formula

$$(11) \quad \binom{2m}{m} \sim \frac{4^m}{\sqrt{\pi m}}.$$

Since $0 \leq c_m \leq 1$, (9), (10) and (11) give

$$\begin{aligned}
 |g_\zeta(\lambda)| &\leq \sum_{m=1}^{\infty} c_m \binom{2m}{m}^{-1/2} \varepsilon^{2m+1} \left| \frac{1-\lambda e^{i\theta}}{1-\varepsilon^2\lambda e^{i\theta}} \right|^m \\
 &\leq C \sum_{m=1}^{\infty} m^{1/4} \varepsilon^{2m+1} \left(\frac{1}{1+\varepsilon^2} \right)^m < \infty
 \end{aligned}$$

for $\lambda \in B_1$, where C is a positive constant. Therefore $g_\zeta \in H^\infty(B_1)$ for $\zeta = e^{i\theta}\zeta_0$, $\theta \in \mathbf{R}$.

3. Proof of Theorem 1. If $n=2$, then Theorem 1 follows from Lemma 1. Suppose $n \geq 3$. By Lemma 1, there exists a holomorphic function g in B_2 such that $g_\zeta \in H^\infty(B_1)$ for all $\zeta \in \partial B_2$ but $g \notin H^\infty(B_2)$.

Define

$$(12) \quad f(z_1, z_2, z_3, \dots, z_n) = g(z_1, z_2).$$

Then f is a holomorphic function in B_n and $f \notin H^\infty(B_n)$. It remains to show that $f_\zeta \in H^\infty(B_1)$ for all $\zeta \in B_n$.

Take any $\zeta = (\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_n) \in \partial B_n$. Put $\zeta' = (\zeta_1, \zeta_2) \in \bar{B}_2$. If $(\zeta_3, \dots, \zeta_n) = (0, \dots, 0)$, then $\zeta' \in \partial B_2$ and $f_\zeta = g_{\zeta'}$, so that, $f_\zeta \in H^\infty(B_1)$. If $(\zeta_3, \dots, \zeta_n) \neq (0, \dots, 0)$, then $\zeta' \in B_2$, so that, $|\lambda\zeta|^2 + |\lambda\zeta_2|^2 \leq |\zeta_1|^2 + |\zeta_2|^2 < 1$ for $\lambda \in \bar{B}_1$. It follows that f_ζ is holomorphic in a neighborhood of $\bar{B}_1$, because $f_\zeta(\lambda) = f(\lambda\zeta) = g(\lambda\zeta_1, \lambda\zeta_2)$ by (12). Hence $f_\zeta \in H^\infty(B_1)$.

4. Lemma 2. Let f be a holomorphic function in a domain Ω in $\mathbb{C}^n$. If $f \circ \psi \in H^\infty(B_1)$ for any holomorphic mapping ψ of B_1 into Ω , then the function f is bounded in Ω .

PROOF. Suppose that f were not bounded in Ω . Then there exists a sequence $\{x_j\}_{j=1,2,\dots}$ of points in Ω such that

$$(13) \quad \lim_{j \rightarrow \infty} |f(x_j)| = \infty.$$

By the continuity of the function f , there exist neighborhoods V_j of the points x_j in Ω such that

$$(14) \quad |f(x)| \geq \frac{1}{2} |f(x_j)|, \quad x \in V_j,$$

for $j=1, 2, \dots$. Moreover, by a theorem of Rudin [4], there exists a holomorphic mapping ψ of B_1 into Ω such that the set $\psi(B_1)$ is dense in Ω . Hence there exist points $y_j \in V_j \cap \psi(B_1)$ and points $z_j \in B_1$ with $y_j = \psi(z_j)$ ($j=1, 2, \dots$).

By (14),

$$|(f \circ \psi)(z_j)| = |f(y_j)| \geq \frac{1}{2} |f(x_j)|,$$

so that, by (13),

$$\lim_{j \rightarrow \infty} |(f \circ \psi)(z_j)| = \infty.$$

Therefore $f \circ \psi \notin H^\infty(B_1)$. Hence the function f is bounded in Ω .

5. Proof of Theorem 2. Suppose that f is an unbounded holomorphic function in the separable, connected complex manifold M of dimension n .

Then there exists a sequence $\{x_j\}_{j=1,2,\dots}$ of points in M such that

$$(15) \quad \lim_{j \rightarrow \infty} |f(x_j)| = \infty.$$

By a theorem of Fornaess-Stout [1], there exists a locally biholomorphic mapping Φ of the open unit polydisc B_1^n onto M . Hence there exists a sequence $\{y_j\}_{j=1,2,\dots}$ of points in B_1^n such that $\Phi(y_j) = x_j$ ($j = 1, 2, \dots$).

The function $f \circ \Phi$ is holomorphic in B_1^n and, by (15),

$$\lim_{j \rightarrow \infty} |(f \circ \Phi)(y_j)| = \infty.$$

Therefore $f \circ \Phi \notin H^\infty(B_1^n)$. It follows from Lemma 2 that there exists a holomorphic mapping ϕ of B_1 into B_1^n with $f \circ \Phi \circ \phi \notin H^\infty(B_1)$. Put $\psi = \Phi \circ \phi$. Then ψ is a holomorphic mapping of B_1 into M and $f \circ \psi \notin H^\infty(B_1)$. This completes the proof.

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Department of Mathematics
Faculty of Science
Shinshu University
Matsumoto, Nagano
Japan