PLURIHARMONIC FUNCTIONS AS THE REAL PARTS OF HOLOMORPHIC FUNCTIONS

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1. Introduction

Let Ω be an arbitrary complex manifold. The real part of a holomorphic function in Ω is a real-valued pluriharmonic function in Ω . But its converse is not necessarily true. In fact, there can happen to exist a real-valued pluriharmonic function in Ω which is not the real part of any holomorphic function in Ω . The purpose of this paper is to provide a necessary and sufficient condition that all the real-valued pluriharmonic functions in Ω are the real parts of holomorphic functions in Ω , in case when Ω is a Riemann domain over a Stein manifold. We shall show that for such Ω the condition is $H^1(\tilde{\Omega}, \mathbf{Z})=0$, where $\tilde{\Omega}$ is the envelope of holomorphy of Ω .

2. Germs of pluriharmonic functions

Let X be a complex manifold of dimension n . Let u be a 2 times continuously differentiable complex-valued function in X . Let $z_1, \dots, z_n$ be a local complex coordinate system in a neighborhood V of a point $x \in X$. Define

$$\partial u = \sum_{j=1}^n \frac{\partial u}{\partial z_j} dz_j, \quad \bar{\partial} u = \sum_{j=1}^n \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j, \quad du = \partial u + \bar{\partial} u$$

and

$$\partial \bar{\partial} u = \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \quad \text{in } V.$$

If $\partial \bar{\partial} u = 0$ in V , then u is said to be pluriharmonic at x . We shall say that the function u is pluriharmonic in X if it is pluriharmonic at every point of X .

Let $\mathfrak{O} = \mathfrak{O}(X)$ and $\mathfrak{P} = \mathfrak{P}(X)$ denote the sheaf of germs of holomorphic func-

tions on X and that of germs of real-valued pluriharmonic functions on X , respectively. If R is a commutative ring and S is a sheaf of R -modules over X , then we shall denote by $H^k(X, S)$ the k -th Čech cohomology module of X with coefficients in S ($k=0, 1, 2, \dots$). $H^0(X, S)=\Gamma(X, S)$ is the module of global sections in S .

Our starting point is the following fact:

LEMMA 1 Let B be an open ball in the complex n -dimensional Euclidean space $\mathbb{C}^n$. Let u be a 2 times continuously differentiable real-valued function in B . Then u is pluriharmonic in B if and only if u is the real part of some holomorphic function B .

PROOF ([3], pp. 271–272). Suppose $\partial\bar{\partial}u=0$ in B .

$$\begin{aligned} \text{Then} \quad d(\sqrt{-1}\partial u - \sqrt{-1}\bar{\partial}u) &= \sqrt{-1}d\partial u - \sqrt{-1}d\bar{\partial}u \\ &= \sqrt{-1}\bar{\partial}\partial u - \sqrt{-1}\partial\bar{\partial}u = -2\sqrt{-1}\partial\bar{\partial}u = 0 \quad \text{in } B. \end{aligned}$$

Since u is real-valued, $\sqrt{-1}\partial u - \sqrt{-1}\bar{\partial}u$ is a real 1-form in B . By the ordinary Poincaré lemma, there is a continuously differentiable real-valued function v in B such that $dv = \sqrt{-1}\partial u - \sqrt{-1}\bar{\partial}u$ and so $\bar{\partial}v = -\sqrt{-1}\bar{\partial}u$ in B . Put $f = u - \sqrt{-1}v$. Then f is a continuously differentiable complex-valued function in B and $\bar{\partial}f = \bar{\partial}u - \sqrt{-1}\bar{\partial}v = 0$ in B . Thus f is holomorphic and $u = \operatorname{Re} f$ in B .

Conversely, suppose $u = \operatorname{Re} f$ for some holomorphic function f in B . Then $u = \frac{1}{2}(f + \bar{f})$, $\bar{\partial}f = 0$ and $\partial\bar{f} = 0$ in B . Thus

$$\partial\bar{\partial}u = \frac{1}{2}\partial\bar{\partial}f + \frac{1}{2}\partial\bar{\partial}\bar{f} = \frac{1}{2}\partial\bar{\partial}f - \frac{1}{2}\bar{\partial}\partial\bar{f} = 0 \quad \text{in } B.$$

It follows from Lemma 1 that the sheaf homomorphism obtained by corresponding a holomorphic function f to its real part $\operatorname{Re} f$,

$$r: \mathfrak{O} \longrightarrow \mathfrak{P}$$

is surjective. In addition, the sheaf homomorphism obtained by corresponding a real number α to a purely imaginary number $\sqrt{-1}\alpha$,

$$i: \mathbf{R} \longrightarrow \mathfrak{O}$$

is injective, where $\mathbf{R}$ is the constant sheaf of the real number field over X . Since a holomorphic function whose real part is vanishing is locally a purely imaginary

constant function, the sequence of sheaves over X

$$0 \longrightarrow \mathbf{R} \xrightarrow{i} \mathfrak{D} \xrightarrow{r} \mathfrak{P} \longrightarrow 0$$

is exact. The induced cohomology sequence

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathbf{R}) \longrightarrow H^0(X, \mathfrak{D}) \longrightarrow H^0(X, \mathfrak{P}) \longrightarrow H^1(X, \mathbf{R}) \\ \longrightarrow H^1(X, \mathfrak{D}) \longrightarrow \dots \end{aligned}$$

is also exact. Hence the homomorphism $H^0(X, \mathfrak{D}) \rightarrow H^0(X, \mathfrak{P})$ is surjective when $H^1(X, \mathbf{R}) = 0$. We have thus obtained

LEMMA 2. If $H^1(X, \mathbf{R}) = 0$, then the real-valued pluriharmonic functions in X are the real parts of holomorphic functions in X .

By the universal-coefficient theorem for cohomology, $H^k(X, \mathbf{R}) \cong H^k(X, \mathbf{Z}) \otimes \mathbf{R}$ ($k=0, 1, 2, \dots$) (See e.g. [5], p. 246.). Hence the following lemma holds:

LEMMA 3. $H^1(X, \mathbf{R}) = 0$ if and only if $H^1(X, \mathbf{Z}) = 0$.

3. A theorem of Docquier-Grauert

Let M be a Stein manifold of dimension n . Let (Ω, ψ) be a Riemann domain over M , that is, Ω is a complex manifold and ψ is a locally biholomorphic mapping of Ω into M . We use the following notations:

$$D = \{(z_1, z_2, \dots, z_n) \in \mathbf{C}^n; |z_1| \leq 1, |z_j| < 1, j=2, \dots, n\},$$

$$\delta D = \{(z_1, z_2, \dots, z_n) \in \mathbf{C}^n; |z_1| = 1, |z_j| < 1, j=2, \dots, n\},$$

$$\mathring{D} = \{(z_1, z_2, \dots, z_n) \in \mathbf{C}^n; |z_j| < 1, j=1, \dots, n\},$$

$$\bar{D} = \{(z_1, z_2, \dots, z_n) \in \mathbf{C}^n; |z_j| \leq 1, j=1, \dots, n\}.$$

We recall some definitions and results from Docquier-Grauert [1]. A filter basis b of open and connected subsets of Ω is called a boundary point of Ω , when

- 1) b does not cluster in Ω ,
- 2) $\psi(b)$ converges to a point $x \in M$,
- 3) for each connected neighborhood $U(x)$ of x b contains just only one connected component V of $\psi^{-1}(U(x))$ and all the elements of b are thus obtained.

We call $\bar{\partial}\Omega = \{b\}$ the boundary of Ω and set $\bar{\Omega} = \Omega \cup \bar{\partial}\Omega$, $\bar{\psi}(y) = \psi(y)$ for $y \in \Omega$,

$\bar{\psi}(b)=x \in M$ for $b \in \bar{\partial}\Omega$, where x is the limit of $\psi(b)$. We can induce a topology in $\bar{\Omega}$ which makes $\bar{\Omega}$ a Hausdorff topological space and the mapping $\bar{\psi}: \bar{\Omega} \rightarrow M$ to be continuous: we take as neighborhoods of points $x \in \Omega$ the neighborhoods of the topology of Ω , and as neighborhoods of points $b_0 \in \bar{\partial}\Omega$ the sets of the form,

$$U(b_0) = V \cup \{b \in \bar{\partial}\Omega \text{ with at least one } V' \in b, V' \subset V\}, V \in b_0.$$

A continuous mapping Φ of the closed polydisc $\bar{D}$ into $\bar{\Omega} = \Omega \cup \bar{\partial}\Omega$ is called an R -mapping, when the following conditions hold:

- 1) $\Phi(\delta D) \subset \subset \Omega$, $\Phi(\bar{D}) \subset \Omega$,
- 2) $\Phi(\bar{D}) \cap \bar{\partial}\Omega \neq \emptyset$,
- 3) $\psi \circ \Phi$ has a biholomorphic extension Ψ of an open neighborhood of $\bar{D}$ into M .

A Riemann domain (Ω, ψ) over M is said to be p_7 -convex if there exist no R -mappings with respect to (Ω, ψ) .

THEOREM D-G (Docquier-Grauert [1], p. 113). A Riemann domain (Ω, ψ) over a Stein manifold M is p_7 -convex if and only if Ω is a Stein manifold.

4. Domain of pluriharmony

Let $(\tilde{\Omega}, \tilde{\psi})$ be the envelope of holomorphy of the Riemann domain (Ω, ψ) over the Stein manifold M , in the sense of Malgrange [4]. Then $\tilde{\Omega}$ is a Stein manifold.

Suppose u is a real-valued pluriharmonic function in Ω . As well as the envelope of holomorphy, by means of Malgrange [4], we can construct the domain of pluriharmony of u as follows. Let $\mathfrak{P} = \mathfrak{P}(M)$ denote the sheaf of germs of real-valued pluriharmonic functions on M . Let $\pi: \mathfrak{P} \rightarrow M$ be the canonical projection. For any point $x \in \Omega$, there exists a neighborhood $U(x)$ of x such that the restriction $\psi|_{U(x)}: U(x) \rightarrow \psi(U(x))$ is biholomorphic. The real-valued function $u \circ (\psi|_{U(x)})^{-1}$ is pluriharmonic in $\psi(U(x)) \subset M$. We define $\phi(y) = [u \circ (\psi|_{U(x)})^{-1}]_{\psi(y)}$ for any point $y \in U(x)$, where $[u \circ (\psi|_{U(x)})^{-1}]_{\psi(y)}$ means the germ of the function $u \circ (\psi|_{U(x)})^{-1}$ at the point $\psi(y)$. Then the mapping $\phi: \Omega \rightarrow \mathfrak{P}$ is locally biholomorphic. Let $\Omega = \bigcup_i \Omega_i$ be the decomposition of connected components of Ω . We denote by Ω'_i the connected component of $\mathfrak{P}$ containing the connected set $\phi(\Omega_i)$ for each i . Put $\Omega' = \bigcup_i \Omega'_i$ and $\psi' = \pi|_{\Omega'}$. Then the pair (Ω', ψ') is a Riemann domain over M and the relation $\psi' \circ \phi = \psi$ holds. We shall call (Ω', ψ') the domain of pluriharmony of u .

We define the pluriharmonic extension u' of u as follows. For any point $\bar{a} \in \Omega' \subset \mathfrak{P}$, we choose a representative v of $\bar{a}$. v is a real-valued pluriharmonic function in a neighborhood of the point $a = \psi'(\bar{a}) = \pi(\bar{a}) \in M$, and $[v]_a = \bar{a}$. Put $u'(\bar{a}) = v(a)$. Then u' is a well-defined real-valued pluriharmonic function in Ω' and the relation $u' \circ \phi = u$ holds.

LEMMA 4. Ω' is a Stein manifold.

PROOF. By Theorem D–G, it suffices to show that (Ω', ψ') is p_γ -convex. We may assume that Ω' is connected. Suppose (Ω', ψ') were not p_γ -convex. Then there exists an R -mapping $\Phi: \bar{D} \rightarrow \Omega' \cup \bar{\partial}\Omega'$. For a sufficiently small positive number ε , we define

$$E = D \cup \{(z_1, \dots, z_n) \in \mathbb{C}^n; 1 < |z_1| < 1 + \varepsilon, |z_j| < 1 + \varepsilon, j = 2, \dots, n\},$$

$$\tilde{E} = \{(z_1, \dots, z_n) \in \mathbb{C}^n; |z_j| < 1 + \varepsilon, j = 1, 2, \dots, n\}.$$

By the definition of R -mapping, we may assume as follows: Φ is a biholomorphic mapping of E into Ω' and there exists a biholomorphic mapping Ψ of $\tilde{E}$ into M such that $\Psi = \bar{\psi}' \circ \Phi$ in $E \cup \bar{D}$.

Choose a point $b \in \Phi(\bar{D}) \cap \bar{\partial}\Omega'$ and set $x = \bar{\psi}'(b) \in M$. If we take a point $z \in \bar{D}$ with $b = \Phi(z)$, then $\Psi(z) = \bar{\psi}' \circ \Phi(z) = \bar{\psi}'(b) = x$. Since the mapping $\Psi: \tilde{E} \rightarrow \Psi(\tilde{E})$ is biholomorphic and $b \in \bar{\partial}\Omega'$, there exist an open and connected neighborhood $U \subset \Psi(\tilde{E})$ of the point x , a connected component $V \in b$ of $\psi'^{-1}(U)$, and an open neighborhood W of the point z such that $W \subset \tilde{E}$, $\Phi(W \cap \bar{D}) \cap \Omega' = V$ and the restriction $\psi'|_V: V \rightarrow \psi'(V) \subset U$ is biholomorphic.

Now the real-valued function $u' \circ \Phi$ is pluriharmonic in E . Since $H^1(E, \mathbb{R}) = 0$, by Lemma 2, there exists a holomorphic function f in E such that $u' \circ \Phi = \operatorname{Re} f$ in E . Since $\tilde{E}$ is the envelope of holomorphy of E , there exists a holomorphic extension $\tilde{f}$ of f to $\tilde{E}$. We note that the real-valued function $\operatorname{Re} \tilde{f} \circ \Psi^{-1}$ is pluriharmonic in $\Psi(\tilde{E}) \supset U$.

Put $\bar{U} = \{[\operatorname{Re} \tilde{f} \circ \Psi^{-1}]_a; a \in U\}$. Then $\bar{U}$ is an open and connected subset of $\mathfrak{P}$ and the restriction $\pi|_{\bar{U}}: \bar{U} \rightarrow U$ is biholomorphic. We define $\Omega'' = \Omega' \cup \bar{U} \subset \mathfrak{P}$.

Since $\operatorname{Re} \tilde{f} = \operatorname{Re} f = u' \circ \Phi$ in E , it follows that $\operatorname{Re} \tilde{f} \circ \Psi^{-1} = u' \circ (\psi'|_V)^{-1}$ in U' where $W' = E \cap \Phi^{-1}(V)$ and $U' = \Psi(W')$. Thus $u'(\bar{a}) = (\operatorname{Re} \tilde{f} \circ \Psi^{-1})(a)$ for any point $\bar{a} \in (\psi'|_V)^{-1}(U') \subset \Omega'$ with $a = \psi'(\bar{a}) \in U'$. By the definition of u' , $\operatorname{Re} \tilde{f} \circ \Psi^{-1}$ is a representative of $\bar{a}$, and so $\bar{a} = [\operatorname{Re} \tilde{f} \circ \Psi^{-1}]_a \in \bar{U}$ for $\bar{a} \in (\psi'|_V)^{-1}(U')$. It follows that $(\psi'|_V)^{-1}(U') \subset \Omega' \cap \bar{U}$. Since the two sets Ω' and $\bar{U}$ are both open and connected, so is Ω'' . By the definition of Ω' , Ω' is a connected component of $\mathfrak{P}$.

But $\Omega' \not\subseteq \Omega''$ because $\bar{U} \not\subset \Omega'$. Thus we have arrived at a contradiction. Hence Ω' is a Stein manifold.

5. Statement and Proof of the main theorem

THEOREM. Let (Ω, ψ) be a Riemann domain over a Stein manifold M . Let $(\tilde{\Omega}, \tilde{\psi})$ denote the envelope of holomorphy of (Ω, ψ) . Then the real-valued pluriharmonic functions in Ω are the real parts of holomorphic functions in Ω if and only if $H^1(\tilde{\Omega}, \mathbf{Z})=0$.

PROOF. Since $\tilde{\Omega}$ is a Stein manifold, $H^1(\tilde{\Omega}, \mathfrak{D})=0$. It follows that the cohomology sequence

$$H^0(\tilde{\Omega}, \mathfrak{D}) \xrightarrow{r} H^0(\tilde{\Omega}, \mathfrak{P}) \longrightarrow H^1(\tilde{\Omega}, \mathbf{R}) \longrightarrow 0$$

is exact. Hence $H^1(\tilde{\Omega}, \mathbf{R})=0$ if and only if the homomorphism $r: H^0(\tilde{\Omega}, \mathfrak{D}) \rightarrow H^0(\tilde{\Omega}, \mathfrak{P})$ is surjective.

By the fact that $(\tilde{\Omega}, \tilde{\psi})$ is the envelope of holomorphy of (Ω, ψ) , there exists a locally biholomorphic mapping $\tilde{\phi}: \Omega \rightarrow \tilde{\Omega}$ such that $\psi = \tilde{\psi} \circ \tilde{\phi}$ and the induced mapping $\lambda: H^0(\tilde{\Omega}, \mathfrak{D}) \rightarrow H^0(\Omega, \mathfrak{D})$ is an isomorphism, where $\lambda(\tilde{f}) = \tilde{f} \circ \tilde{\phi}$ for $\tilde{f} \in H^0(\tilde{\Omega}, \mathfrak{D})$. We claim that the induced mapping $\lambda^*: H^0(\tilde{\Omega}, \mathfrak{P}) \rightarrow H^0(\Omega, \mathfrak{P})$ is also an isomorphism, where $\lambda^*(\tilde{u}) = \tilde{u} \circ \tilde{\phi}$ for $\tilde{u} \in H^0(\tilde{\Omega}, \mathfrak{P})$. To see this it suffices to show that λ^* is surjective.

Suppose $u \in H^0(\Omega, \mathfrak{P})$. Let (Ω', ψ') be the domain of pluriharmonicity of u . Then there exists a locally biholomorphic mapping $\phi': \Omega \rightarrow \Omega'$ such that $\psi = \psi' \circ \phi'$. Denote by u' the pluriharmonic extension of u to (Ω', ψ') . Then $u = u' \circ \phi'$. The mapping ϕ' induces the homomorphism $\lambda': H^0(\Omega', \mathfrak{D}) \rightarrow H^0(\Omega, \mathfrak{D})$, $\lambda'(f') = f' \circ \phi'$ for $f' \in H^0(\Omega', \mathfrak{D})$. Thus we obtain the $\mathbf{C}$ -algebra homomorphism

$$\lambda^{-1} \circ \lambda': H^0(\Omega', \mathfrak{D}) \longrightarrow H^0(\tilde{\Omega}, \mathfrak{D}).$$

By Lemma 4, Ω' is a Stein manifold. $\tilde{\Omega}$ is also Stein. From these facts, there exists a holomorphic mapping $\chi: \tilde{\Omega} \rightarrow \Omega'$ such that $\chi \circ \tilde{\phi} = \phi'$ (See [2], Chap. V, §7.). Define $\tilde{u} = u' \circ \chi \in H^0(\tilde{\Omega}, \mathfrak{P})$. Then $\tilde{u} \circ \tilde{\phi} = u' \circ \chi \circ \tilde{\phi} = u' \circ \phi' = u$. It follows that $\lambda^*: H^0(\tilde{\Omega}, \mathfrak{P}) \rightarrow H^0(\Omega, \mathfrak{P})$ is an isomorphism.

From the two isomorphisms $H^0(\tilde{\Omega}, \mathfrak{P}) \cong H^0(\Omega, \mathfrak{P})$ and $H^0(\tilde{\Omega}, \mathfrak{D}) \cong H^0(\Omega, \mathfrak{D})$, we know that the mapping $H^0(\tilde{\Omega}, \mathfrak{D}) \rightarrow H^0(\tilde{\Omega}, \mathfrak{P})$ is surjective if and only if the mapping $H^0(\Omega, \mathfrak{D}) \rightarrow H^0(\Omega, \mathfrak{P})$ is surjective. Hence we can conclude that $H^1(\tilde{\Omega}, \mathbf{R})=0$ if and only if the mapping $r: H^0(\Omega, \mathfrak{D}) \rightarrow H^0(\Omega, \mathfrak{P})$ is surjective.

By Lemma 3, this completes the proof.

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