FIRST-ORDER OPTIMALITY CONDITIONS IN NONDIFFERENTIABLE PROGRAMMING PROBLEMS

By

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Introduction

F. H. Clarke [2] introduced the concept of generalized gradient in a general class of functions, which is an extension of subdifferential of a convex function. F. H. Clarke [3] and J. B. Hiriart-Urruty [5] gave necessary optimality conditions in a locally Lipschitz mathematical programming problem by using the generalized gradient.

When a differentiable function is given, the generalized gradient of the function may be a set including its gradient in usual sense and may not coincide with the later one. On account of this fact, the necessary conditions given by F. H. Clarke [3] and J. B. Hiriart-Urruty [5] seem not to be sufficiently precise. The motive of this paper is to make more precise their necessary conditions.

In Section 1, after summarizing well-known definitions in convex analysis a new derivative is introduced to locally Lipschitz functions and a gradient is defined by using the derivative. In Section 2 an infinitesimal cone is defined by using the new derivative introduced in Section 1 and some relations between this infinitesimal cone and other well-known infinitesimal cones are discussed. In Section 3, a mathematical programming problem with locally Lipschitz inequality constraints is considered in a Banach space, and necessary optimality conditions of Fritz John type are given in a general abstract form. In Section 4 it is shown that under the calmness condition (cf. [3]) necessary optimality conditions in Section 3 become to be of Kuhn-Tucker type. A new constraint qualification is also proposed. The last section is devoted to the sufficiency of the optimality conditions.

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1. Gradients of functions

Let X be a Banach space, and let X^* be the topological dual space of X . The norm of an element ξ of X^* is denoted by $\|\xi\|^*$; however, all statements involving a topology on X^* are given with respect to the weak* topology, unless otherwise stated.

For a convex subset B of X^* , $\delta^*(\cdot | B)$ is the support function of B defined by:

$$\delta^*(x | B) = \sup_{x^* \in B} \langle x, x^* \rangle.$$

Let $f: X \rightarrow \mathbb{R}$ be a convex function. An element ζ of X^* is said to be a subgradient of f at a point x , if

$$f(z) \geq f(x) + \langle z - x, \zeta \rangle \quad \forall z \in X. \quad (1.1)$$

The set of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial_s f(x)$. The one-sided directional derivative of f at x is defined to be the limit

$$f'(x; d) = \lim_{\lambda \downarrow 0} [f(x + \lambda d) - f(x)] \lambda^{-1}. \quad (1.2)$$

For each d , the difference quotient in the definition of $f'(x; d)$ is a non-decreasing function of $\lambda > 0$, so that $f'(x; d)$ exists. If f is continuous at x_0 , then

$$f'(x; d) = \delta^*(d | \partial_s f(x_0)) \quad \forall d, \quad (1.3)$$

(cf. [1], Proposition 2.3).

We shall consider henceforth a locally Lipschitz function $f: X \rightarrow \mathbb{R}$ throughout this section. This means that any point x in X admits a neighborhood N such that, for some constant K , for all y and z in N , we have

$$|f(y) - f(z)| \leq K \|y - z\|.$$

K is called a Lipschitz constant of f at x .

Definitions of the generalized directional derivative and the generalized gradient of f at x are given by F. H. Clarke [3] in the context of Banach space as follow: the generalized directional derivative of f at x is the limit

$$f^\circ(x; d) = \limsup_{\substack{y \rightarrow x \\ \lambda \downarrow 0}} [f(y + \lambda d) - f(y)] \lambda^{-1}, \quad (1.4)$$

and the generalized gradient of f at x , which is denoted by $\partial f(x)$, is the set of all elements $\zeta \in X^*$ such that

$$f^\circ(x; v) \geq \langle v, \zeta \rangle \quad \forall v \in X. \quad (1.5)$$

$f^\circ(x; \cdot)$ is a positively homogeneous convex function and for any v in X it holds that

$$f^\circ(x; v) = \delta^*(v | \partial f(x)). \quad (1.6)$$

$\partial f(x)$ is a nonempty convex weak* compact subset of X^* . If f is convex, then $\partial f(x) = \partial_s f(x)$. For various properties of the generalized gradients, the reader is referred to the papers of F. H. Clarke [2], [3].

DEFINITION 1. For each x and d in X , we define $f^\square(x; d)$ and $\partial^* f(x)$ by:

$$f^\square(x; d) = \limsup_{\lambda \downarrow 0} [f(x + \lambda d) - f(x)] \lambda^{-1}, \quad (1.7)$$

$$\partial^* f(x) = \{\zeta \in X^* | f^\square(x; d) \geq \langle d, \zeta \rangle \quad \forall d \in X\}. \quad (1.8)$$

It follows from definitions that for each x and d in X ,

$$f^\circ(x; d) \geq f^\square(x; d).$$

Thus we have $\partial^* f(x) \subset \partial f(x)$. It may be possible that $\partial^* f(x)$ is empty. If f is convex, then it follows from (1.3) that

$$\partial^* f(x) = \partial_s f(x) = \partial f(x).$$

If f is Fréchet differentiable at x and $Df(x)$ is the Fréchet differential of f at x , we have $\partial^* f(x) = \{Df(x)\}$.

The following is a consequence of (1.8).

PROPOSITION 1. $\delta^*(d | \partial^* f(x)) \leq f^\square(x; d)$.

PROPOSITION 2. The function $f^\square(x; \cdot)$ is a positively homogeneous Lipschitz function and satisfies

$$|f^\square(x; v)| \leq K \|v\|.$$

(K is the Lipschitz constant of f at x)

PROOF. For sufficiently small $\lambda > 0$, we have

$$|[f(x + \lambda v) - f(x)] \lambda^{-1}| \leq K \|v\|.$$

Therefore, $|f^\square(x; v)| \leq K\|v\|$. That the function is positively homogeneous is evident. Let d_1 and d_2 be any two distinct points in X . Then

$$\begin{aligned} f^\square(x; d_1) - f^\square(x; d_2) &\leq \limsup_{\lambda \downarrow 0} [f(x + \lambda d_1) - f(x + \lambda d_2)] \lambda^{-1} \\ &\leq K\|d_1 - d_2\|. \end{aligned}$$

This inequality is valid for any d_1 and d_2 in X , so $f^\square(x; \cdot)$ is of Lipschitz.

PROPOSITION 3. $\partial^*f(x)$ is a convex w^* -compact subset of X^* .

PROOF. It follows from Proposition 2 that

$$\begin{aligned} \partial^*f(x) &\subset \{\zeta \in X^* \mid \langle v, \zeta \rangle \leq K\|v\| \quad \forall v \in X\} \\ &= \{\zeta \in X^* \mid \|\zeta\|^* \leq K\} \end{aligned}$$

Since $\partial^*f(x)$ is closed, $\partial^*f(x)$ is w^* -compact.

PROPOSITION 4. If $f(x; \cdot)$ is convex, then it holds that

$$f^\square(x; d) = \delta^*(d \mid \partial^*f(x)).$$

PROOF. We set $g(\cdot) = f^\square(x; \cdot)$.

It follows from Proposition 2 that g is continuous convex function. Hence $g'(0; d) = \delta^*(d \mid \partial_s g(0))$. Definition 1 implies $\partial^*f(x) = \partial_s g(0)$, so $g'(0; d) = \delta^*(d \mid \partial^*f(x))$. The result follows from the fact that $g'(0; d) = f^\square(x; d)$.

2. Various cones

If $A \subset X$, then cone A is the conical hull of A , i.e.,

$$\text{cone } A = \{\lambda a \mid \lambda > 0 \text{ and } a \in A\}.$$

The polar cone of a nonempty subset A of X is given by

$$A^\circ = \{x^* \in X^* \mid \langle a, x^* \rangle \leq 0 \quad \forall a \in A\}.$$

If A is empty, we have $A^\circ = X^*$. A° is evidently a closed convex cone.

Throughout this section, we shall deal with a nonempty subset C of X and a point $x_0 \in \text{cl}C$, where “cl” denotes the “closure”. Let d_c be the distance function of C :

$$d_c(x) = \inf \{\|x - c\| \mid c \in C\}.$$

It is easy to show that d_c is a Lipschitz function and its Lipschitz constant is equal to 1.

DEFINITION 2. The tangent cone to C at x_0 , denoted by $\mathcal{T}(C; x_0)$, is given by

$$\mathcal{T}(C; x_0) = \{v \in X \mid d_c^\circ(x_0; v) \leq 0\}.$$

The polar cone of $\mathcal{T}(C; x_0)$ is called the normal cone to C at x_0 , and denoted by $N(C; x_0)$.

DEFINITION 3. The cone $T(C; x_0)$ of adherent displacements for C at x_0 is defined by

$$T(C; x_0) = \{v \in X \mid v = \lim_{n \rightarrow \infty} \lambda_n(x_n - x_0), \lambda_n > 0, x_n \in C, x_n \longrightarrow x_0\}.$$

For various properties of these cones, the reader is referred to F. H. Clarke [2] and J. B. Hiriart-Urruty [6].

DEFINITION 4. We define closed cones $\mathcal{T}^*(C; x_0)$ and $N^*(C; x_0)$ as follows:

$$\mathcal{T}^*(C; x_0) = \{v \in X \mid d_c^\square(x_0; v) \leq 0\},$$

$$N^*(C; x_0) = [\mathcal{T}^*(C; x_0)]^\circ.$$

Since $d_c^\square(x_0; v)$ is nonnegative, we have $\mathcal{T}^*(C; x_0) = \{v \in X \mid d_c^\square(x_0; v) = 0\}$.

PROPOSITION 5. $v \in \mathcal{T}^*(C; x_0)$ if and only if for every $\{\lambda_n\} \subset \mathbb{R}_+^*$ converging to 0, there exists a sequence $\{v_n\}$ converging to v such that $x_0 + \lambda_n v_n \in C$ for all n .

PROOF. Let $v \in \mathcal{T}^*(C; x_0)$, and let $\{\lambda_n\}$ be a sequence as in the statement of Proposition. There exists a sequence $\{x_n\} \subset C$ such that

$$\|x_n - (x_0 + \lambda_n v)\| \leq d_c(x_0 + \lambda_n v) + \lambda_n^2.$$

We define $v_n = (x_n - x_0)\lambda_n^{-1}$. Then $x_0 + \lambda_n v_n = x_n \in C$. Since v is an element of $\mathcal{T}^*(C; x_0)$, $d_c(x_0 + \lambda_n v)\lambda_n^{-1}$ converges to 0. Consequently we have

$$\|v_n - v\| = \|x_n - x_0 - \lambda_n v\|\lambda_n^{-1} \leq d_c(x_0 + \lambda_n v)\lambda_n^{-1} + \lambda_n.$$

Therefore v_n converges to v .

Conversely, if the announced property holds for v , we have

$$0 \leq [d_c(x_0 + \lambda_n v)]\lambda_n^{-1} \leq \|x_0 + \lambda_n v - x_0 - \lambda_n v_n\|\lambda_n^{-1} = \|v - v_n\|.$$

Thus $[d_c(x_0 + \lambda_n v)]\lambda_n^{-1}$ converges to 0 and we have $d_c^\square(x_0; v) = 0$.

PROPOSITION 6. $v \in \mathcal{T}^*(C; x_0)$ if and only if there exist an $\varepsilon > 0$ and a mapping $t \rightarrow r(t)$ from the interval $[0, \varepsilon]$ into X such that

$$\begin{aligned} x_0 + tv + r(t) &\in C \quad \forall t \in [0, \varepsilon], \\ \|r(t)\|t^{-1} &\longrightarrow 0 \quad \text{as } t \downarrow 0. \end{aligned}$$

PROOF. Let $v \in \mathcal{T}^*(C; x_0)$. Then for each $t > 0$, there exists $x(t) \in C$ such that

$$\|x(t) - (x_0 + tv)\| \leq d_c(x_0 + tv)t^2.$$

Since v is an element of $\mathcal{T}^*(C; x_0)$, $\lim_{t \downarrow 0} d_c(x_0 + tv)t^{-1} = 0$. We set $v(t) = (x(t) - x_0)t^{-1}$ and $r(t) = t(v(t) - v)$. Then we have $x_0 + tv + r(t) = x_0 + tv(t) = x(t) \in C$ and

$$\|r(t)\|t^{-1} = \|v(t) - v\| = \|x(t) - x_0 - tv\|t^{-1} \leq d_c(x_0 + tv)t^{-1} + t.$$

Thus, $\lim_{t \downarrow 0} \|r(t)\|t^{-1} = 0$.

Conversely, if the announced property holds, then we have

$$0 \leq d_c(x_0 + tv)t^{-1} \leq \|x_0 + tv - (x_0 + tv + r(t))\|t^{-1} = \|r(t)\|t^{-1}.$$

Therefore we have $d_c^\square(x_0; v) = 0$

PROPOSITION 7. Let $C \subset X$ and let $x_0 \in \text{cl } C$, then we have

$$\mathcal{T}(C; x_0) \subset \mathcal{T}^*(C; x_0) \subset T(C; x_0) \subset \text{cl cone}(C - x_0).$$

PROOF. (a) If v belongs to $\mathcal{T}(C; x_0)$, it follows from definitions that $d_c^\square(x_0; d) \leq d_c^\circ(x_0; d) \leq 0$. Therefore v is an element of $\mathcal{T}^*(C; x_0)$.

(b) Let $v \in \mathcal{T}^*(C; x_0)$, and let $\{\lambda_n\} \subset \mathbb{R}_+^*$ be a sequence converging to 0. It is easily seen that

$$\lim_{n \rightarrow \infty} [d_c(x_0 + \lambda_n v)]\lambda_n^{-1} = 0. \quad (2.1)$$

For every n , there exists a sequence $\{x_n\} \subset C$ such that $\|x_n - (x_0 + \lambda_n v)\| \leq d_c(x_0 + \lambda_n v) + \lambda_n^2$, and we have

$$\|(x_n - x_0)\lambda_n^{-1} - v\| = \|x_n - (x_0 + \lambda_n v)\|\lambda_n^{-1} \leq d_c(x_0 + \lambda_n v) + \lambda_n.$$

It follows from (2.1) that $v = \lim_{n \rightarrow \infty} (x_n - x_0)\lambda_n^{-1}$. Since x_n converges to x_0 , we have $v \in T(C; x_0)$.

(c) Let $v \in T(C; x_0)$. Then there exist a sequence $\{\lambda_n\} \subset R_+^*$, and $\{x_n\} \subset C$ converging to x_0 such that $v = \lim_{n \rightarrow \infty} \lambda_n(x_n - x_0)$. Since $\lambda_n(x_n - x_0) \in \text{cone}(C - x_0)$, we have $v \in \text{cl cone}(C - x_0)$.

We will mention the fact that $\mathcal{T}(C; x_0) \subset T(C; x_0)$ has been proved by J. B. Hiriart-Urruty [6].

PROPOSITION 8. *Let $C \subset X$ and let $x_0 \in \text{cl } C$. If C is convex, we have $\mathcal{T}(C; x_0) = \mathcal{T}^*(C; x_0) = T(C; x_0) = \text{cl cone}(C; x_0)$.*

PROOF. Let $y \in \text{cone}(C - x_0)$. There exist $\alpha > 0$ and $z \in C$ such that $y = \alpha(z - x_0)$. Since C is convex, d_c is a convex function. Therefore we have $\partial^* d_c(x_0) = \partial_s d_c(x_0) = \partial d_c(x_0)$.

It follows from (1.6) and Proposition 1 that

$$\begin{aligned} d_c^\circ(x_0; y) &= \delta^*(y | \partial d_c(x_0)) = \delta^*(y | \partial^* d_c(x_0)) \\ &\leq d^\square(x_0; y) \\ &= \limsup_{\lambda \downarrow 0} [d_c((1 - \lambda\alpha)x_0 + \lambda\alpha z)]\lambda^{-1}. \end{aligned}$$

For sufficient small $\lambda > 0$, we have $(1 - \lambda\alpha)x_0 + \lambda\alpha z \in \text{cl } C$. Therefore $d_c^\circ(x_0; y) \leq 0$ and $y \in \mathcal{T}(C; x_0)$. Hence the relation $\text{cone}(C - x_0) \subset \mathcal{T}(C; x_0)$ has been proved. Since $\mathcal{T}(C; x_0)$ is closed, we have $\text{cl cone}(C - x_0) \subset \mathcal{T}(C; x_0)$. Hence the desired result can be shown by Proposition 7.

3. Necessary optimality conditions

For a given nonnegative integer m , let $\langle 1, m \rangle$ denote the set $\{1, 2, \dots, m\}$. Let f and $g_i: (i \in \langle 1, m \rangle)$ be locally Lipschitz functions from X to R , and let Q_0 be a nonempty subset of X .

We set $Q_1 = \{x \in X | g_i(x) \leq 0 \quad \forall i \in \langle 1, m \rangle\}$. Consider the following problem:

$$\left(\begin{array}{l} \text{minimize } f(x) \\ \text{subject to} \\ x \in Q_0 \cap Q_1. \end{array} \right. \quad (3.1)$$

Our Purpose of this section is to give Fritz John type optimality conditions for the Problem (3.1).

THEOREM 1. *Let x_0 be a local solution of problem (3.1). We set*

$$\begin{aligned}
I &= \{i \in \langle 1, m \rangle \mid g_i(x_0) = 0\}, \\
F_0 &= \{d \in X \mid f_i^\square(x_0; d) < 0\}, \\
G_i &= \{d \in X \mid g_i^\square(x_0; d) < 0\} \quad (i \in I), \quad \text{and} \\
G_0 &= \bigcap_{i \in I} G_i.
\end{aligned}$$

Then we have $F_0 \cap G_0 \cap T(Q_0; x_0) = \phi$.

PROOF. Let us prove by contradiction. Suppose that there exists $d \in X$ such that $d \in F_0 \cap G_0 \cap T(Q_0; x_0)$. It follows from the definition of $T(Q_0; x_0)$ that there exist a sequence $\{\lambda_n\} \subset R_+^*$, and a sequence $\{x_n\} \subset Q_0$ converging to x_0 such that

$$d = \lim_{n \rightarrow \infty} (x_n - x_0) \lambda_n^{-1}. \quad (3.2)$$

Since $d \in F_0 \cap G_0$, we have $d \neq 0$. Therefore we may suppose that λ_n converges to 0.

(a) We shall first prove that there exists a natural number n_0 such that $f(x_n) < f(x_0)$ for all $n > n_0$.

Let K be the Lipschitz constant of f at x_0 . For sufficiently large n we have

$$| [f(x_n) - f(x_0 + \lambda_n d)] \lambda_n^{-1} | \leq K \| (x_n - x_0) \lambda_n^{-1} - d \|.$$

It follows from (3.2) that

$$\lim_{n \rightarrow \infty} [f(x_n) - f(x_0 + \lambda_n d)] \lambda_n^{-1} = 0. \quad (3.3)$$

Since $x_0 \in F_0$, we have

$$\limsup_{n \rightarrow \infty} [f(x_0 + \lambda_n d) - f(x_0)] \lambda_n^{-1} \leq f^{\text{cl}}(x_0; d) < 0. \quad (3.4)$$

(3.3) and (3.4) imply $\lim_{n \rightarrow \infty} [f(x_n) - f(x_0)] \lambda_n^{-1} < 0$, from which the desired result follows.

(b) For each $i \in \langle 1, m \rangle$, we show that there exist a number n_i such that $g_i(x_n) < g_i(x_0)$ for all $n > n_i$. We distinguish two cases:

(i) $i \notin I$; in this case the result can be deduced from the continuity of g_i .

(ii) $i \in I$; in this case we can show the result in the same manner as (a), since $d \in G_i$.

If we set $N = \max \{n_0, n_1, n_2, \dots, n_m\}$, then for all $n > N$ one has $x_n \in Q_0 \cap Q_1$ and $f(x_n) < f(x_0)$.

This contradicts the fact that x_0 is a local solution.

COROLLARY 1. *Let Q be a nonempty subset of X and let f be a locally Lipschitz function. If $x_0 \in Q$ is a local minimum of f on Q , then*

$$f^\square(x_0; d) \geq 0 \quad \forall d \in T(Q; x_0).$$

PROOF. It follows from Theorem 1 that

$$\{d \in X \mid f^\square(x_0; d) < 0\} \cap T(Q; x_0) = \emptyset.$$

Since $f^\circ(x_0; d) \geq f^\square(x_0; d)$ for every $d \in X$,

Corollary 1 implies

$$f^\circ(x_0; d) \geq 0 \quad \forall d \in T(Q; x_0).$$

This result coincides with one proved by J. B. Hiriart Urruty [5, Theorem 2].

For the sake of convenience, henceforth, we set $g_0 \equiv f$. For each $i \in \{0\} \cup \langle 1, m \rangle$, we consider a convex w^* -compact subset B_i of X^* satisfying the following hypothesis.

$$(H) \quad \delta^*(d \mid B_i) \geq g_i^\square(x_0; d) \quad \forall d \in X.$$

If we put $B_i = \partial g_i(x_0)$, then it follows from (1.6) that the hypothesis (H) is satisfied. Similarly, if $g_i^\square(x_0; \cdot)$ is convex and we put $B_i = \partial^* g_i(x_0)$, then Proposition 4 implies that the hypothesis (H) is satisfied. Especially, if g_i is differentiable at x_0 , then $B_i = \{Dg_i(x_0)\}$ satisfies the hypothesis (H). When B_i satisfies the hypothesis (H), B_i must contain $\partial^* g_i(x_0)$.

We now derive Fritz John type optimality condition for the problem (3.1).

THEOREM 2. *Let x_0 be a local solution of the problem (3.1), let $B_i (i \in \{0\} \cup \langle 1, m \rangle)$ be convex w^* -compact subsets of X^* satisfying the hypothesis (H), and let M be a closed convex cone included in $T(Q_0; x_0)$. Then there exist numbers $\lambda_0, \lambda_i (i \in \langle 1, m \rangle)$ such that*

- (a) $\lambda_i \geq 0 \quad (i \in \{0\} \cup \langle 1, m \rangle),$
- (b) $(\lambda_0, \lambda_1, \dots, \lambda_m) \neq (0, 0, \dots, 0),$
- (c) $\lambda_i g_i(x_0) = 0 \quad (i \in \langle 1, m \rangle),$
- (d) $0 \in \sum_{i=0}^m \lambda_i B_i + M^\circ.$

PROOF. Let I, F_0 , and G_0 be the sets given in Theorem 1. Define F and G as follows:

$$F = \{d \in X \mid \delta^*(d \mid B_0) < 0\},$$

$$G = \{d \in X \mid \delta^*(d \mid B_i) < 0 \quad \forall i \in I\}.$$

Since the hypothesis (H) is satisfied, we have $F \subset F_0$, and $G \subset G_0$. It follows from Theorem 1 that

$$F \cap G \cap M = \phi. \quad (3.5)$$

It is easily seen that

$$F \cap G = \{d \in X \mid \delta^*(d \mid \text{co} \{ \bigcup_{i \in I_0} B_i \}) < 0\},$$

where $I_0 = \{0\} \cup I$. Since M is a closed convex cone, we have

$$M = \{d \in X \mid \delta^*(d \mid M^\circ) \leq 0\}.$$

It follows from (3.5) that, for each $d \in X$, we have

$$\delta^*(d \mid \text{co} \{ \bigcup_{i \in I_0} B_i \}) \geq 0 \quad \text{or} \quad \delta^*(d \mid M^\circ) = +\infty.$$

Therefore we have $\delta^*(d \mid \text{co} \{ \bigcup_{i \in I_0} B_i \} + M^\circ) \geq 0$. Since $\text{co} \{ \bigcup_{i \in I_0} B_i \} + M^\circ$ is the sum of a convex compact subset and a convex closed subset of X^* , $\text{co} \{ \bigcup_{i \in I_0} B_i \} + M^\circ$ is a convex closed subset of X^* . It follows from the separation theorem that

$$0 \in \text{co} \{ \bigcup_{i \in I_0} B_i \} + M^\circ.$$

Consequently there exist numbers λ_i ($i \in I_0$) such that

$$\lambda_i \geq 0 \quad (i \in I_0), \quad \sum_{i \in I_0} \lambda_i = 1, \quad \text{and}$$

$$0 \in \sum_{i \in I_0} \lambda_i B_i + M^\circ$$

Set $\lambda_i = 0$ for every $i \notin I_0$. Then we have the desired results.

The smaller the convex w^* -compact set B_i is, the more strengthened is the optimality condition in Theorem 2. Similarly the larger the convex cone M is, the more strengthened is the condition.

COROLLARY 2. *If x_0 is a local solution of problem (3.1), then there exist numbers λ_0, λ_i ($i \in \langle 1, m \rangle$) such that*

- (a) $\lambda_i \geq 0 \quad \forall i \in \{0\} \cup \langle 1, m \rangle$,
- (b) $(\lambda_0, \lambda_1, \dots, \lambda_m) \neq (0, 0, \dots, 0)$,
- (c) $\lambda_i g_i(x_0) = 0 \quad \forall i \in \langle 1, m \rangle$,

$$(d) \quad 0 \in \lambda_0 \partial f(x_0) + \sum_{i=1}^m \lambda_i \partial g_i(x_0) + N(Q_0; x_0).$$

PROOF. Take B_i to be $\partial g_i(x_0)$ for every $i \in \{0\} \cup \langle 1, m \rangle$ and M to be $\mathcal{T}(Q_0; x_0)$.

This corollary has been proved by F. H. Clarke [3]. However it should be noted that B_i can be taken smaller than $\partial g_i(x_0)$ in Theorem 2.

EXAMPLE 1. Let C be a nonempty convex w^* -compact subset of X^* , and let $f: X \rightarrow R$ be defined by $f(x) = -\delta^*(x | -C)$. Then $\partial f(0) = C$ and $f^\square(0; d) = -\delta^*(x | -C)$. Let $\zeta_0 \in C$ and we set $B_0 = \{\zeta_0\}$. Then B_0 satisfies the hypothesis (H).

EXAMPLE 2. Let $f: R \rightarrow R$ be defined by

$$f(x) = \begin{cases} x^2 \sin x^{-1} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $\partial f(0) = [-1, 1]$. If we set $B_0 = \{0\}$, then B_0 satisfies the hypothesis (H). Obviously the set $\{0\}$ is strictly smaller than the set $\partial f(0)$.

COROLLARY 3. Let Q be a nonempty subset of X , and let $f: X \rightarrow R$ be a locally Lipschitz function. Let B be a convex w^* -compact subset of X^* such that

$$\delta^*(d | B) \geq f^\square(x_0; d) \quad \forall d \in X,$$

and let M be a closed convex cone included in $T(Q; x_0)$. If x_0 is a local minimum of f on Q , then we have

$$B \cap (-M^\circ) \neq \phi.$$

PROOF. It follows from Theorem 2 that there exists $\lambda_0 > 0$ such $0 \in \lambda_0 B + M^\circ$. Since M° is cone, we have $0 \in B + M^\circ$.

If we set $B = \partial f(x_0)$ in Corollary 3, we have $\partial f(x_0) \cap (-M^\circ) \neq \phi$. This condition has been proved by J. B. Hiriart-Urruty [5].

4. Constraint qualification

We consider the following problem:

$$\left(\begin{array}{l} \text{minimize } f(x) \\ \text{subject to} \\ x_0 \in Q_0, g_i(x) \leq s_i \quad i \in \langle 1, m \rangle, \end{array} \right. \quad (4.1)$$

where f , g_i , and Q_0 are as given in the problem (3.1) and $s=(s_1, \dots, s_m)$ is a vector in R^m . We set $g_0=f$.

Let x_0 be a local solution of problem (4.1), let $B_i (i \in \{0\} \cup \langle 1, m \rangle)$ be a convex w^* -compact subset of X^* satisfying the hypothesis (H), and let M be a closed convex cone included in $T(Q_0; x_0)$. It follows from Theorem 2 that there exist numbers $\lambda_0, \lambda_i (i \in \langle 1, m \rangle)$ such that

$$\lambda_0 \geq 0 \quad (i \in \{0\} \cup \langle 1, m \rangle), \quad (4.2)$$

$$(\lambda_0, \lambda_1, \dots, \lambda_m) \in (0, 0, \dots, 0), \quad (4.3)$$

$$\lambda_i(g_i(x_0) - s_i) = 0 \quad (i \in \langle 1, m \rangle), \quad (4.4)$$

$$0 \in \sum_{i=0}^m \lambda_i B_i + M^\circ. \quad (4.5)$$

Let $\eta > 0$ and we define $\phi_{(x_0, \eta)}: R^m \rightarrow [-\infty, +\infty]$ as follow

$$\phi_{(x_0, \eta)}(s) = \inf \{f(x) \mid x \in Q_0, \|x - x_0\| < \eta, g_i(x) \leq s_i \quad i \in \langle 1, m \rangle\}$$

If the infimum is over the empty set, we have $\phi_{(x_0, \eta)}(s) = +\infty$.

The following notation was introduced by F. H. Clarke [3].

DEFINITION 5. Problem (4.1) is said to be calm in η -neighborhood of x_0 if $\phi_{(x_0, \eta)}(s)$ is finite and if

$$\liminf_{s' \rightarrow s} [\phi_{(x_0, \eta)}(s') - \phi_{(x_0, \eta)}(s)] \|s' - s\|^{-1} > -\infty.$$

F. H. Clarke proved that when problem (4.1) is calm, there exist numbers $\lambda_i (i \in \langle 1, m \rangle)$ such that

$$\lambda_i \geq 0 \quad (i \in \langle 1, m \rangle),$$

$$\lambda_i(g_i(x_0) - s_i) = 0 \quad (i \in \langle 1, m \rangle),$$

$$0 \in \partial f(x_0) + \sum_{i=1}^m \lambda_i \partial g_i(x_0) + N(Q_0; x_0).$$

Under the same condition, we will prove that there exist $\lambda_0 > 0$ and $\lambda_i \geq 0 (i \in \langle 1, m \rangle)$ satisfying (4.4) and (4.5).

LEMMA 1. Let l be a positive integer, let $h_i: X \rightarrow R (i \in \langle 1, l \rangle)$ be locally Lipschitz functions, and let $x_0 \in X$. We set

$$h(x) = \max \{h_i(x) \mid i \in \langle 1, l \rangle\},$$

$$I(x_0) = \{i \in \langle 1, l \rangle \mid h(x_0) = h_i(x_0)\}.$$

If we take a convex w^* -compact subset A_i ($i \in I(x_0)$) so that

$$\delta^*(d \mid A_i) \geq h_i^\square(x_0; d) \quad \forall d \in X,$$

then we have

$$\delta^*(d \mid \text{co} \{ \bigcup_{i \in I(x_0)} A_i \}) \geq h^\square(x_0; d) \quad \forall d \in X.$$

PROOF. It is obvious that h is locally Lipschitz. Since we have $\delta^*(d \mid \text{co} \{ \bigcup_{i \in I(x_0)} A_i \}) \geq \max_{i \in I(x_0)} h_i^\square(x_0; d)$, it suffices to show that

$$\max_{i \in I(x_0)} h_i^\square(x_0; d) \geq h^\square(x_0; d).$$

There exists a sequence $\{\lambda_n\} \subset \mathbb{R}_+^*$ converging to 0 such that

$$h^\square(x_0; d) = \lim_{n \rightarrow \infty} [h(x_0 + \lambda_n d) - h(x_0)] \lambda_n^{-1}.$$

Without loss of generality, we may suppose that there exists $i_0 \in \langle 1, l \rangle$ such that

$$h(x_0 + \lambda_n d) = h_{i_0}(x_0 + \lambda_n d) \quad \forall n.$$

Since h_{i_0} and h are continuous, by tending n to infinity we have $h(x_0) = h_{i_0}(x_0)$; that is, $i_0 \in I(x_0)$. Consequently we have

$$\begin{aligned} h^\square(x_0; d) &= \lim_{n \rightarrow \infty} [h_{i_0}(x_0 + \lambda_n d) - h_{i_0}(x_0)] \lambda_n^{-1} \\ &\leq h_{i_0}^\square(x_0; d) \leq \max_{i \in I(x_0)} h_i^\square(x_0; d). \end{aligned}$$

THEOREM 3. Let x_0 be a local solution of the problem (4.1), let B_i ($i \in \{0\} \cup \langle 1, m \rangle$) be a convex w^* -compact subset of X^* satisfying the hypothesis (H), and let M be a closed convex cone included in $T(Q_0; x_0)$. If the problem (4.1) is calm in η -neighborhood of x_0 for sufficiently small $\eta > 0$, then there exist numbers λ_i ($i \in \langle 1, m \rangle$) such that

$$\lambda_i \geq 0 \quad (i \in \langle 1, m \rangle), \quad (4.6)$$

$$\lambda_i (g_i(x_0) - s_i) = 0 \quad (i \in \langle 1, m \rangle), \quad (4.7)$$

$$0 \in B_0 + \sum_{i=1}^m \lambda_i B_i + M^\circ. \quad (4.8)$$

PROOF. By the proof of Theorem 2 in [3], there exists $\rho > 0$ such that x_0 is a local solution of the following problem:

$$\begin{cases} \text{minimize } h(x) \\ \text{subject to} \\ x \in Q_0, \end{cases} \quad (4.9)$$

where $h(x) = f(x) + \rho \sum_{i=1}^m \max \{0, g_i(x) - s_i\}$.

The hypothesis (H) for B_i implies that

$$\delta^*(d \mid \rho B_i) \geq (\rho g_i)^\square(x_0; d). \quad (4.10)$$

For each $i \in \langle 1, m \rangle$, let us define $h_i: X \rightarrow R$ as follows:

$$h_i(x) = \max \{0, \rho g_i(x) - \rho s_i\}. \quad (4.11)$$

Then we have $h(x) = f(x) + \sum_{i=1}^m h_i(x)$. If $g_i(x_0) < s_i$, it follows from (4.11) that

$$\delta^*(d \mid \{0\}) \geq h_i^\square(x_0; d) \quad \forall d.$$

If $g_i(x_0) = s_i$, then it follows from (4.10), (4.11), and Lemma 1 that

$$\delta^*(d \mid \text{co} \{\{0\} \cup \rho B_i\}) \geq h_i^\square(x_0; d) \quad \forall d.$$

For each $i \in \langle 1, m \rangle$, we define convex w^* -compact subset C_i as follows:

$$C_i = \begin{cases} \{0\} & \text{if } g_i(x_0) < s_i \\ \text{co} \{\{0\} \cup \rho B_i\} & \text{if } g_i(x_0) = s_i. \end{cases} \quad (4.12)$$

Then we have that

$$\begin{aligned} \delta^*(d \mid B_0 + \sum_{i=1}^m C_i) &\geq f^\square(x_0; d) + \sum_{i=1}^m h_i^\square(x_0; d) \\ &\geq (f + \sum_{i=1}^m h_i)^\square(x_0; d) = h^\square(x_0; d). \end{aligned}$$

Now we apply Theorem 2 to the problem (4.9). Then there exists a positive number μ_0 such that

$$0 \in \mu_0(B_0 + \sum_{i=1}^m C_i) + M^\circ$$

Since M° is cone, we have $0 \in B_0 + \sum_{i=1}^m C_i + M^\circ$, which together with (4.12) implies the assertions (4.6)–(4.8).

Next we shall give Kuhn-Tucker type optimality conditions for the problem (3.1).

THEOREM 4. *Let x_0 be a local solution of problem (3.1), let B_0 and B_i ($i \in \langle 1, m \rangle$) be convex w^* -compact subsets of X^* satisfying the hypothesis (H), and let M be a closed convex cone included in $T(Q_0; x_0)$. We suppose that the following condition (U) is satisfied:*

$$(U) \quad \exists d_0 \in M; \delta^*(d_0 | B_i) < 0 \quad \forall i \in J(x_0)$$

where $J(x_0) = \{i \in \langle 1, m \rangle \mid g_i(x_0) = 0\}$.

Then there exist numbers λ_i ($i \in \langle 1, m \rangle$) such that

$$\lambda_i \geq 0, \quad i \in \langle 1, m \rangle \quad (4.13)$$

$$\lambda_i g_i(x_0) = 0, \quad i \in \langle 1, m \rangle \quad (4.14)$$

$$0 \in B_0 + \sum_{i=1}^m \lambda_i B_i + M^\circ. \quad (4.15)$$

PROOF. It follows from Theorem 2 that there exist numbers μ_0, μ_i ($i \in \langle 1, m \rangle$) such that

$$\begin{aligned} \mu_i &\geq 0 \quad (i \in \{0\} \cup \langle 1, m \rangle), (\mu_0, \mu_1, \dots, \mu_m) \neq (0, 0, \dots, 0), \\ \mu_i g_i(x_0) &= 0 \quad (i \in \langle 1, m \rangle), \end{aligned} \quad (4.16)$$

$$0 \in \mu_0 B_0 + \sum_{i=1}^m \mu_i B_i + M^\circ.$$

Let us suppose that $\mu_0 = 0$. Then we have $0 \in \sum_{i \in J(x_0)} \mu_i B_i + M^\circ$, which implies that

$$\delta^*(d_0 | \sum_{i \in J(x_0)} \mu_i B_i) \geq \delta^*(d_0 | \sum_{i \in J(x_0)} \mu_i B_i + M^\circ) \geq 0.$$

Since μ_i , $i \in J(x_0)$, are not all zero, the above contradicts condition (U), hence $\mu_0 \neq 0$. Hence it follows from (4.16) that

$$0 \in B_0 + \sum_{i=1}^m \mu_i \mu_0^{-1} B_i + M^\circ.$$

The smaller the set B_i becomes as well as the larger the convex cone M

becomes, the more weakened is the condition (U). If we let B_0, B_i ($i \in \langle 1, m \rangle$), and M be $\partial f(x_0)$, $\partial g_i(x_0)$, and $\mathcal{T}(Q_0; x_0)$, respectively, (U) is equivalent to the following condition:

$$\exists d_0 \in \mathcal{T}(Q_0; x_0); g_i(x_0; d_0) < 0 \quad \forall i \in J(x_0).$$

This condition has been given by J. B. Hiriart-Urruty [5].

5. Sufficient optimality conditions

Our purpose, in this section, is to give some sufficient conditions for the optimality of a solution in the problem (3.1). We recall the definition of quasi-convex function.

DEFINITION 6. Let $S \subset X$ and $x_0 \in S$. $f: S \rightarrow R$ is said to be quasiconvex at x_0 relative to S if

$$f((1-\lambda)x_0 + \lambda x_1) \leq f(x_0) \quad \forall \lambda \in [0, 1],$$

whenever $x_1 \in S$, $f(x_1) \leq f(x_0)$, and $(1-\lambda)x_0 + \lambda x_1 \in S$. If f is quasiconvex at each point of S relative to S , then we shall simply say that f is quasiconvex on S .

DEFINITION 7. Let $S \subset X$ and $x_0 \in S$. $f: S \rightarrow R$ is said to be strictly quasiconvex at x_0 relative to S if

$$f((1-\lambda)x_0 + \lambda x_1) < f(x_0) \quad \forall \lambda \in (0, 1)$$

whenever $x_1 \in S$, $f(x_1) < f(x_0)$, and $(1-\lambda)x_0 + \lambda x_1 \in S$. If f is strictly quasiconvex at each point of S relative to S , then we shall simply say that f is strictly quasiconvex on S .

Let S be a convex subset of X and let $f: S \rightarrow R$ be lower semi-continuous. It is well known that if f is strictly quasiconvex on S , then f is quasiconvex on S . Let $S \subset X$, $x_0 \in S$ and let f be a real valued function defined on an open set containing S .

Allowing the limit (1.7) to take $+\infty$ or $-\infty$, we consider the following condition:

(C) For every $x_1 \in S$, $f^\square(x_0; x_1 - x_0) \geq 0$ implies $f(x_1) \geq f(x_0)$.

If f is differentiable at x_0 , the condition (C) is equivalent to that f is pseudo-convex at x_0 .

PROPOSITION 9. Let S be a convex subset of X and let f be a real valued

function on an open set containing S . If f is convex, then f satisfies the condition (C) for any x_0 in S .

PROOF. If $x_1 \in S$, we have

$$\begin{aligned} f^\square(x_0; x_1 - x_0) &= f'(x_0; x_1 - x_0) \\ &\leq [f(x_0 + 1(x_1 - x_0)) - f(x_0)]1^{-1} = f(x_1) - f(x_0). \end{aligned}$$

PROPOSITION 10. Let S be a convex subset of X and let f be a locally Lipschitz function defined on an open set containing S . If f satisfies the condition (C) for any x_0 in S , then f is strictly quasiconvex on S .

PROOF. Let us prove by contradiction. We suppose that, f satisfies the condition (C) for any x_0 in S , but f is not strictly quasiconvex on S . Then there exist $x_1 \in S$, $x_2 \in S$, and $\lambda_0 \in (0, 1)$ such that

$$f(x_1) < f(x_2) \leq f((1 - \lambda_0)x_1 + \lambda_0 x_2). \quad (5.1)$$

Let us define $h: [0, 1] \rightarrow R$ as follows:

$$h(t) = f((1 - t)x_1 + tx_2).$$

Since h is locally Lipschitz, there exists $\lambda^* \in [0, 1]$ such that

$$h(\lambda^*) = \max_{\lambda \in [0, 1]} h(\lambda). \quad (5.2)$$

We may assume from (5.1) that $\lambda^* \in (0, 1)$. Let $x^* = (1 - \lambda^*)x_1 + \lambda^*x_2$. Then we have

$$f^\square(x^*; x_1 - x^*) \leq 0. \quad (5.3)$$

If $f^\square(x^*; x_1 - x^*) = 0$, it follows from the condition (C) and (5.1) that $f(x_2) > f(x_1) \geq f(x^*)$. This contradicts (5.2). Therefore we have $f(x^*; x_1 - x^*) < 0$. It follows from (5.1) that there exists $\varepsilon_0 \in (0, 1)$ such that

$$f(x_1) < f(x^* + \varepsilon(x_1 - x^*)) < f(x^*) \quad \forall \varepsilon \in (0, \varepsilon_0].$$

We set $\lambda_0 = (1 - \varepsilon_0)\lambda^*$. Then we have

$$h(0) < h(\lambda) < h(\lambda^*) \quad \forall \lambda \in [\lambda_0, \lambda^*]. \quad (5.4)$$

Since h is locally Lipschitz on $[0, 1]$, h is differentiable at almost all points of $[0, 1]$. Suppose that $h'(\lambda) \leq 0$ for all points $\lambda \in [\lambda_0, \lambda^*]$, at which h is differentiable. Then we have

$$h(\lambda^*) - h(\lambda_0) = \int_{\lambda_0}^{\lambda^*} h'(\lambda) d\lambda \leq 0.$$

This contradicts (5.4). Therefore there exists $\lambda_1 \in [\lambda_0, \lambda^*]$ such that $h'(\lambda_1) > 0$. We set $x_3 = (1 - \lambda_1)x_1 + \lambda_1 x_2$. Then we have

$$\begin{aligned} f^\square(x_3; x_1 - x_3) &= \limsup_{\lambda \downarrow 0} [h(\lambda_1 + \lambda \lambda_1) - h(\lambda_1)] \lambda^{-1} \\ &= \lambda_1 h'(\lambda_1) > 0. \end{aligned}$$

It follows from the condition (C) that $f(x_1) \geq f(x_3)$. Consequently we have $h(0) \geq h(\lambda_1)$. This contradicts (5.4).

We can summarize the relative strength of various convexity conditions of locally Lipschitz function defined on an open convex set by the following diagram:

convex $\Rightarrow$ condition (C) $\Rightarrow$ strictly quasiconvex $\Rightarrow$ quasiconvex.

Now we consider the problem (3.1) and let x_0 be a local solution of the problem (3.1). If $f^\square(x_0; \cdot)$, $g_i^\square(x_0; \cdot)$ ($i \in \langle 1, m \rangle$), and $d_{Q_0}^\square(x_0; \cdot)$ are all convex, we can take B_0 , B_i ($i \in \langle 1, m \rangle$), and M as $\partial^* f(x_0)$, $\partial^* g_i(x_0)$ ($i \in \langle 1, m \rangle$), and $\mathcal{T}^*(Q_0; x_0)$, respectively in Theorem 2. Then there exist numbers λ_0, λ_i ($i \in \langle 1, m \rangle$) such that

$$\lambda_i \geq 0 \quad (i \in \{0\} \cup \langle 1, m \rangle), \quad (5.5)$$

$$(\lambda_0, \lambda_1, \dots, \lambda_m) \neq (0, 0, \dots, 0), \quad (5.6)$$

$$\lambda_i g_i(x_0) = 0 \quad (i \in \langle 1, m \rangle), \quad (5.7)$$

$$0 \in \lambda_0 \partial^* f(x_0) + \sum_{i=1}^m \lambda_i \partial^* g_i(x_0) + N^*(Q_0; x_0). \quad (5.8)$$

In connection with (5.8) we consider the following conditions.

$$\lambda_0 f^\square(x_0; d) + \sum_{i=1}^m \lambda_i g_i^\square(x_0; d) \geq 0 \quad \forall d \in \mathcal{T}^*(Q_0; x_0). \quad (5.9)$$

$$\lambda_0 f^\square(x_0; y - x_0) + \sum_{i=1}^m \lambda_i g_i^\square(x; y - x_0) \leq 0 \quad \forall y \in Q_0 \cap Q_1. \quad (5.10)$$

PROPOSITION 11. *If (5.5) holds, (5.8) implies (5.9). If Q_0 is convex, (5.9) implies (5.10).*

PROOF. (5.8) implies

$$\delta^*(d \mid \lambda_0 \partial^* f(x_0) + \sum_{i=1}^m \lambda_i \partial^* g_i(x_0) + N^*(Q_0; x_0)) \geq 0 \quad \forall d.$$

(5.9) follows from (5.5) and Proposition 1.

If Q_0 is convex, it follows from Proposition 8 that

$$Q_0 \cap Q_1 - x_0 \subset \text{cl cone}(Q_0 - x_0) = \mathcal{F}^*(Q_0; x_0).$$

Therefore (5.9) implies (5.10).

Conversely it is not difficult to show that, if $f^\square(x_0; \cdot)$, $g_i^\square(x_0; \cdot)$, and $d_{Q_0}^\square(x_0; \cdot)$ are convex, and (5.5) holds then (5.9) implies (5.8).

THEOREM 5. *Let f , g_i ($i \in \langle 1, m \rangle$), Q_0 and Q_1 be as given in the problem (3.1). Let $x_0 \in Q_0 \cap Q_1$ and let $J(x_0)$ denote the following set:*

$$J(x_0) = \{i \in \langle 1, m \rangle \mid g_i(x_0) = 0\}.$$

We suppose that x_0 is an interior point of Q_0 and f satisfies the condition (C) at x_0 with $S = Q_0 \cap Q_1$ and g_i ($i \in J(x_0)$) is quasiconvex at x_0 relative to Q_0 . Then in order that x_0 is a global optimal solution of the problem (3.1), it is sufficient that there exist numbers $\lambda_0 > 0$ and λ_i ($i \in \langle 1, m \rangle$) satisfying (5.5), (5.7), (5.10).

PROOF. Let $y \in Q_0 \cap Q_1$. Since x_0 is an interior point of Q_0 , for sufficiently small $\lambda > 0$, we have $(1 - \lambda)x_0 + \lambda y \in Q_0$. $y \in Q_1$ implies that

$$g_i(y) \leq 0 = g_i(x_0) \quad \forall i \in J(x_0).$$

Since g_i ($i \in J(x_0)$) is quasiconvex at x_0 relative to Q_0 , for sufficiently small $\lambda > 0$ we have:

$$g_i((1 - \lambda)x_0 + \lambda y) \leq g_i(x_0) = 0 \quad \forall i \in J(x_0).$$

So we have:

$$g_i^\square(x_0; y - x_0) \leq 0 \quad \forall i \in J(x_0).$$

Hence from (5.5) and (5.7) we have $\sum_{i=1}^m \lambda_i g_i^\square(x_0; y - x_0) \leq 0$. Hence it follows from (5.10) and $\lambda_0 > 0$ that $f^\square(x_0; y - x_0) \geq 0$. The condition (C) implies that x_0 is global optimal.

THEOREM 6. *Let f , g_i ($i \in \langle 1, m \rangle$), Q_0 , and Q_1 be as given in the problem (3.1). Let $x_0 \in Q_0 \cap Q_1$, and let $J(x_0)$ be as defined in Theorem 5. We suppose*

that Q_0 is convex and f satisfies the condition (C) at x_0 relative to $Q_0 \cap Q_1$, and that g_i ($i \in J(x_0)$) is quasiconvex at x_0 relative to Q_0 . Then in order that x_0 is a global optimal solution of the problem (3.1), it is sufficient that there exist numbers $\lambda_0 > 0$ and λ_i ($i \in \langle 1, m \rangle$) satisfying (5.5), (5.7), (5.10). The above assertion still holds, if we replace (5.10) with (5.8) or (5.9).

PROOF. Follow the same line as in the proof of Theorem 5. The last assertion follows from Proposition 11.

Remark. In the proofs of Theorem 5 and Theorem 6, we do not use the Lipschitz properties of f and g_i .

COROLLARY 4. Let Q be a convex subset of X and let $x_0 \in Q_0$. We suppose that $f: X \rightarrow \mathbb{R}$ be a locally Lipschitz function satisfying the condition (C) at x_0 relative to Q . Then $x_0 \in Q$ is a global minimum of f on Q if and only if

$$f^\square(x_0; d) \geq 0 \quad \forall d \in \mathcal{T}^*(Q; x_0).$$

PROOF. The necessity follows from Corollary 1 and Proposition 6.

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