CONJUGACY CLASSES IN CLASSICAL GROUPS

By

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§ 0. Introduction

Let $G$ be a simple algebraic group over an algebraically closed field. Conjugacy classes in $G$ have been studied extensively by several mathematicians, see e.g., Springer and Steinberg [4]. Recently Burgoyne and Cushman in [1] considered the case of classical groups over $\mathbb{R}$ or $\mathbb{C}$, which will be explained below, and found a systematic way to describe all conjugacy classes of these groups.

The purpose of this paper is to supplement [1]. Let $G$ be a classical group, and let $L$ be the Lie algebra of $G$. In [1], they first describe all orbits in $L$ under the adjoint action of $G$, and using the methods and results, they finally develop a method to study conjugacy classes in $G$. But their argument about the latter part is too brief, and in particular the group version of Proposition 2, which is essential for the theory, is not explained completely in the paper.

In this paper, making some changes of their definitions, we shall establish a complete theory of conjugacy classes of classical groups, more or less along their lines.

In our terminology, a "classical group" implies one of the following groups. We retain their notation in the paper [1].

Let $V$ be a finite dimensional complex vector space.

1) $GL(V)$: the totality of all linear endomorphisms of $V$ onto itself.
2) $O(V, \tau)$: the isometry group of a non-degenerate alternative bi-linear form $\tau$.
3) $Sp(V, \tau)$: the isometry group of a non-degenerate alternative bi-linear form $\tau$.

Let $\sigma_+$ and $\sigma_-$ be anti-linear maps of $V$ onto itself where $\sigma_+^2 = 1$ and $\sigma_-^2 = 1$, respectively.

4) $GL(V, \sigma_+) = \{x \in GL(V); \sigma_+ x = x \sigma_+\}$
5) \(GL(V, \sigma_\pm) = \{ x \in GL(V); \sigma_- x = x \sigma \}\)

6) \(O(V, \tau, \sigma_\pm) \equiv \sigma_\pm x\) where \(\tau^\sigma(u, v) = \tau(\sigma u, \sigma v) = \overline{\tau(u, v)}\)

7) \(O(V, \tau, \sigma_\pm) = \{ x \in O(V, \tau); x \sigma_- = \sigma_- x\}\) where \(\tau^\sigma(u, v) = \overline{\tau(u, v)}\)

8) \(Sp(V, \tau, \sigma_+) = x \in Sp(V, \tau); x \sigma_+ = \sigma_+ x\) where \(\tau^\sigma(u, v) = \overline{\tau(u, v)}\)

9) \(Sp(V, \tau, \sigma_\pm) = \{ x \in Sp(V, \tau); x \sigma_- = \sigma_- x\}\) where \(\tau^\sigma(u, v) = \overline{\tau(u, v)}\)

10) \(GL(V, \tau^\sigma)\); the isometry group of a hermitian form \(\tau^*\).

\textbf{Note} You can find the details of the notations in the appendix 1 of the paper [1].

\section{Definitions and Some Preliminary Results}

Retaining the notation in [1], let \(G(V, \sigma, \tau)\) denote one of the classical groups and \(L(V, \sigma, \tau)\) the Lie algebra of \(G(V, \sigma, \tau)\).

Let \(A\) and \(B\) in \(G(V, \sigma, \tau)\) and \(G(V', \sigma', \tau')\) respectively. If there exists a complex linear isomorphism \(\phi\) of \(V\) onto \(V'\) such that

\[\phi A = B \phi, \quad \phi \sigma = \sigma' \phi\]

\[\tau(u, v) = \tau' (\phi u, \phi v) \quad \text{for any} \quad u, v \in G(V, \sigma, \tau),\]

then we write \((A, V, \sigma, \tau) \sim (B, V', \sigma', \tau')\). It turns out that \(\sim\) is an equivalence relation, and we call the equivalence class to which \((A, V, \sigma, \tau)\) belongs the type of \((A, V, \sigma, \tau)\).

\textbf{Proposition 1.} Let \(A\) and \(B\) be in \(G(V, \sigma, \tau)\). Then there exists a \(g \in G(V, \sigma, \tau)\) such that \(g A g^{-1} = B\), if and only if \((A, V, \sigma, \tau)\) and \((B, V, \sigma, \tau)\) belong to the same type.

Prove follows from the definition.

By Proposition 1 the determination of conjugacy classes is equivalent to the classification of types. In order to classify types, we shall first define their decompositions and study them.

Let \(\Delta\) be a type, and let \((A, V, \sigma, \tau)\) be a representative of \(\Delta\). Suppose that \(V = V_1 + V_2\) is an orthogonal disjoint sum of proper \(\Delta\)-invariant and \(\sigma\)-invariant subspaces. Then \(G(V_i, \sigma, \tau)\) is one of the classical groups and contains \(A | V_i\).

Let \(\Delta_i\) denote the type \((A_i, V_i, \sigma, \tau)\). Then we write \(\Delta = \Delta_1 + \Delta_2\). It turns out that \(\Delta_1 + \Delta_2 = \Delta_2 + \Delta_1\) and \((\Delta_1 + \Delta_2) + \Delta_3 = \Delta_1 + (\Delta_2 + \Delta_3)\). Therefore we can write \(\Delta_1 + \Delta_2 + \Delta_3 + \cdots + \Delta_n\) with no danger of confusion.
Let $K$ be a perfect field. Let $A$ be in $GL(n, K)$. Then there exist unique elements $S$ and $U$ in $GL(n, K)$ such that $A = SU = US$, $S$ is semi-simple and $U$ is unipotent. We call $A = SU$ the Jordan decomposition of $A$. It is known that if $A$ belongs to an algebraic group $G$, then $S$ and $U$ belong to $G$.

Let $A$ be in $G(V, \sigma, \tau)$, and let $A = SU$ be its Jordan decomposition. The fact that $A$ commutes with $\sigma$ implies that $S$ commutes with $\sigma$. So we have that $S$ and $U$ are contained in $G(V, \sigma, \tau)$. Because $U$ is unipotent, there is a unique nilpotent element $N$ such that $\exp N = U$. The entries of $\exp tN$ are polynomials functions of $t$ and, for any integer $n$, $\exp nN$ is contained in $G(V, \sigma, \tau)$, and so we have that $N$ is in $L(V, \sigma, \tau)$. We shall call $A = S \exp N$ its Jordan decomposition. (See [3])

Let $(A, V, \sigma, \tau)$ be in a type, $\Delta$, and $A = S \exp N$ its Jordan decomposition.

We can find the integer $m$ such that $N^m \neq 0$ and $N^{m+1} = 0$. We call $m$ the height of $\Delta$ and denote it by $ht \Delta$.

We call $\Delta$ a uniform type if $\ker N^m = NV$, and if in particular $ht \Delta = 0$ we call it semi-simple.

If $\Delta$ is uniform, we shall define a new semi-simple type $\bar{\Delta}$ as follows. Set $V = V/\ker N^m$, $\bar{A}v = Av$, $\bar{\sigma}v = \bar{v}$, where $\bar{v}$ implies $v$ mod. $NV$ for $v \in V$, and we put $\bar{\tau}(\bar{u}, \bar{v}) = \tau(u, N^mv)$. Considering that $NV = \ker N^m$ we have that $\bar{\tau}$ is non-degenerate on $\bar{V}$. Since $\bar{Av} = S\bar{v}$ for $v \in V$, $\bar{A}$ is semi-simple on $\bar{V}$. It is easy to see that $\bar{A} \in G(\bar{V}, \bar{\sigma}, \bar{\tau})$ and that $G(\bar{V}, \bar{\sigma}, \bar{\tau})$ is one of the classical groups. Let $\bar{\Delta}$ be the type to which $(\bar{A}, \bar{V}, \bar{\sigma}, \bar{\tau})$ belongs.

Remark Even if $\tau$ is alternative (symmetric), $\bar{\tau}$ may not be always alternative (symmetric).

Finally in this section we shall prove some preliminary lemmas based on some results in the paper [1].

Let $A$ be in $G(V, \sigma, \tau)$. And let $A = S \exp N$ be its Jordan decomposition.

**Lemma 1.** Let $U$ be any proper subspace of $V$ which is $S$-invariant and $\sigma$-invariant.

Then there exists a $S$-invariant $\sigma$-invariant complement to $U$.

(Proof) See the page 344 in [1].

**Lemma 2.** Suppose that $(A, V, \sigma, \tau)$ is contained in a uniform type $\Delta$.

Then there exists a $S$-invariant and $\sigma$-invariant subspace $H$ such that
$V = H + NH + \cdots + N^mH$ (direct sum)

where $m = ht\Delta$, and $\dim N^iH = \dim H$ ($0 \leq i \leq m$).

(Proof) Since $NV$ is $S$-invariant and $\sigma$-invariant, by lemma 1, we can find a $S$-invariant $\sigma$-invariant complement $H$ to $NV$. Then we have

$$V = H + NV = H + N(H + NV) = H + NH + N^2V$$

$$= H + NH + \cdots + N^mH.$$

Next suppose that $\sum_{k=0}^{m} N^k h_k = 0$, where $h_kH$. Since $H + NV$ is direct sum by Lemma 1, we have that $h_0 = 0$ and $\sum_{k=1}^{m} N^k h_k = N \sum_{k=1}^{m} N^{k-1} h_k = 0$. Then $\sum_{k=1}^{m} N^{k-1} h_k = h_1 + N \sum_{k=2}^{m} N^{k-2} h_k \in \ker N \subset \ker N^m$. Because $\ker N^m = NV$, we have $h_1 = 0$ and $N \sum_{k=2}^{m} N^{k-2} h_k = 0$. Continuing this procedure we have,

$$h_0 = h_1 = \cdots = h_m = 0.$$

**Lemma 3.** Let $m$ be the height of $(A, V, \sigma, \tau)$, and let $U$ be a proper $A$-invariant and $\sigma$-invariant subspace of $V$.

Suppose that $(A \mid U, U)$ is uniform, then there exists a complement to $U$ in $V$ which is $A$-invariant and $\sigma$-invariant.

(Proof) See the page 344 in [1].

§ 2. Decomposition of Types

In this section we shall prove some propositions about the decompositions of types, and show the uniqueness of the decompositions. Some of the following propositions can be proved in the same way as the corresponding ones in [1].

**Proposition 2.** If $\Delta$ is a uniform type it is uniquely determined by the height of $\Delta$ and $\bar{\Delta}$.

This proposition is essential in the whole theory. It is proved in a similar way as in Proposition 2 in the paper [1]. We only need to replace the equality $S^*\tilde{\tau}_j = -\tau S$ by $S^*\tilde{\tau}_j = \tilde{\tau}_j S^{-1}$.

In order to prove more propositions, we need two lemmas.
LEMMA 4. If $\Delta$ is not uniform, then there exist two types, $\Delta_1$ and $\Delta_0$ such that

\[ ht\Delta = ht\Delta_1 > ht\Delta_0 \]

$\Delta_1$ is uniform

and $\Delta = \Delta_1 + \Delta_0$.

(Proof) See the first half of the proof of Proposition 4 in [1].

LEMMA 5. For any semi-simple type $\Delta$ and a given positive integer $m$, there exists a uniform type $\Delta'$ such that

$\Delta' = \Delta$ and $ht\Delta' = m$.

Note By Proposition 2, such a type $\Delta'$ is uniquely determined.

(Proof) Let $(S, F, \sigma, \tau)$ be in $\Delta$. We put

\[ V = F \oplus \cdots \oplus F \quad (m + 1\text{-times direct sum}) \]

\[ N: V \ni (f_0, f_1, \ldots, f_m) \longrightarrow (0, f_0, f_1, \ldots, f_{m-1}) \in V. \]

So we may write

\[ V = F \oplus NF \oplus \cdots \oplus N^m F. \]

Let us define an anti-linear map $\bar{\sigma}$ and a linear map $\bar{S}$ by

\[ \bar{S}(f_0, f_1, \ldots, f_m) = (Sf_1, Sf_2, \ldots, Sf_m) \]

\[ \bar{\sigma}(f_0, f_1, \ldots, f_m) = (\sigma f_1, \sigma f_2, \ldots, \sigma f_m). \]

We set $\bar{\tau}(\sum_{r=0}^{m} N^r f_r, \sum_{s=0}^{m} N^s f'_s) = \sum_{r+s=m} (-1)^{r+s} \tau(f_r, f'_s)$ for $f_r, f'_s \in F$. Then $\bar{\tau}$ is a bi-linear map on $V$. From the definition it follows that

$\bar{\tau}$ is non-degenerate

$\bar{\sigma}^2 = \pm 1$ and $\bar{\tau} \bar{\sigma} = \bar{\sigma} \bar{\tau}$.

So $G(V, \bar{\sigma}, \bar{\tau})$ is one of the classical groups. And the direct computation shows that

$\bar{\sigma} N = N \bar{\sigma}, \quad \bar{S} \bar{\sigma} = \bar{\sigma} \bar{S}$
|\( \tilde{\tau}(\mathbf{S}u, \mathbf{S}v) = \tilde{\tau}(u, v) \n |
and \( \tilde{\tau}(Nu, v) + \tilde{\tau}(u, Nv) = 0 \) for \( u, v \in V \).

In fact the first three equalities are easy to prove and for the last equality we shall prove as follows.

Set \( u = (u_0, u_1, \ldots, u_m), v = (v_0, v_1, \ldots, v_m) \in V \).

\[
\tilde{\tau}(Nu, v) = \tilde{\tau}(u, Nv) = \sum_{r+s=m}(\tau((Nu)_r, v_s) + \sum_{r+s=m}(\tau(u_r, (Nv)_s)
\]
\[
= \sum_{r+s=m}(\tau(u_{r-1}, v_s) + \sum_{r+s=m}(\tau(u_r, v_{s-1})
\]

(where \( u_{-1} = v_{-1} = 0 \))
\[
= \sum_{r+s=m-1, 0 \leq s \leq m}(\tau(u_r, v_s)
\]
\[
+ \sum_{s \leq m-1, 0 \leq s \leq m-1}(\tau(u_r, v_s)
\]

\( = 0. \)

Therefore we have that \( S \in G(V, \tilde{\sigma}, \tilde{\tau}), N \in L(V, \tilde{\sigma}, \tilde{\tau}) \) and \( S \exp N \in G(V, \tilde{\sigma}, \tilde{\tau}) \).

Let \( A \) be \( S \exp N \) and \( A' \) the type which contains \( (A, V, \tilde{\sigma}, \tilde{\tau}) \). Because of the uniqueness of the Jordan decomposition, \( SN = NS' \) implies that \( S \exp N \) is the Jordan decomposition of \( A \).

By the inverse procedure of the proof of Proposition 2 we can show that

\[ A' = A \quad \text{and} \quad htA' = m. \]

**Proposition 3.** If \( A \) is indecomposable then \( A \) is uniform and \( A \) is indecomposable.

(Proof) Lemma 4 implies that \( A \) is uniform. Suppose that \( A = A'_1 + A'_2 \).

Obviously \( A'_i \) is a semi-simple type. Then, by Lemma 5, we can find types, say \( A'_1 \) and \( A'_2 \), such that

\[ A'_i = A' \quad \text{and} \quad htA'_i = htA \quad \text{where} \quad i = 1, 2 \]

then we can show that

\[ A = A'_1 + A'_2 \]

in a similar way as in the latter half of the proof of Proposition 3 in [1].
PROPOSITION 4. If $\Delta$ is not uniform, there exist the unique types $\Delta_1$ and $\Delta_0$ such that

$$\Delta = \Delta_1 + \Delta_0$$

$\Delta_1$ is uniform

and $ht\Delta = ht\Delta_1 > ht\Delta_0$

(Proof) See the proof of Proposition 4 in [1].

PROPOSITION 5 If $\Delta$ is semi-simple then its decomposition into indecomposable types is unique.

In order to prove Proposition 5 we must classify the semi-simple indecomposable types.

But its classification was carried out by Burgoyne and Cushman by means of the Cayley transform. (see [1]). Since any decomposed factor of a semi-simple type is semi-simple, using their results, we can prove Proposition 5 in the same way as Proposition 5 in [1]. In order to do so we must replace the definition of the form $\theta$ there by the following. Let $\Delta$ be a semi-simple indecomposable type. And set $(A, V, \sigma, \tau) \in \Delta$. Set the bilinear form $\theta$ on $V_\sigma$ by

$$\theta_+(u, v) = \tau_+(u, v) \quad \text{if } \tau \text{ is symmetric}$$

$$\quad = \tau_+(u, T v) \quad \text{if } \tau \text{ is alternating}$$

for $u, v \in V_\sigma^+$

$$\theta_-(u, v) = \tau_-(u, v) \quad \text{if } \tau \text{ is alternating}$$

$$\quad = \tau_-(u, v) \quad \text{if } \tau \text{ is symmetric}$$

for $u, v \in V_\sigma^- = V$

where $T$ is the inverse image of $S$ by the Cayley transform. (see appendix 2 in [1]).

Thus we get the theorem.

THEOREM. The decomposition of a type into indecomposable ones is unique.

(Proof) See the page 343 in [1].
§ 3. Determination of the Conjugacy classes

Let $\mathcal{A}'$ be a semi-simple indecomposable type and $m$ a positive integer. By Proposition 2 and Lemma 5 we can find the unique type $\mathcal{A}$ such that

$$\mathcal{A} = \mathcal{A}' \quad \text{and} \quad ht\mathcal{A} = m$$

We denote the type $\mathcal{A}$ by $\mathcal{A}'(m)$. By Proposition 2, 3, we can denote every indecomposable type by the form $\mathcal{A}'(m)$ where $\mathcal{A}'$ is a semi-simple indecomposable type and $m$ is a positive integer.

Conversely, for any semi-simple indecomposable type $\mathcal{A}'$ and a given positive integer $m$, $\mathcal{A}'(m)$ is indecomposable. It follows from the fact that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are uniform and if $ht\mathcal{A}_1 = ht\mathcal{A}_2$, $\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}_1 + \mathcal{A}_2$.

Thus we have determined all the indecomposable types, since we know all the semi-simple indecomposable types. Now we can compute the conjugacy classes. Let $\mathcal{A}$ be in $G = G(V, \sigma, \tau)$, and let $\mathcal{A}$ denote the type $(\mathcal{A}, V, \sigma, \tau)$. Then we get the unique decomposition

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \cdots + \mathcal{A}_n.$$ 

Then we have

$$\dim \mathcal{A} = \dim \mathcal{A}_1 + \dim \mathcal{A}_2 + \cdots + \dim \mathcal{A}_n$$

$$\ind \mathcal{A} = \ind \mathcal{A}_1 + \ind \mathcal{A}_2 + \cdots + \ind \mathcal{A}_n \quad (\ast)$$

(The notations, "ind" and "dim", are defined in a similar way as in the algebra case in the page 348 in [1]). Conversely, suppose that $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are indecomposable types belonging to the same family of the groups in §0, and satisfy the condition $(\ast)$. Then we can construct the type $\mathcal{A}$ such that

$$\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_n.$$ 

And $\mathcal{A}$ contains some $(\mathcal{A}, V, \sigma, \tau)$ with $A \in G$.

Thus we have a complete description of the conjugacy classes of $G$.

References

339–362.

Added in proof
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