ON THE NEBENHÜLLE

By

Makoto ABE

To the memory of Professor Shawich Sato

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0. Introduction

E. Bedford – J. E. Fornaess [1] and K. Diederich – J. E. Fornaess [3], [4], [5] studied the sufficient conditions that a bounded domain of holomorphy in $\mathbb{C}^n$ with smooth boundary coincides with its Nebenhülle. On the contrary, we shall study the necessary condition that a bounded domain in $\mathbb{C}^n$ coincides with its Nebenhülle. The main purpose of the present paper is to show that such a domain has a kind of disk property. The well-known Hartogs triangle does not have this property.

1. Preliminaries

Let Ω be a domain in $\mathbb{C}^n$. We denote by $\mathcal{N}(\Omega)$ the interior of the intersection of all domains of holomorphy containing $\bar{\Omega}$, and call it the Nebenhülle of the domain Ω . $\mathcal{N}(\Omega)$ is a domain of holomorphy. The notation $\mathcal{N}(\Omega)$ is the same as in K. Diederich – J. E. Fornaess [3].

If Ω is a bounded domain in $\mathbb{C}^n$ such that $\mathcal{N}(\Omega) = \Omega$, $\bar{\Omega}$ has a neighbourhood basis consisting of domains of holomorphy. This is stated in H. Behnke – P. Thullen [2] and proved in S. Sato [6]. The converse is of course true.

PROPOSITION 1. *The following two conditions are equivalent for any domain Ω in $\mathbb{C}^n$.*

- i) *For any compact set $K \subset \Omega$ there exists a domain of holomorphy Ω' such that $\bar{\Omega} \subset \Omega'$ and $\hat{K}_{\Omega'} \in \Omega$.*
- ii) *For any compact set $K \subset \Omega$ there exists a Runge pair (Ω'', Ω') of domains of holomorphy such that $\bar{\Omega} \subset \Omega'$ and $K \in \Omega'' \in \Omega$.*

PROOF. i)→ii). Put $L = \hat{K}_{\Omega'}$. Since $\hat{L}_{\Omega'} = L$, L has a neighbourhood basis consisting of polyhedral domain with respect to $\mathcal{O}(\Omega')$, the family of all holomorphic functions on Ω' . Thus we can find a polyhedral domain Ω'' with respect to $\mathcal{O}(\Omega')$ such that $L \in \Omega'' \in \Omega$. (Ω'', Ω') is a Runge pair.

ii)→i). Since (Ω'', Ω') is a Runge pair, $\hat{K}_{\Omega'} = \hat{K}_{\Omega''} \in \Omega'' \subset \Omega$.

For simplicity we shall say that a domain in $\mathbf{C}^n$ has Property (N) if one of the equivalent conditions in Proposition 1 is satisfied. Such a domain is a domain of holomorphy. Further the following proposition holds.

PROPOSITION 2. *If Ω is a domain in $\mathbf{C}^n$ that has Property (N), then $(\Omega, \mathcal{N}(\Omega))$ is a Runge pair.*

PROOF. For any compact set $K \subset \Omega$ we can find a Runge pair (Ω'', Ω') of domains of holomorphy such that $\bar{\Omega} \subset \Omega'$ and $K \in \Omega'' \in \Omega$. Since $K \subset \Omega'' \subset \Omega \subset \mathcal{N}(\Omega) \subset \Omega'$, $\hat{K}_{\Omega''} \subset \hat{K}_{\Omega} \subset \hat{K}_{\mathcal{N}(\Omega)} \subset \hat{K}_{\Omega'}$. And since (Ω'', Ω') is a Runge pair, $\hat{K}_{\Omega''} = \hat{K}_{\Omega'}$. It follows that $\hat{K}_{\Omega} = \hat{K}_{\mathcal{N}(\Omega)}$. Hence $(\Omega, \mathcal{N}(\Omega))$ is a Runge pair.

The following basic lemma is used in the proof of Propositions 5 and 6.

LEMMA 3. *Let X and Y be two complex manifolds or spaces and $\psi: X \rightarrow Y$ be a holomorphic map. Then for any subset K of X , $\psi(\hat{K}_X) \subset \psi(K)\hat{Y}$.*

2. Property (N)

PROPOSITION 4. *If Ω is a bounded domain in $\mathbf{C}^n$ such that $\mathcal{N}(\Omega) = \Omega$, then Ω has Property (N).*

PROOF. Let K be a compact set contained in Ω . Put $\varepsilon = d(K, \Omega^c)$, then $\varepsilon > 0$. Here we denote by d the ordinary Euclidean distance. Since $\partial\Omega$ is compact, there exist finitely many points $p_1, p_2, \dots, p_l \in \partial\Omega$ such that $\partial\Omega \subset \bigcup_{j=1}^l B\left(p_j, \frac{\varepsilon}{2}\right)$, where $B\left(p_j, \frac{\varepsilon}{2}\right) = \left\{x \in \mathbf{C}^n; d(x, p_j) < \frac{\varepsilon}{2}\right\}$. We can find a point $q_j \in B\left(p_j, \frac{\varepsilon}{2}\right) \setminus \bar{\Omega}$ for each j since $\bar{\Omega} = \Omega$. $\bar{\Omega} \subset \Omega \cup \left(\bigcup_{j=1}^l B\left(p_j, \frac{\varepsilon}{2}\right)\right) \setminus \{q_1, q_2, \dots, q_l\}$. There exists a domain of holomorphy Ω' such that $\bar{\Omega} \subset \Omega' \subset \Omega \cup \left(\bigcup_{j=1}^l B\left(p_j, \frac{\varepsilon}{2}\right)\right) \setminus \{q_1, q_2, \dots, q_l\}$. Let $p \in \Omega' \setminus \Omega$. It holds that for some j $d(p, q_j) \leq d(p, p_j) + d(p_j, q_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Also $q_j \notin \Omega'$. Thus $d(p, \Omega'^c) < \varepsilon$. On the other hand $d(K, \Omega'^c) \geq d(K, \Omega^c) = \varepsilon$. Since Ω' is a domain of holomorphy, $p \notin \hat{K}_{\Omega'}$. Thus we obtain that $\hat{K}_{\Omega'} \subset \Omega$.

COROLLARY. *If Ω is a bounded domain in $\mathbb{C}^n$ such that $\mathcal{N}(\Omega) = \Omega$, then for any compact set $K \subset \Omega$ we obtain that $\hat{K}_\Omega = \{x \in \bar{\Omega}; |f(x)| \leq \sup |f(y)| \text{ for all } f \in \mathcal{O}(\bar{\Omega})\}$. Especially Ω is holomorphically convex with respect to $\mathcal{O}(\bar{\Omega})$, the family of all holomorphic functions in neighbourhoods of $\bar{\Omega}$.*

PROOF. By Proposition 4 there exists a Runge pair (Ω'', Ω') of domains of holomorphy such that $K \subset \Omega'' \Subset \Omega \Subset \Omega'$. Put $L = \{x \in \bar{\Omega}; |f(x)| \leq \sup |f(y)| \text{ for all } f \in \mathcal{O}(\bar{\Omega})\}$. Since $\mathcal{O}(\Omega') \subset \mathcal{O}(\bar{\Omega})$, $L \subset \hat{K}_{\Omega'}$. Since (Ω'', Ω') is a Runge pair, $\hat{K}_{\Omega''} = \hat{K}_{\Omega'}$. Thus $L \subset \hat{K}_{\Omega''} \subset \hat{K}_\Omega$. On the other hand $\hat{K}_\Omega \subset L$.

3. Disk Property

Throughout this section we denote by Δ the unit disk in $\mathbb{C}$. Let Ω be a domain in $\mathbb{C}^n$. For simplicity we shall say that Ω has disk property if it satisfies the condition that if $\psi: \bar{\Delta} \rightarrow \mathbb{C}^n$ is a continuous map holomorphic in Δ such that $\psi(\bar{\Delta}) \subset \bar{\Omega}$ and $\psi(\partial\Delta) \subset \Omega$, then $\psi(\bar{\Delta}) \subset \Omega$. If Ω is a domain in $\mathbb{C}^n$ that has disk property, we can show that Ω has the Oka-pseudoconvexity, it follows that Ω is a domain of holomorphy.

PROPOSITION 5. *If Ω is a domain in $\mathbb{C}^n$ that has Property (N), then Ω has disk property.*

PROOF. Let $\psi: \bar{\Delta} \rightarrow \mathbb{C}^n$ be a continuous map holomorphic in Δ such that $\psi(\bar{\Delta}) \subset \bar{\Omega}$ and $\psi(\partial\Delta) \subset \Omega$. Since $\psi(\partial\Delta) \subset \Omega$ and ψ is continuous, there exists a number $r \in (0, 1)$ such that $\psi(\{z \in \mathbb{C}; r \leq |z| \leq 1\}) \subset \Omega$. $\psi(\{z \in \mathbb{C}; |z| = r\})$ is a compact set contained in Ω . By hypothesis there exists a domain of holomorphy Ω' such that $\bar{\Omega} \subset \Omega'$ and $\psi(\{z \in \mathbb{C}; |z| = r\})^\wedge \subset \Omega$. Since $\psi(\{z \in \mathbb{C}; |z| \leq r\}) \subset \Omega'$ and ψ is continuous, there exists a number $r' \in (r, 1)$ such that $\psi(D) \subset \Omega'$, where $D = \{z \in \mathbb{C}; |z| < r'\}$. $\psi|_D: D \rightarrow \Omega'$ is a holomorphic map. By Lemma 3, $\psi(\{z \in \mathbb{C}; |z| \leq r\}) = \psi(\{z \in \mathbb{C}; |z| = r\})^\wedge \subset \psi(\{z \in \mathbb{C}; |z| = r\})^\wedge \subset \Omega$. Thus we obtain that $\psi(\bar{\Delta}) \subset \Omega$.

PROPOSITION 6. *If Ω is a bounded domain in $\mathbb{C}^n$ such that $(\Omega, \mathcal{N}(\Omega))$ is a Runge pair, then Ω has disk property. Especially if Ω is a bounded domain in $\mathbb{C}^n$ such that $\mathcal{N}(\Omega) = \Omega$ then Ω has disk property.*

PROOF. $\mathcal{N}(\Omega)$ is a bounded domain and $\mathcal{N}(\mathcal{N}(\Omega)) = \mathcal{N}(\Omega)$ (see S. Sato [5]). By Propositions 4 and 5, $\mathcal{N}(\Omega)$ has disk property. Let $\psi: \bar{\Delta} \rightarrow \mathbb{C}^n$ be a continuous map holomorphic in Δ such that $\psi(\bar{\Delta}) \subset \bar{\Omega}$ and $\psi(\partial\Delta) \subset \Omega$. There

exists a number $r \in (0, 1)$ such that $\psi(\{z \in \mathbb{C}; r \leq |z| \leq 1\}) \subset \Omega$. Since $\mathcal{N}(\Omega)$ has disk property, $\psi(\Delta) \subset \mathcal{N}(\Omega)$. By Lemma 3, $\psi(\{z \in \mathbb{C}; |z| \leq r\}) = \psi(\{z \in \mathbb{C}; |z| = r\})^\circ \subset \psi(\{z \in \mathbb{C}; |z| = r\})^\circ_{\mathcal{N}(\Omega)} = \psi(\{z \in \mathbb{C}; |z| = r\})^\circ_\Omega \subset \Omega$. It follows that $\psi(\Delta) \subset \Omega$.

From Propositions 2, 4 and 6 we get the following inclusions for a bounded domain Ω in $\mathbb{C}^n$:

$$\begin{aligned} \mathcal{N}(\Omega) &= \Omega \\ \implies \Omega &\text{ has Property (N)} \\ \implies (\Omega, \mathcal{N}(\Omega)) &\text{ is a Runge pair} \\ \implies \Omega &\text{ has disk property} \\ \implies \Omega &\text{ is a domain of holomorphy.} \end{aligned}$$

Finally we give an example of a domain of holomorphy that does not have disk property.

EXAMPLE. Let $n \geq 2$. $\Omega_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n; |z_1| < \dots < |z_n| < 1\}$ is called Hartogs triangle of dimension n . This is a domain of holomorphy and $\mathring{\Omega}_n = \Omega_n$. Let $\psi: \bar{\Delta} \rightarrow \mathbb{C}^n$, $\psi(z) = \left(\left(\frac{z}{2} \right)^n, \left(\frac{z}{2} \right)^{n-1}, \dots, \frac{z}{2} \right)$. $\psi(\bar{\Delta} \setminus \{0\}) \subset \Omega_n$. But $\psi(0) = (0, \dots, 0) \notin \Omega_n$. It follows that Ω_n does not have disk property. Especially $\Omega_n \not\subseteq \mathcal{N}(\Omega_n)$. H. Behnke and P. Thullen [2] proved directly that $\Omega_2 \not\subseteq \mathcal{N}(\Omega_2)$.

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