DOMAINS OVER A K -COMPLETE MANIFOLD

By

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§0. Introduction

In §§1 and 2 we shall point out that the Levi's problem is solved for unramified domains over a K -complete complex manifold. Let X be a K -complete complex manifold. For an unramified domain $\pi: D \rightarrow X$ over X such that $\pi(D) \in X$ the following six conditions are equivalent:

- i) D is a Stein manifold,
- ii) There exists a plurisubharmonic exhaustion function on D ,
- iii) $\pi: D \rightarrow X$ is locally Stein,
- iv) $\pi: D \rightarrow X$ is p_γ -convex in the sense of F. Docquier-H. Grauert [8],
- v) There exist a complex Lie group L and a family $\mathcal{F}$ of holomorphic maps from D into L such that $\pi: D \rightarrow X$ is an envelope of holomorphy with respect to $\mathcal{F}$,
- vi) There exists a holomorphic function g on D such that $\pi: D \rightarrow X$ is a domain of holomorphy of g .

The proof of this fact is essentially due to A. Andreotti-R. Narasimhan [3], G. Elenewajg [9] and A. Takeuchi [17].

As an application of it we shall prove a variant of the theorem of Cartan-Behnke-Stein in §3. Let X be a K -complete complex manifold and $\pi: D \rightarrow X$ be an unramified domain over X . D is a Stein manifold if the following two conditions are satisfied:

- i) D is a Cousin-I manifold,
- ii) For every non-constant holomorphic function f on X the set $\{y \in D \mid (f \circ \pi)(y) = 0\}$ is a Stein subvariety of D .

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§1. Levi's problem in a K -complete manifold

Let $\pi: D \rightarrow \mathbb{C}^n$ be an unramified domain over $\mathbb{C}^n$ and ψ be a real-valued

continuous function on D . For any $x \in D$ we put

$$W\psi(x) = \inf_{\substack{\lambda \in \mathbb{C}^n \\ |\lambda|=1}} \liminf_{r \rightarrow +0} \frac{1}{2\pi r^2} \int_0^{2\pi} (\psi \circ (\pi|_{B^*})^{-1}(\pi(x) + re^{i\theta}\lambda) - \psi(x)) d\theta,$$

where B^* is a neighbourhood of x such that $\pi|_{B^*}: B^* \rightarrow \pi(B^*)$ is biholomorphic. If $W\psi > 0$ on D , then ψ is strongly plurisubharmonic on D .

Let X be a complex manifold and g be a Hermitian metric on X . For any $x \in X$ and $r > 0$ we put $B(x, r; g) = \{y \in X \mid d(x, y) < r\}$, where d is a distance on X deduced from g . Let $\pi: D \rightarrow X$ be an unramified domain over X . For any $x \in D$ we put $b_D(x) = \sup \{r > 0 \mid \text{there exists a neighbourhood } B^* \text{ of } x \text{ such that } \pi|_{B^*}: B^* \rightarrow B(\pi(x), r; g) \text{ is biholomorphic}\}$ and call it the boundary distance to D at x .

The following proposition is a slight modification of Lemme principal of G. Elencwajg [9].

PROPOSITION 1. *Let U be an open set in $\mathbb{C}^n$ containing 0. Let g be a C^∞ Kähler metric on U such that $g_{ij}(0) = \delta_{ij}$, $dg_{ij}(0) = 0$ for all $i, j = 1, \dots, n$. Then there exist a neighbourhood $V \Subset U$ of 0 and constants $\varepsilon, K > 0$ such that for any unramified Stein domain $\pi: D \rightarrow U$ over U it holds that $W(-\log b_D)(x) \geq -K$ if $x \in D$, $\pi(x) \in V$ and $b_D(x) < \varepsilon$.*

PROOF. We can take an open set U' with $0 \in U' \Subset U$ such that there exists a geodesic combining any two points in U' and that conditions a), b), c) and d) of G. Elencwajg [9], p. 301, are satisfied on U' . By the approximation with real-analytic metrics and by Lemme 3 of G. Elencwajg [9] it is sufficient to prove the proposition when g is a real-analytic Hermitian metric which satisfies conditions a), b), c) and d) of G. Elencwajg [9]. We take sufficiently small open set V with $0 \in V \Subset U'$. There exists a positive number ε such that for every unramified Stein domain $\pi: D \rightarrow U$ over U it holds that $B(\pi(p), b_D(p); g) \Subset U'$ if $\pi(p) \in V$, $b_D(p) < \varepsilon$. There exists a sequence $(D_j)_{j=1}^\infty$ of strictly pseudoconvex domains with C^∞ smooth boundary such that $D_1 \Subset D_2 \Subset \dots$ and $D = \bigcup_{j=1}^\infty D_j$. If $p \in D_k$, $\pi(p) \in V$ and $b_D(p) < \varepsilon$, then $b_{D_k}(x) = d(\pi(x), \pi(\partial D_k) \cap B(\pi(x), b_D(x); g))$ for any x near p . Therefore we can find a positive constant K independent of D and $(D_j)_{j=1}^\infty$ such that $W(-\log b_{D_k})(p) \geq -K$. This part of the proof is the same with the corresponding part of the proof of Lemme principal of G. Elencwajg [9]. Since $(b_{D_j})_{j=1}^\infty$ converges to b_D , it holds that $W(-\log b_D)(p) \geq -K$ if $\pi(p) \in V$, $b_D(p) < \varepsilon$.

An unramified domain $\pi: D \rightarrow X$ over a complex manifold X is called locally Stein when for any $x \in X$ there exists a neighbourhood U of x such that $\pi^{-1}(U)$ is a Stein manifold. Using Proposition 1 we can prove the following theorem by the method developed in A. Takeuchi [17].

THEOREM 2. *Let X be a complex manifold on which a continuous strictly plurisubharmonic function exists. If $\pi: D \rightarrow X$ be a locally Stein unramified domain over X such that $\pi(D) \in X$, then D is a Stein manifold.*

PROOF. Since there exists a C^∞ strictly plurisubharmonic function on X (Satz 4.2 of R. Richberg [15]), X has a C^∞ Kähler metric. Using Proposition 1 we can find a continuous plurisubharmonic function Φ_0 on D such that for any $a \in \mathbf{R}$ there exists a positive number ρ with $\{x \in D \mid \Phi_0(x) < a\} \subset \{x \in D \mid b_D(x) > \rho\}$ (cf. Lemme 7 of A. Takeuchi [17]). According to the proof of Lemme 8 of A. Takeuchi [17] we can find a continuous strictly plurisubharmonic exhaustion function on D_a for any $a \in \mathbf{R}$, where $D_a = \{x \in D \mid \Phi_0(x) < a\}$. Therefore D_a is a Stein manifold for any $a \in \mathbf{R}$. By Proposition 1 of J. Brun [7] D is a Stein manifold.

A complex manifold X is said to be K -complete when for any $p \in X$ there exist finitely many holomorphic functions $f_1, f_2, \dots, f_k$ on X such that p is an isolated point of the set $\{x \in X \mid f_i(x) = f_i(p), i = 1, 2, \dots, k\}$.

PROPOSITION 3. *Let X be a K -complete complex manifold. If $\pi: D \rightarrow X$ be a locally Stein unramified domain over X such that $\pi(D) \in X$, then D is a Stein manifold.*

PROOF. Let X' be an open set of X with $\pi(D) \in X' \subset X$. There exists a C^∞ strictly plurisubharmonic function on X' by Lemma 5 of A. Andreotti–R. Narasimhan [3]. Therefore D is a Stein manifold by Theorem 2.

E. Sakai put some questions on Levi's problem in §2 of [16]. Let X be a holomorphically separable complex manifold on which global holomorphic functions give local coordinates at any point. If D is a domain of X which is locally Stein or on which there exists a plurisubharmonic exhaustion function, then is D a Stein domain? This question is affirmative. Since such a manifold X is K -complete, every relatively compact locally Stein domain D of X is a Stein domain by Proposition 3. Here the supposition of the relative compactness is indispensable. A K -complete complex manifold which admits a plurisubharmonic

exhaustion function is a Stein manifold by Proposition 1 of J. Brun [7] and Proposition 3 (see also Theorem 3 of A. Andreotti–R. Narasimhan [3]).

§2. Envelope of holomorphy

PROPOSITION 4. *Let X be a second countable complex manifold and $\pi: D \rightarrow X$ be an unramified domain over X . If D is a Stein manifold, then there exists a holomorphic function g on D such that $\pi: D \rightarrow X$ is a domain of holomorphy of g .*

PROOF. We denote by $\tilde{\partial}D$ the boundary of $\pi: D \rightarrow X$ defined in F. Docquier–H. Grauert [8]. There exists a divergent sequence $(p_n)_{n=1}^\infty \subset D$ such that $\overline{(p_n)_{n=1}^\infty} \supset \tilde{\partial}D$. We denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{D,x}$ for any $x \in D$, where $\mathcal{O}_D$ is the sheaf of the germs of holomorphic functions on D . By the theorem B of H. Cartan there exists a holomorphic function f on D such that $f(p_1)=1$, $f_{p_n} \in \mathfrak{m}_{p_n}^n$ ($n=2, 3, \dots$). Using f we can find g by the method of the proof of Theorem 7.5.2 of J. Kajiura [12], p. 134.

PROPOSITION 5. *Let X be a complex manifold and $\pi: D \rightarrow X$ be an unramified domain over X . If there exist a complex Lie group L and a family $\mathcal{F}$ of holomorphic maps from D into L such that $\pi: D \rightarrow X$ is an envelope of holomorphy with respect to $\mathcal{F}$, then $\pi: D \rightarrow X$ is p_γ -convex in the sense of F. Docquier–H. Grauert [8].*

PROOF. The proof of Lemma 3 of K. Adachi–M. Suzuki–M. Yoshida [2] remains valid for any complex manifold X .

We now have the following theorem.

THEOREM 6. *Let X be a K -complete complex manifold. The following six conditions are equivalent for any unramified domain $\pi: D \rightarrow X$ over X such that $\pi(D) \Subset X$.*

- i) D is a Stein manifold.
- ii) There exists a plurisubharmonic exhaustion function on D .
- iii) $\pi: D \rightarrow X$ is locally Stein.
- iv) $\pi: D \rightarrow X$ is p_γ -convex.

v) There exist a complex Lie group L and a family $\mathcal{F}$ of holomorphic maps from D into L such that $\pi: D \rightarrow X$ is an envelope of holomorphy with respect to $\mathcal{F}$.

vi) *There exists a holomorphic function g on D such that $\pi: D \rightarrow X$ is a domain of holomorphy of g .*

PROOF. i) $\rightarrow$ ii). Satz 5 of F. Docquier–H. Grauert [8].
 ii) $\rightarrow$ i). See the comments after Proposition 3.
 i) $\rightarrow$ iii). Trivial.
 iii) $\rightarrow$ i). Proposition 3.
 i) $\rightarrow$ vi). Proposition 4. X is second countable by Satz A of H. Grauert [10].
 vi) $\rightarrow$ v). Trivial.
 v) $\rightarrow$ iv). Proposition 5.
 iv) $\rightarrow$ iii). Let $x \in X$. We take a neighbourhood U of x biholomorphic to an open ball in $\mathbb{C}^n$. Since $\pi|_{\pi^{-1}(U)}: \pi^{-1}(U) \rightarrow U$ is p_7 -convex, $\pi^{-1}(U)$ is a Stein manifold by Satz 10 of F. Docquier–H. Grauert [8].

§3. Cousin-I domain

A complex manifold X is said to be Cousin-I if and only if any additive Cousin distribution has a solution in X (see, for example, J. Kajiwaru [12], p. 108). Every Cousin-I domain in $\mathbb{C}^2$ is a Stein domain, which is the well-known theorem of Cartan-Behnke-Stein. There are various generalizations of it, for example, [1], [4], [5], [6] and [13]. As an application of Theorem 6 we prove the following variant.

THEOREM 7. *Let X be a K -complete complex manifold and $\pi: D \rightarrow X$ be an unramified domain over X such that $\pi(D) \Subset X$. Suppose that the following two conditions are satisfied.*

- i) *D is a Cousin-I manifold.*
- ii) *For every non-constant holomorphic function f on X the set $\{y \in D \mid (f \circ \pi)(y) = 0\}$ is a Stein subvariety of D .*
Then D is a Stein manifold.

PROOF. By Theorem 6 it is sufficient to prove that $\pi: D \rightarrow X$ is p_7 -convex. Suppose that $\pi: D \rightarrow X$ is not p_7 -convex. Then there exist H, P, ϕ and ψ with following properties. $H = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_j| < 1 \ (j=1, \dots, n)\} \cup \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid 1-\varepsilon < |z_1| < 1+\varepsilon, |z_j| < 1+\varepsilon \ (j=2, \dots, n)\}$, $P = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_j| < 1+\varepsilon \ (j=1, \dots, n)\}$, $0 < \varepsilon < 1$. $\phi: \bar{H} \rightarrow \bar{D} = D \cup \bar{\partial}D$ is a continuous map, where $\bar{\partial}D$ is the boundary of $\pi: D \rightarrow X$ defined in F. Docquier–H. Grauert [8]. There exists a $b = (b_1, \dots, b_n)$ such that $\phi(b) \in \bar{\partial}D$, $|b_1| < 1-\varepsilon$ and $\max_{2 \leq j \leq n} |b_j| = 1$. $\psi: P \rightarrow X$ is an

injective holomorphic map such that $\pi \circ (\phi|_H) = \psi|_H$. Let $\lambda: D \rightarrow D'$, $\pi': D' \rightarrow X$ be an envelope of holomorphy of $\pi: D \rightarrow X$. It holds that $\pi' \circ (\lambda \circ (\phi|_H)) = \psi|_H$. Since $\pi': D' \rightarrow X$ is p_7 -convex by Proposition 5, there exists a holomorphic map $\phi': P \rightarrow D'$ such that $\phi'|_H = \lambda \circ (\phi|_H)$. $\phi'(b)$ is a boundary point of $\lambda(D)$ in D' . Take a non-constant holomorphic function f on X such that $f(\pi'(\phi'(b))) = 0$. Take an irreducible component A of the subvariety $\{y' \in \phi'(P) | (f \circ \pi')(y') = 0\}$ such that $\phi'(b) \in A$, then $\lambda(D) \cap A \supset \phi'(H) \cap A \neq \emptyset$. We have a sequence $(q_v)_{v=1}^\infty \subset D$ and a boundary point p_0 of $\lambda(D)$ in D' such that $\lim_{v \rightarrow \infty} \lambda(q_v) = p_0$ and $\lambda(q_v) \in A$ for all $v = 1, 2, \dots$. Since $V = \{y \in D | (f \circ \pi)(y) = 0\}$ is Stein and $(q_v)_{v=1}^\infty \subset V$, there exists a holomorphic function g on V which satisfies $\lim_{v \rightarrow \infty} |g(q_v)| = +\infty$. By Theorem 1 of G. Berg [6] there exists a holomorphic function G such that $G|_V = g$. There exists a holomorphic function G' on D' such that $G = G' \circ \lambda$. It holds that $|G'(p_0)| = \lim_{v \rightarrow \infty} |G(q_v)| = +\infty$. It is a contradiction.

We may replace the condition ii) of Theorem 7 by the following condition:

ii') *There exists a non-constant holomorphic function f on X such that the set $\{y \in D | (f \circ \pi)(y) = c\}$ is a Stein subvariety of D for every $c \in \mathbb{C}$.*

Since every one dimensional reduced complex space without compact irreducible components is a Stein space by Theorem 1 of R. Narasimhan [14], the condition ii) of Theorem 7 is always satisfied if X is of dimension two. Then we have the following corollary.

COROLLARY. *Let X be a K -complete complex manifold of dimension two and $\pi: D \rightarrow X$ is an unramified domain over X such that $\pi(D) \in X$. If D is Cousin-I, then D is a Stein manifold.*

If X is a Stein manifold, then we can drop the condition of the relative compactness in Theorem 7 and in the corollary to it. We have the following theorem. Compare also Theorem 5 of G. Berg [6].

THEOREM 8. *Let X be a Stein manifold and $\pi: D \rightarrow X$ be an unramified domain over X . Suppose that the following two conditions are satisfied.*

- i) *D is a Cousin-I manifold.*
- ii) *For every non-constant holomorphic function f on X the zero set of which is non-singular the set $\{y \in D | (f \circ \pi)(y) = 0\}$ is a Stein submanifold. Then D is a Stein manifold.*

PROOF. In the proof of Theorem 7 we can take a non-constant holomorphic

function f on X such that $f(\pi'(\phi'(b)))=0$ and the set $\{x \in X | f(x)=0\}$ is non-singular. By the same reasoning of the proof of Theorem 7 $\pi: D \rightarrow X$ is p_7 -convex. Thus by Satz 10 of F. Docquier–H. Grauert [8] D is a Stein manifold.

COROLLARY. *Let X be a Stein manifold of dimension two and $\pi: D \rightarrow X$ be an unramified domain over X . If D is Cousin-I, then D is a Stein manifold.*

The same reasoning of the proof of Theorem 7 gives also the following theorem, which is considered as a generalization of Theorem 1 of S. Hitotumatu [11].

THEOREM 9. *Let $\pi: D \rightarrow \mathbb{C}^n$ be an unramified domain over $\mathbb{C}^n$. Suppose that the following two conditions are satisfied.*

- i) D is a Cousin-I manifold.
 - ii) *For every non constant linear function f of n variables without constant term the set $\{y \in D | (f \circ \pi)(y)=0\}$ is a Stein submanifold of D .*
- Then D is a Stein manifold.*

PROOF. Put $X = \mathbb{C}^n$. In the proof of Theorem 7 we can take a linear function of n variables without constant term as a non-constant holomorphic function on X such that $f(\phi'(\pi'(b)))=0$.

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