NOTE ON THE THEOREM OF CARTAN-BEHNKE-STEIN

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0. Introduction

Every Cousin-I domain in $\mathbb{C}^2$ is a Stein domain. This fact is the well-known theorem of Cartan-Behnke-Stein ([4], [5]). The purpose of this note is to generalize this theorem to K -complete complex spaces. Complex spaces are always supposed to have a countable basis of topology, to be reduced and to be of finite dimension.

Let X be a K -complete connected normal complex space. Suppose that the following conditions are satisfied.

- i) X is a Cousin-I space, that is, every additive Cousin problem is universally solvable on it.
- ii) For every non-constant holomorphic function f on X , the set $\{x \in X \mid f(x)=0\}$ is a Stein subvariety of X .

Then X coincides with its K -envelope $H(X)$ in the sense of H. Kerner [10]. This result will be proved in §1. The condition ii) is always satisfied if X is of dimension two. If we replace the condition i) by the stronger condition that $H^1(X, \mathcal{O}_X)=0$, then we conclude that X is a Stein space. More precisely we have the following result.

Let X be a K -complete complex space. Suppose that the following conditions are satisfied.

- i) $H^1(X, \mathcal{O}_X)=0$, where we denote by $\mathcal{O}_X$ the sheaf of germs of holomorphic functions on X .
- ii) For every holomorphic function f on X which is not constant on any positive dimensional irreducible component of X , the set $\{x \in X \mid f(x)=0\}$ is a Stein subvariety of X .

Then X is a Stein space. In the proof, we use the methods of J. E. Fornaess-R. Narasimhan [6] and B. Jennane [8].

1. Cousin-I spaces

Throughout this note complex spaces are always supposed to have a countable basis of topology, to be reduced and to be of finite dimension. It is the aim of this section to generalize the theorem of Cartan-Behnke-Stein to K -complete connected normal complex spaces. G. Berg [3] used the following Lemma 1 to show that a Cousin-I domain in a two dimensional Stein space is a domain of holomorphy. We shall also use it for our purpose.

LEMMA 1. *Let X be a Cousin-I space and f be a holomorphic function on X which is not constant on any positive dimensional irreducible component. If the set $\{x \in X | f(x)=0\}$ is a Stein subvariety of X , then every holomorphic function on $\{x \in X | f(x)=0\}$ is extended holomorphically to X .*

LEMMA 2. *Let X be a K -complete complex space. Suppose that the following conditions are satisfied.*

- i) X is a Cousin-I space.
- ii) *For every holomorphic function f on X which is not constant on any positive dimensional irreducible component of X , the set $\{x \in X | f(x)=0\}$ is a Stein subvariety of X .*

Then X is holomorphically separable.

PROOF. Let $\Phi: X \rightarrow \mathbb{C}^n$ be a realization of X as a ramified region. Such a realization exists by H. Grauert [7]. Let x_0 and x_1 be two distinct points in X . In case that $\Phi(x_0) \neq \Phi(x_1)$, there exists a holomorphic function f on $\mathbb{C}^n$ such that $f(\Phi(x_0)) \neq f(\Phi(x_1))$. $f \circ \Phi$ is holomorphic on X . In case that $\Phi(x_0) = \Phi(x_1)$, we put $s_0 = \Phi(x_0) = \Phi(x_1)$. Take an appropriate non-constant holomorphic function g such that $g(s_0) = 0$. $g \circ \Phi$ is not constant on any positive dimensional component of X , and therefore $Y = \{x \in X | (g \circ \Phi)(x) = 0\}$ is Stein. Since $x_0, x_1 \in Y$ and $x_0 \neq x_1$, there exists a holomorphic function h on Y such that $h(x_0) \neq h(x_1)$. By Lemma 1 there exists a holomorphic function H on X such that $H|_Y = h$. $H(x_0) \neq H(x_1)$. It follows that X is holomorphically separable.

We shall use the following Lemma 3 due to H. Kerner [10].

LEMMA 3. *Let X be a K -complete connected normal complex space. Then $X = H(X)$ if and only if $X = \tilde{H}(X, \Phi, \mathbb{C}^n)$ for any realization $\Phi: X \rightarrow \mathbb{C}^n$ of X as a ramified domain. Here we denote by $\tilde{H}(X, \Phi, \mathbb{C}^n)$ the Φ -envelope of the*

Riemann domain $\Phi: X \rightarrow \mathbb{C}^n$.

Using the above lemmas, we will prove our main theorem of this section.

THEOREM 4. *Let X be a K -complete connected normal complex space. Suppose that the following conditions are satisfied.*

- i) X is a Cousin-I space.
- ii) For every non-constant holomorphic function f on X , the set $\{x \in X \mid f(x)=0\}$ is a Stein subvariety of X .

Then X coincides with its K -envelope $H(X)$ in the sense of H. Kerner [10].

PROOF. We may assume that $n \geq 2$. Let $\Phi: X \rightarrow \mathbb{C}^n$ be an arbitrary realization of X as a ramified Riemann domain and $\tilde{H}(X, \Phi, \mathbb{C}^n)$ be its Φ -envelope. Then we have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota} & \tilde{H}(X, \Phi, \mathbb{C}^n) \\ \searrow \Phi & & \swarrow \tilde{\Phi} \\ & \mathbb{C}^n & \end{array}.$$

Since X is holomorphically separable by Lemma 2, X can be considered as a subdomain of $\tilde{H}(X, \Phi, \mathbb{C}^n)$. Assume that $\tilde{H}(X, \Phi, \mathbb{C}^n) - X \neq \emptyset$. Then there exists a boundary point p of X in $\tilde{H}(X, \Phi, \mathbb{C}^n)$. There exists a non-constant holomorphic function f on $\mathbb{C}^n$ such that $f(\tilde{\Phi}(p)) = 0$. Put $Y = \{x \in \tilde{H}(X, \Phi, \mathbb{C}^n) \mid (f \circ \tilde{\Phi})(x) = 0\}$. We may assume that $Y \cap X \neq \emptyset$. Then there exist a point $q \in Y \cap \partial X$ and a sequence $(x_n)_{n=1}^{\infty}$ in $Y \cap X$ converging to q . Since $Y \cap X$ is Stein, there exists a holomorphic function g on $Y \cap X$ such that $\lim_{n \rightarrow \infty} |g(x_n)| = +\infty$. By Lemma 1 there exists a holomorphic function G on X such that $G|_{Y \cap X} = g$. Then we have $\lim_{n \rightarrow \infty} |G(x_n)| = +\infty$. On the other hand, by the property of the Φ -envelope, G can be continued to a holomorphic function $\tilde{G}$ on $\tilde{H}(X, \Phi, \mathbb{C}^n)$. Then we have $\lim_{n \rightarrow \infty} |G(x_n)| = |\tilde{G}(q)| < +\infty$. This is a contradiction. Thus we have $X = \tilde{H}(X, \Phi, \mathbb{C}^n)$. It follows that $X = H(X)$ by Lemma 3.

Every one dimensional complex space without compact irreducible component is a Stein space by R. Narasimhan [11]. Therefore the condition ii) of Theorem 4 is always satisfied if X is of dimension two. Then we have the following corollary which is also considered as a generalization of the result of J. Kajiwaru [9].

COROLLARY. *Every two dimensional K -complete connected normal Cousin-I space coincides with its K -envelope.*

The following theorem is also a corollary to Theorem 4.

THEOREM 5. Let $\Phi: X \rightarrow \mathbb{C}^2$ be an unramified Riemann domain over $\mathbb{C}^2$. If X is a Cousin-I manifold, then X is a Stein manifold.

PROOF. By the construction of $H(X)$, $H(X)$ coincides with the envelope of holomorphy of $(X, \Phi, \mathbb{C}^2)$ (see H. Kerner [10]). Thus $H(X)$ is Stein.

2. Complex spaces with a vanishing cohomology set

For a complex space X we always denote by $\mathcal{O}_X$ the sheaf of germs of holomorphic functions on X . The condition that $H^1(X, \mathcal{O}_X) = 0$ is stronger than the condition that X is a Cousin-I space. J. E. Fornaess–R. Narasimhan [6] proved the following theorem.

Let X be a relatively compact domain in a Stein space S . Suppose that the following conditions are satisfied.

- i) $H^1(X, \mathcal{O}_X) = 0$.
- ii) For every holomorphic function f on S which is not constant on any positive dimensional irreducible component of S , the set $\{x \in X \mid f(x) = 0\}$ is a Stein subvariety of X .
- iii) X is locally Stein.

Then X is a Stein space.

B. Jennane and others ([1], [2], [8]) pointed out that the condition iii) can be removed. We shall show in this section that a K -complete space X which satisfies the condition i) in the above theorem and condition ii) in Lemma 2 is a Stein space, making use of the methods in J. E. Fornaess–R. Narasimhan [6] and B. Jennane [8].

LEMMA 6. Let X be a Cousin-I space and f be a holomorphic function on X which is not constant on any positive dimensional irreducible component. Put $Y_t = \{x \in X \mid f(x) = t\}$ and suppose that Y_t be a Stein subvariety for every $t \in \mathbb{C}$. Then for every compact set $K \subset X$ there exist $\varepsilon > 0$ and an open set $U \subset X$ such that $U \cap Y_t$ is a Runge open set of Y_t and $K \cap Y_t \subset U \cap Y_t$ for $t \in \mathbb{C}$, $|t| < \varepsilon$.

PROOF. See the proof of Lemma 2.4 of [6]. Instead of Lemma 2.2 of [6] we use Lemma 1.

LEMMA 7. Let X be a complex space such that $H^1(X, \mathcal{O}_X) = 0$. Let f be a holomorphic function on X which is not constant on any positive dimensional

irreducible component. Put $Y_t = \{x \in X \mid f(x) = t\}$ and suppose that Y_t be a Stein subvariety for every $t \in \mathbb{C}$. Then for every compact set $K \subset X$ there exist $\varepsilon > 0$ and a compact set $L \subset X$, $K \subset L$, such that the following conditions are satisfied: for $t \in \mathbb{C}$, $|t| < \varepsilon$, there exists $M_t > 0$ such that for any $\tau > 0$ and for any holomorphic function g on Y_t there exists a holomorphic function G on X which satisfies $G|_{Y_t} = g$ and $|G|_K \leq M_t |g|_{L \cap Y_t} + \tau$.

PROOF. We can find a countable Stein open covering $\mathfrak{U} = (U_i)_{i=1}^\infty$ of X . The proof of this lemma is the same as that of Lemme 4 in [8]. We shall use Lemma 6 and Lemma 2.3.2 of [6].

THEOREM 8. Let X be a K -complete complex space. Suppose that the following conditions are satisfied.

- i) $H^1(X, \mathcal{O}_X) = 0$.
- ii) For every holomorphic function f on X which is not constant on any positive dimensional irreducible component of X , the set $\{x \in X \mid f(x) = 0\}$ is a Stein subvariety of X .

Then X is a Stein space.

PROOF. In the proof, we use the method of Jennane [8]. Let $\Phi: X \rightarrow \mathbb{C}^n$ be a realization of X as a ramified region. Such a realization exists by H. Grauert [7]. It suffices to prove that X is holomorphically convex. Let K be a compact set in X . $\Phi(K)_{\mathbb{C}^n}$ is a compact set in $\mathbb{C}^n$ and we have $\Phi(\hat{K}_X) \subset \Phi(K)_{\mathbb{C}^n}$. Take an arbitrary $s \in \mathbb{C}^n$ and an appropriate non-constant holomorphic function f on $\mathbb{C}^n$ such that $f(s) = 0$. $f \circ \Phi$ is a holomorphic function on X which is not constant on any positive dimensional irreducible component. Put $Y_t = \{x \in X \mid (f \circ \Phi)(x) = t\}$ for $t \in \mathbb{C}$. By Lemma 6 there exists a positive number ε' and a compact set $L \subset X$ such that $(K \cap Y_t)_{Y_t}^\wedge \subset L$ for $t \in \mathbb{C}$, $|t| < \varepsilon'$. Take $p \in Y_t - (L \cap Y_t)_{Y_t}^\wedge$. There exists a holomorphic function g on Y_t such that $|g(p)| > 1$ and $|g|_{L \cap Y_t} < 1/2M_t$. By Lemma 7 there exist a holomorphic function G on X such that $G|_{Y_t} = g$ and $|G|_K \leq M_t |g|_{L \cap Y_t} + \frac{1}{2} < 1$. Since $|G(p)| = |g(p)| > 1$, $p \notin \hat{K}_X$. Therefore $\hat{K}_X \cap Y_t \subset (L \cap Y_t)_{Y_t}^\wedge$ for $t \in \mathbb{C}$, $|t| < \varepsilon'$. By Lemma 6 there exists a positive number ε with $0 < \varepsilon < \varepsilon'$ such that $\bigcup_{|t| < \varepsilon} (L \cap Y_t)_{Y_t}^\wedge \subset X$. Then we have $\bigcup_{|t| < \varepsilon} (\hat{K}_X \cap Y_t) = \hat{K}_X \cap \{x \in X \mid |(f \circ \Phi)(x)| < \varepsilon\} \subset X$. Thus for every $s \in \mathbb{C}^n$ there exists a neighborhood U_s of s such that $\hat{K}_X \cap \Phi^{-1}(U_s) \subset X$. Since $\Phi(K)_{\mathbb{C}^n}$ is compact, there exist finitely many open sets $U_1, U_2, \dots, U_m$ of $\mathbb{C}^n$ such that $\Phi(\hat{K}_X) \subset \bigcup_{j=1}^m U_j$ and $\hat{K}_X \cap \Phi^{-1}(U_j) \subset X$ ($1 \leq j \leq m$). So we obtain that $\hat{K}_X \subset X$. It follows that X is holomorphically

convex.

The condition ii) in Theorem 8 is always satisfied if X is of dimension two. Then we have the following corollary.

COROLLARY. *If X is two dimensional K -complete space such that $H^1(X, \mathcal{O}_X) = 0$, then X is a Stein space.*

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