A NECESSARY CONDITION FOR THE EXISTENCE OF PEAK FUNCTIONS

By

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0. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a domain with smooth C^2 -boundary; i.e. for any $p \in \text{b}\Omega$ there exists a real-valued C^2 -function r in a neighborhood U of p such that $\Omega \cap U = \{x \in U; r(x) < 0\}$ and $dr \neq 0$ on $\text{b}\Omega \cap U$, where $\text{b}\Omega$ is the boundary of Ω . A point $p \in \text{b}\Omega$ is a peak point if there exist a neighborhood W of p and $f \in A^0(\bar{\Omega} \cap W)$ such that $f(p) = 1$ and $|f(x)| < 1$ for $x \in (\bar{\Omega} \cap W) \setminus \{p\}$, where $A^0(\bar{\Omega} \cap W)$ is the set of all functions holomorphic on $\bar{\Omega} \cap W$ and continuous on $\bar{\Omega} \cap W$. We call the function f satisfying the above conditions a peak function at p .

In [2] R. F. Basener gave the following necessary condition for the existence of peak functions at $p \in \text{b}\Omega$:

(C₁) p is a limit of strongly pseudoconvex boundary points of Ω . In this paper we shall show that the following condition is also necessary for the existence of peak functions at $p \in \text{b}\Omega$:

(C₂) $\text{b}\Omega$ cannot contain a germ of positive dimensional complex variety at p . We shall also show that if $\Omega \subset \mathbb{C}^n$ is a pseudoconvex domain with smooth real-analytic boundary then the condition (C₂) is precisely stronger than the condition (C₁). Furthermore, we shall note that in the case $n=2$, the condition (C₂) is necessary and sufficient for the existence of peak functions.

1. A necessary condition

This section is devoted to the following theorem.

THEOREM 1. *Let $\Omega \subset \mathbb{C}^n$ be a domain with smooth C^1 -boundary. If $p \in \text{b}\Omega$ is a peak point, then the condition (C₂) is satisfied at p .*

PROOF. Let r be a defining function of Ω near p . We can take local holomorphic coordinates $z=(z_1, \dots, z_n)$ with origin at p such that $r(z)=\operatorname{Re} z_n + o(|z|)$. Put $z=(z', z_n)$. We may further assume, by using the implicit function theorem that r has the form

$$r(z)=\operatorname{Re} z_n + R(z', \operatorname{Im} z_n),$$

where R is independent of $\operatorname{Re} z_n$. Let U be a neighborhood of p such that the above coordinates $(z_1, \dots, z_n)$ and the function r can be taken there. Let U_0 be a neighborhood of p which is relatively compact in U . For sufficiently small $\varepsilon \geq 0$, we define the translation $F_\varepsilon: U_0 \rightarrow U, (z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-1}, z_n - \varepsilon)$. Then, we have $F_\varepsilon(b\Omega \cap U_0) \subset \Omega$.

Suppose that the condition (C_2) were not satisfied at $p \in b\Omega$. Then, there exists a closed complex variety V of a neighborhood W of p such that $p \in V \subset b\Omega$ and $\dim_p V > 0$. Shrinking U_0 if necessary, we may further assume that V is a closed complex variety of U_0 . Since $V \subset b\Omega$, it follows that $F_\varepsilon(V) = V_\varepsilon \subset \Omega$ and V_ε is a closed complex variety of $F_\varepsilon(U_0)$. Set $x_\varepsilon = F_\varepsilon(0)$, then $\dim_{x_\varepsilon} V_\varepsilon = \dim_0 V$. Let W_ε be an irreducible component of V_ε such that $\dim W_\varepsilon > 0, x_\varepsilon \in W_\varepsilon$. There exist a neighborhood $U_1 \Subset U_0$ of p and $\varepsilon_0 > 0$ such that $W_\varepsilon \cap U_1$ is a closed complex variety of U_1 and $W_\varepsilon \cap bU_1 \neq \emptyset$ for all ε with $\varepsilon_0 \geq \varepsilon \geq 0$. We set $\delta = \bigcup_{0 \leq \varepsilon \leq \varepsilon_0} W_\varepsilon \cap bU_1$. Then δ is compact in $\bar{\Omega}$ and does not contain p . Let f be a peak function at p . We may assume that f is defined on $\bar{\Omega} \cap U$. There exists a positive number $M < 1$ such that $|f| < M$ on δ . But there exists a positive number $\varepsilon_1 (< \varepsilon_0)$ such that $|f(x_{\varepsilon_1})| > M$, because $f(x_\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$. If we set $M_1 = \max \{|f(x)|; x \in W_{\varepsilon_1} \cap \bar{U}_1\}$, then $M_1 > M$. Since $|f| < M$ on $W_{\varepsilon_1} \cap bU_1$, M_1 can be attained at an interior point of $W_{\varepsilon_1} \cap U_1$. By the maximum principle, we conclude that $|f| \equiv M_1$ on $W_{\varepsilon_1} \cap U_1$. Hence $|f| = M_1$ on $W_{\varepsilon_1} \cap bU_1$ by the continuity of f , which leads to a contradiction.

2. The sequence of complex varieties

In this section we shall express the result of Forneaess which was used in Kohn [6]. We give its proof, for we need a strict expression.

The following theorem of Ephraim [5] is used in our arguments.

Let V be a germ of complex variety at $0 \in \mathbb{C}^n$, and let $\{g_1, \dots, g_s\}$ be generators of the complex ideal of V at 0 . Regard V as a real-analytic germ in $\mathbb{R}^{2n}$ and let I be its real-analytic ideal at 0 . Then $g_1, \dots, g_s, \bar{g}_1, \dots, \bar{g}_s$ generate $I \otimes_{\mathbb{R}} \mathbb{C}$ if

and only if V is irreducible.

Suppose that S is a real-analytic variety of $\mathbb{C}^n$ with real codimension 1. Let r be a defining function of S near $q \in S$. Then, the Taylor expansion

$$r(z) = \sum a_{IJ}(z-q)^I(\bar{z}-\bar{q})^J$$

of r at q converges on some $\Delta(q; \delta)$, polydisc of radius $\delta > 0$ centered at q . Let $\tilde{r}$ be the function defined on $\Delta(q; \delta) \times \Delta(\bar{q}; \delta)$ by

$$\tilde{r}(z, w) = \sum a_{IJ}(z-q)^I(w-\bar{q})^J.$$

Then we have $r(z) = \tilde{r}(z, \bar{z})$ and $\overline{\tilde{r}(z, \bar{w})} = \tilde{r}(w, z)$.

Let M be an irreducible complex subvariety of S .

PROPOSITION 1 (Diederich and Fornaess [4]). *There exists a closed complex variety V of $\Delta(q; \delta)$ such that $V \subset \Delta(q; \delta) \cap S$ and $M \cap \Delta(q; \delta) \subset V$.*

PROOF. Let $I_q(M)$ be the ideal of germs of holomorphic functions at q that vanish on M . Assume that $I_q(M)$ is generated by $\{\varphi_1, \dots, \varphi_s\}$. Then, there exists a neighborhood U of q such that $U \subset \Delta(q; \delta)$, φ_j is holomorphic in U and $\hat{M} = M \cap U = \{x \in U; \varphi_1(x) = 0, \dots, \varphi_s(x) = 0\}$. Using Ephraïm's theorem and shrinking a neighborhood U , we obtain for the defining function r of S

$$r(z) = \sum \gamma_f(z, \bar{z})\varphi_f(z) + \sum \overline{\gamma_f(z, \bar{z})}\varphi_f(\bar{z}),$$

where $\gamma_f(z, \bar{z})$ is a real-analytic function in U . For fixed $z_0 \in \hat{M}$

$$\tilde{r}(z, \bar{z}_0) = \sum \gamma_f(z, \bar{z}_0)\varphi_f(z) + \sum \gamma_f(z_0, \bar{z})\overline{\varphi_f(z_0)} = 0$$

for all $z \in \hat{M}$. If we set $V_1 = \bigcap_{z_0 \in \hat{M}} \{z \in \Delta(q; \delta); \tilde{r}(z, \bar{z}_0) = 0\}$, then V_1 is a closed complex variety of $\Delta(q, \delta)$. Since $\tilde{r}(z, \bar{w}) \equiv 0$ on $V_1 \times \hat{M}$, $\tilde{r}(w, \bar{z}) \equiv 0$ on $\hat{M} \times V_1$. Hence $V_2 = \bigcap_{z_0 \in V_1} \{z \in \Delta(q; \delta); \tilde{r}(z, \bar{z}_0) = 0\}$ is also a closed complex variety, and $\hat{M} \subset V_2 \subset V_1$. From $\tilde{r}(z, \bar{w}) \equiv 0$ on $V_2 \times V_1$, it follows $r|_{V_2} = 0$. For fixed $z_0 \in V_1$, $\tilde{r}(z, \bar{z}_0)|_{\hat{M}} = 0$. Since $\hat{M}$ is an open subset of $M \cap \Delta(q, \delta)$, $\tilde{r}(z, \bar{z}_0)|_{M \cap \Delta(q, \delta)} = 0$ by the identity theorem. Hence, V_2 is the desired variety.

PROPOSITION 2 (Diederich and Fornaess [4]). *Suppose that the Taylor expansion of r at q converges on a neighborhood U of q . Then there exists a neighborhood $U' \subset U$ of q such that, for any irreducible complex variety W in $U' \cap S$, there exists a closed complex variety W' of U' with $W \subset W' \subset U' \cap S$. Furthermore, for any irreducible complex variety $W \subset U' \cap S$ we have $\tilde{r}(z, \bar{w}) = 0$*

whenever $z, w \in W$.

PROOF. There exists $\delta > 0$ such that $U \subset \Delta(q; \delta)$. Put $U' = \Delta(q; \delta/3)$. Then for all $x \in W \subset U' \cap S$, $U' \subset \Delta(x; 2/3\delta) \subset \Delta(q; \delta)$. By Proposition 1 there exists a closed complex variety W' of $\Delta(x; 2/3\delta)$ such that $W \subset W' \subset \Delta(x; 2/3\delta) \cap S$. Then, $W' \cap U'$ is a closed complex variety of U' . We can also take such W' as satisfying $\tilde{r}(z, \bar{w})|_{W \times W'} = 0$ from the proof of Proposition 1. Hence the proof is complete.

PROPOSITION 3 (Fornaess, see Theorem 6.23 in Kohn [6]). *Let $\{W_k\}$ be a sequence of complex varieties with $W_k \subset S$ and $\dim W_k \geq m$. If $\dim_{q_k} W_k \geq m$, $q_k \in W_k$ and x_0 is a cluster point of $\{q_k\}$, then there exists a complex variety W such that $x_0 \in W$, $\dim_{x_0} W \geq m$ and $W \subset S$.*

PROOF. Let U' be a bounded neighborhood of x_0 with the properties of Proposition 2. So we may assume that W_k is a closed irreducible complex variety of U' with $\dim W_k \geq m$. And also assume that x_0 is a cluster point of any subsequence of $\{W_k\}$. Let C_0 be the set of all cluster points of $\{W_k\}$. Since C_0 is totally bounded, there exists finite number of points $x_1^{(1)}, \dots, x_{r_1}^{(1)} \in C_0$ such that $C_0 \subset \bigcup_{i=1}^{r_1} O(x_i^{(1)}, 1)$, where $O(x_i^{(1)}, 1)$ is an open ball in $\mathbb{C}^n$ of radius 1 centered at $x_i^{(1)}$. We can choose a subsequence $\{W_k^{11}\} \subset \{W_k\}$ such that for all k there exists $x_{1k}^{(1)} \in W_k^{11}$ with $x_{1k}^{(1)} \rightarrow x_1^{(1)}$ ($k \rightarrow \infty$). We set $p_1^{(1)} = x_1^{(1)}$. When there exist cluster points of $\{W_k^{11}\}$ in $O(x_2^{(1)}, 1)$, we take $p_2^{(1)}$ from these points. Then there exists a subsequence $\{W_k^{12}\} \subset \{W_k^{11}\}$ such that for all k there exists $p_{2k}^{(1)} \in W_k^{12}$ with $p_{2k}^{(1)} \rightarrow p_2^{(1)}$ ($k \rightarrow \infty$). If there exists no cluster point of W_k^{11} in $O(x_2^{(1)}, 1)$, we set $p_2^{(1)} = p_1^{(1)}$ and $\{W_k^{12}\} = \{W_k^{11}\}$. Repeating this procedure r_1 times, we have a subsequence $\{W_k^{(1)}\} \subset \{W_k\}$ and its cluster points $\{p_1^{(1)}, \dots, p_{r_1}^{(1)}\}$ such that for all k there exists $p_{ik}^{(1)} \in W_k^{(1)}$ with $p_{ik}^{(1)} \rightarrow p_i^{(1)}$ ($k \rightarrow \infty$), $i = 1, \dots, r_1$. Let C_1 be the set of all cluster points of $\{W_k^{(1)}\}$. From the construction of $\{W_k^{(1)}\}$, if $C_1 \cap O(x_i^{(1)}, 1) \neq \emptyset$ then $p_i^{(1)} \in O(x_i^{(1)}, 1)$. Since C_1 is totally bounded, there exist finite number of points $x_1^{(2)}, \dots, x_{r_2}^{(2)} \in C_1$ such that $C_1 \subset \bigcup_{i=1}^{r_2} O(x_i^{(2)}, 1/2)$. Similarly, we obtain a subsequence $\{W_k^{(2)}\} \subset \{W_k^{(1)}\}$ and its cluster points $\{p_1^{(2)}, \dots, p_{r_2}^{(2)}\}$ such that for all k there exists $p_{ik}^{(2)} \in W_k^{(2)}$ with $p_{ik}^{(2)} \rightarrow p_i^{(2)}$ ($k \rightarrow \infty$), $i = 1, \dots, r_2$. Proceeding inductively, we obtain sequences $\{W_k^{(1)}\} \supset \{W_k^{(2)}\} \supset \dots$, $C_1 \supset C_2 \supset \dots$, and the set $C' = \{p_1^{(1)}, \dots, p_{r_1}^{(1)}, p_1^{(2)}, \dots, p_{r_2}^{(2)}, \dots\}$, where C_j is the set of all cluster points of $\{W_k^{(j)}\}$. Using diagonalization we finally obtain a sequence $\{W_k^{(k)}\}$ such that for any $p_i^{(j)} \in C'$ there exists $p_{ik}^{(j)} \in W_k^{(k)}$ ($k = 1, 2, \dots$) with $p_{ik}^{(j)} \rightarrow p_i^{(j)}$ ($k \rightarrow \infty$). Let C

be the set of all cluster points of $\{W_k^{(k)}\}$. Take any point $p \in C$. For all $\varepsilon > 0$ there exists a integer $m > 1$ with $1/m < \varepsilon/2$. Since $C \subset C_m$, $p \in C_m$. Then p belongs to some $O(x_i^{(m)}, 1/m)$. Hence $O(x_i^{(m)}, 1/m) \cap C_m \neq \emptyset$. So there exists $p_i^{(m)} \in O(x_i^{(m)}, 1/m)$. It follows that $p_i^{(m)} \in O(p, \varepsilon)$. This means that C' is dense in C . Set $C' = \{p^{(k)}\}$ by changing indices. Since $\tilde{r}(z, \bar{w}) = 0$ for all $z, w \in W_k^{(k)}$, $\tilde{r}(p^{(i)}, \bar{p}^{(j)}) = 0$. By the density of C' , $\tilde{r}(p, \bar{p}') = 0$ for all $p, p' \in C$. If we set $W' = \bigcap_{p' \in C} \{p \in U'; \tilde{r}(p, \bar{p}') = 0\}$, then W' is a closed complex variety of U' and $C \subset W'$. Since $\tilde{r}(c, \bar{w}') = \tilde{r}(\bar{w}', c) = 0$ for all $w' \in W'$ and $c \in C$, $W = \bigcap_{w' \in W'} \{w \in U'; \tilde{r}(w, \bar{w}') = 0\}$ is also a closed complex variety of U' . And $x_0 \in C \subset W \subset W'$. For any $w \in W$, $\tilde{r}(w, \bar{w}) = 0$. Hence $W \subset U' \cap S$. It remains to show that $\dim_{x_0} W \geq m$.

Consider the stratification $\hat{W}_0 \subset \hat{W}_1 \subset \dots \subset \hat{W}_l = W$, where $\hat{W}_j$ is the set of all singular points of $\hat{W}_{j+1}$ for $j = 1, \dots, l-1$. Let d be the smallest integer such that $C \setminus W_d$ does not cluster at x_0 . Then we have $x_0 \in W_d$. Let $W_d = W^1 \cup \dots \cup W^s$ be the decomposition of $\hat{W}_d$ into irreducible germs of $\hat{W}_d$ at x_0 . Then $(W^i \setminus \hat{W}_{d-1}) \cap C$ clusters at x_0 for some $i \in \{1, \dots, s\}$. We can take $p \in (W^i \setminus \hat{W}_{d-1}) \cap C$ and a neighborhood U'' of p such that $U'' \cap W^i$ consists of regular points of W^i and $U'' \cap C$ is contained in W^i . Let $\eta_1, \dots, \eta_n$ be holomorphic coordinates with origin at p such that $\Delta \subset U''$ and $W^i \cap \Delta = \{(\eta_1, \dots, \eta_n) \in \Delta; \eta_{t+1} = 0, \dots, \eta_n = 0\}$, where $\Delta = \{(\eta_1, \dots, \eta_n) \in \mathbb{C}^n; |\eta_j| < 1, j = 1, \dots, n\}$ and $t = \dim W^i$. We have $\eta_{t+1} = 0, \dots, \eta_n = 0$ on $C \cap \Delta$. Let $\tilde{\Delta} = \{\eta; |\eta_j| < 1/2, j = 1, \dots, n\}$. Then, for sufficiently large k we have $W_k^{(k)} \cap \tilde{\Delta} \subset \{\eta \in \tilde{\Delta}; |\eta_j| < 1/4, j = t+1, \dots, n\}$. If $\dim W_k^{(k)} > t$, there exists $\xi \in \{(\eta_1, \dots, \eta_t) \in \mathbb{C}^t; |\eta_j| < 1/2, j = 1, \dots, t\}$ such that $\dim(W_k^{(k)} \cap \{\eta \in \tilde{\Delta}; \eta_1 = \xi_1, \dots, \eta_t = \xi_t\}) > 0$. Since $W_k^{(k)}$ is a closed complex variety of $\tilde{\Delta}$, $W_k^{(k)} \cap \{\eta \in \tilde{\Delta}; \eta_1 = \xi_1, \dots, \eta_t = \xi_t\} \not\subset \{\eta \in \tilde{\Delta}; \eta_1 = \xi_1, \dots, \eta_t = \xi_t, |\eta_j| < 1/4, j = t+1, \dots, n\}$. This is the contradiction. Hence we conclude $m \leq \dim W_k^{(k)} \leq t = \dim W^i$, which completes the proof.

3. The relation between (C_1) and (C_2) in the real analytic case

Let $T_p^{10}(\mathfrak{b}\Omega)$ be the complex tangent space of $\mathfrak{b}\Omega$ at $p \in \mathfrak{b}\Omega$. We denote by $\langle, \rangle_p$ the pairing of the space of $(1, 1)$ -forms at p and its dual space.

DEFINITION. For any $p \in \mathfrak{b}\Omega$ we denote the Levi null space at p by $N_p = \{L \in T_p^{10}(\mathfrak{b}\Omega); \langle \partial \bar{\partial} r, L \wedge \bar{L} \rangle_p = 0\}$, where r is a defining function of $\mathfrak{b}\Omega$ near p . We set $\text{codim}_e N_p = \dim_e T_p^{10}(\mathfrak{b}\Omega) - \dim_e N_p$.

The following proposition is a special case of the result in the author [1].

PROPOSITION 4. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth real-analytic boundary. If $\text{codim}_{\bullet} N_p = k$, $p \in \text{b}\Omega$ and $\text{codim}_{\bullet} N_q \leq k$ in a neighborhood of p , then there exists a local complex foliation of $\text{b}\Omega$ at p whose leaves have complex dimension $n - (k + 1)$.*

THEOREM 2. *Let $\Omega \subset \mathbb{C}^n$ be a pseudocnvex domain with smooth real-analytic boundary. Then the condition (C_2) implies the condition (C_1) .*

PROOF. Let p be a boundary point of Ω . It is sufficient to show the following; if there exists a neighborhood U of p such that $\text{b}\Omega \cap U$ has no strongly pseudoconvex boundary point, then there exists a complex variety V such that $p \in V \subset \text{b}\Omega$ and $\dim_p V > 0$.

Let $\text{codim}_{\bullet} N_p = k$, $k < n - 1$. The proof conveniently splits into two cases. Case 1. $\text{codim}_{\bullet} N_q \leq k$ in a neighborhood of p . In this case there exists a local complex foliation of $\text{b}\Omega$ at p by Proposition 4.

Case 2. There exists a sequence $\{q_j\} \subset \text{b}\Omega$ with following properties; (a) $q_j \rightarrow p$, (b) $\text{codim}_{\bullet} N_{q_j} = m_j$, $k < m_j < n - 1$, (c) $\text{codim}_{\bullet} N_q \leq m_j$ in some neighborhood of q_j . By (b) and (c) there exists a local complex foliation of $\text{b}\Omega$ at q_j . In particular, there exists a complex submanifold W_j such that $q_j \in W_j \subset \text{b}\Omega$, $\dim_{q_j} W_j \geq 1$. From (a) and Proposition 3, we conclude that there exists a complex variety V such that $p \in V \subset \text{b}\Omega$ and $\dim_p V \geq 1$.

EXAMPLE. We shall give an example which shows that the condition (C_2) is precisely stronger than the condition (C_1) . We consider the following pseudoconvex domain in $\mathbb{C}^2$.

$$\Omega = \{(z, w) \in \mathbb{C}^2; r(z, w) = \text{Re } w + |zw|^2 < 0\}.$$

Let $p = (z, w) \in \text{b}\Omega$. Then the sign of the Levi-form restricted to $T_p^{1,0}(\text{b}\Omega)$ is the same as

$$L = r_{z\bar{z}}|r_w|^2 - r_{z\bar{w}}r_{\bar{z}w}r_w - r_{\bar{z}w}r_zr_{\bar{w}} + r_{w\bar{w}}|r_z|^2.$$

Since we have $L = |w|^2/4$, $\text{b}\Omega$ is strongly pseudocnvex except the set $\{w = 0\}$, which lies on $\text{b}\Omega$.

4. The case $n = 2$

Let Ω be a pseudoconvex domain in $\mathbb{C}^2$ with smooth real-analytic boundary. When Ω is bounded, Bedford and Fornaess showed in [3] that there exists a

global peak function at every boundary point of Ω . Bounded pseudoconvex domains with smooth real-analytic boundary have two important properties in our arguments. First, $b\Omega$ has no germs of positive dimensional complex variety. Second, Ω has the global regularity of $\bar{\partial}$ -problem. By using the former they constructed a local peak function. By the latter they obtained a global peak function. We say merely a local peak function a peak function in this paper. So, if the condition (C_2) is satisfied at $p \in b\Omega$ then we can construct a (local) peak function at p .

Furthermore, we note that the following proposition holds for real-analytic hypersurfaces of $\mathbb{C}^n$.

PROPOSITION 5. *If M is a real-analytic hypersurface of $\mathbb{C}^n$, then the following (a) and (b) are equivalent;*

- (a) *M cannot contain a germ of $(n-1)$ -dimensional complex manifold at $p \in M$,*
- (b) *M cannot contain a germ of $(n-1)$ -dimensional complex variety at $p \in M$.*

PROOF. It is sufficient to show that (a) implies (b). Suppose that there exists a $(n-1)$ -dimensional complex variety V with $p \in V \subset M$. We may assume that V is irreducible at p . We claim that p is a regular point of V . Suppose that p were a singular point of V . Then there exist a neighborhood U of p and a holomorphic function f in U such that $V \cap U = \{x \in U; f(x)=0\}$ and f generates $I_p(V)$. Let $z=(z_1, \dots, z_n)$ be holomorphic coordinates of $\mathbb{C}^n$ with origin at p . Since p is a singular point of V , it follows that $\partial f / \partial z_1(0)=0, \dots, \partial f / \partial z_n(0)=0$. Let r be a defining function of M near p . Shrinking U if necessary, we can write by Ephraim's theorem, $r=\alpha f + \bar{\alpha} \bar{f}$, where α is a real-analytic function in U . Hence we can see $(dr)_p=0$. This contradicts that r is a defining function of the hypersurface M near p . Thus p is a regular point of V , and the proposition is proved.

Theorem 1, Proposition 5 and the remark at the beginning of this section, we obtain the following proposition.

PROPOSITION 6. *Let Ω be a pseudoconvex domain with smooth real-analytic boundary. Then $p \in b\Omega$ is a peak point if and only if $b\Omega$ cannot contain a germ of 1-dimensional complex manifold at p .*

REMARK. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth real-analytic boundary. Then the $\bar{\partial}$ -Neumann problem is subelliptic for $(0, q)$ -forms at

$p \in b\Omega$ if and only if $b\Omega$ cannot contain a germ of q -dimensional complex variety at p (see Kohn [7]). Hence the following three conditions are equivalent in the case $n=2$.

- (i) $p \in b\Omega$ is a peak point,
- (ii) $b\Omega$ cannot contain a germ of 1-dimensional complex manifold at p ,
- (iii) the $\bar{\partial}$ -Neumann problem is subelliptic for $(0, 1)$ -forms at p .

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