**STEINNESS OF A DOMAIN WITH A VANISHING
COHOMOLOGY SET IN A PRODUCT MANIFOLD
OF THE ONE DIMENSIONAL COMPLEX
PROJECTIVE SPACE AND A ONE
DIMENSIONAL COMPLEX TORUS**

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1. Introduction

J. Kajiwara and H. Kazama [8] showed that a domain Ω in a two dimensional Stein manifold is a Stein manifold if there exists a positive dimensional complex Lie group L with $H^1(\Omega, \mathcal{A}_L)=0$, where $\mathcal{A}_L$ is the sheaf of germs of holomorphic mappings in L over Ω . Similar results are known. For a domain in the two-dimensional complex projective space $\mathbf{P}_2$ or in the product manifold $\mathbf{P} \times \mathbf{P}$ of two one-dimensional complex projective spaces $\mathbf{P}$, there is a result of J. Kajiwara and K. Watanabe [10]. J. Kajiwara [7] proved that a domain Ω in a product manifold of an open Riemann surface and a compact Riemann surface with genus $g \geq 1$ is a Stein manifold if there exists a positive dimensional complex Lie group L with $H^1(\Omega, \mathcal{A}_L)=0$. J. Kajiwara and M. Takase [9] showed that a domain Ω in a product manifold $\mathbf{T} \times \mathbf{T}$ of two one-dimensional complex tori $\mathbf{T}$ is a Stein manifold if there exists a complex linear group L whose Lie algebra contains a non-zero purely integral element such that $H^1(\Omega, \mathcal{A}_L)=0$.

In this paper we shall show that a similar result holds for a domain in a product manifold $X=S \times R_1 \times R_2 \times \cdots \times R_q$, where S is a Stein manifold and each R_i is $\mathbf{P}$ or $\mathbf{T}$. Our argument is slightly different from the proof of J. Kajiwara and M. Takase [9]. And our result contains the result of J. Kajiwara and M. Takase [9].

2. Distance function of a pseudoconvex domain in a Kähler manifold

Let X be a Kähler manifold with Kähler metric g . We denote by $T_x(X)$ the tangent space of X at x . Let R be the Riemannian curvature tensor of X determined by g and J be the complex structure of X . The concept of holomorphic bisectional curvature was introduced by S. I. Goldberg and S. Kobayashi [5].

DEFINITION 1. *Given two J -invariant planes p and p' in $T_x(X)$, the holomorphic bisectional curvature $H(p, p')$ is defined by*

$$H(p, p') = R(V, JV, W, JW),$$

where V and W are unit vectors in p and p' respectively. If $H(p, p')$ is positive (non-negative) for all planes p, p' in $T_x(X)$ invariant by J and for all points $x \in X$, then X is called a Kähler manifold with positive (non-negative) holomorphic bisectional curvature.

DEFINITION 2. *A domain Ω in X is pseudoconvex if for every x on the boundary $\partial\Omega$ of Ω , there exists a neighbourhood U of x in X such that $\Omega \cap U$ is a Stein manifold.*

For $x, y \in X$, we denote by $d(x, y)$ the distance relative to g between x and y . We define the boundary distance $d(x)$ by $d(x) = d(x, \partial\Omega) = \inf_{y \in \partial\Omega} d(x, y)$. Using G. Elencwajg's argument [4] directly, the author proved the following proposition.

PROPOSITION 1. *Let $X_i (i=1, 2, \dots, q)$ be a Kähler manifold with positive holomorphic bisectional curvature or a Kähler manifold of zero curvature. If Ω is a relatively compact pseudoconvex domain in the product manifold $X = X_1 \times X_2 \times \dots \times X_q$, then there exists a neighbourhood U of $\partial\Omega$ such that $-\log d$ is plurisubharmonic in $\Omega \cap U$.*

This proposition is contained in the following Theorem of R. E. Greene and H. Wu [6] which was proved more geometrically. So we omit the proof of Proposition 1.

THEOREM. *Let X be a Kähler manifold with non-negative holomorphic bisectional curvature. If Ω is a relatively compact pseudoconvex domain in X , then there exists a neighbourhood U of $\partial\Omega$ such that $-\log d$ is plurisubharmonic in $\Omega \cap U$.*

The n -dimensional complex projective space $\mathbf{P}_n$ is a Kähler manifold with positive holomorphic bisectional curvature by the Fubini-Study metric (cf. S. Kobayashi and K. Nomizu [11]). Let $\mathbf{T}_n$ be an n -dimensional complex torus. Since the metric $ds^2 = \sum_{\alpha=1}^n dz^\alpha d\bar{z}^\alpha$ on $\mathbf{C}^n$ induces a Kähler metric on $\mathbf{T}_n$, $\mathbf{T}_n$ is a Kähler manifold of curvature zero.

3. K-completeness of a domain with a vanishing cohomology set

Let L be a complex linear group. That is, L is a complex Lie subgroup of $GL(m, \mathbf{C})$. Let $\mathcal{L}(m, \mathbf{C})$ be the Lie algebra of $GL(m, \mathbf{C})$. We say that $X \in \mathcal{L}(m, \mathbf{C})$ is purely integral if there exists $P \in GL(m, \mathbf{C})$ such that

$$P^{-1}XP = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_m \end{bmatrix}, \lambda_j \in \mathbf{Z}.$$

The following Lemma is due to J. Kajiwarra and M. Takase [9].

LEMMA 1. *Let L be a complex linear group whose Lie algebra contains a non-zero purely integral element. Let M be a complex manifold with $H^1(M, \mathcal{A}_L) = 0$, where $\mathcal{A}_L$ is the sheaf of germs of holomorphic mappings in L over M . Let h be a holomorphic function on an open set U in M . If $V = \{x \in U; h(x) = 0\}$ is closed in M , then there exist two positive integers n, m , and holomorphic functions $f_1, f_2, \dots, f_m$ on M , holomorphic functions $g_1, g_2, \dots, g_m$ on U satisfying the following conditions:*

- (1) $|f_1|^2 + |f_2|^2 + \dots + |f_m|^2 > 0$ on $M \setminus V$,
- (2) $|g_1|^2 + |g_2|^2 + \dots + |g_m|^2 > 0$ on U ,
- (3) $f_j = h^n g_j$ on $U, j = 1, 2, \dots, m$.

DEFINITION 3. *A domain Ω in a complex manifold M is said to be K-complete if for every $x \in \Omega$ there exists a holomorphic map f on Ω into some $\mathbf{C}^N$ such that x is an isolated point of $f^{-1}(f(x))$.*

LEMMA 2. *Let S be a Stein manifold and R_i ($i = 1, 2, \dots, q$) be a Riemann surface. Let Ω be a domain in the product manifold $S \times R_1 \times R_2 \times \dots \times R_q$. If*

there exists a complex linear group L whose Lie algebra contains a non-zero purely integral element such that $H^1(\Omega, \mathcal{A}_L) = 0$, then Ω is K -complete.

PROOF. Let $\pi_k: S \times R_1 \times R_2 \times \cdots \times R_q \rightarrow R_k$ be a projection. Take any $z = (s, x_1, x_2, \dots, x_q) \in \Omega$. Applying Lemma 1 to $V = \pi_k^{-1}(x_k) \cap \Omega$, we obtain a holomorphic map $f^k = (f_1^k, f_2^k, \dots, f_{n_k}^k): \Omega \rightarrow \mathbb{C}^{n_k}$ such that $(f^k)^{-1}(0) = \pi_k^{-1}(x_k) \cap \Omega$. We set $f = (f^1, f^2, \dots, f^q): \Omega \rightarrow \mathbb{C}^{n_1 + n_2 + \cdots + n_q}$, then we have $f^{-1}(0) = (S \times \{x_1\} \times \{x_2\} \times \cdots \times \{x_q\}) \cap \Omega$. Since Ω is Stein, there exist holomorphic functions $g_1, g_2, \dots, g_l$ such that $\{s' \in S; g_1(s') = 0, g_2(s') = 0, \dots, g_l(s') = 0\} = \{s\}$. If we set $F = (g_1, g_2, \dots, g_l, f): \Omega \rightarrow \mathbb{C}^{l + n_1 + n_2 + \cdots + n_q}$, then we have $F^{-1}(0) = \{z\}$. This completes the proof.

4. Domain with a vanishing cohomology set

The following theorem includes the result of J. Kajiwara and M. Takase [9].

THEOREM 1. Let S be a Stein manifold and R_i ($i = 1, 2, \dots, q$) be $\mathbf{P}$ or $\mathbf{T}$. Let Ω be a pseudoconvex domain in $S \times R_1 \times R_2 \times \cdots \times R_q$. If there exists a complex linear group L whose Lie algebra contains a non-zero purely integral element such that $H^1(\Omega, \mathcal{A}_L) = 0$, then Ω is a Stein manifold.

PROOF. First we prove the case that S is a Stein open set in $\mathbb{C}^n$. Let $\psi(s)$ be a C^∞ strongly plurisubharmonic exhaustion function of S . We write $x \in S \times R_1 \times R_2 \times \cdots \times R_q$ such as $x = (s, \gamma)$, $s \in S$, $\gamma \in R_1 \times R_2 \times \cdots \times R_q$. For any $\alpha \in \mathbf{R}$ we set $\Omega_\alpha = \{x = (s, \gamma) \in \Omega; \psi(s) < \alpha\}$. Then Ω_α is pseudoconvex and $\Omega_\alpha \Subset S \times R_1 \times R_2 \times \cdots \times R_q$. S has the flat metric which is the restriction of the natural flat metric of $\mathbb{C}^n$. By Proposition 1 there exists a neighbourhood U of $\partial\Omega_\alpha$ such that $-\log d$ is plurisubharmonic in $\Omega_\alpha \cap U$. Construct a continuous plurisubharmonic exhaustion function of Ω_α from $-\log d$. By Lemma 2 Ω_α is K -complete. Hence Ω_α is a Stein manifold (A. Andreotti and R. Narasimhan [1]). Since any Ω_α is a Stein manifold, Ω itself is a Stein manifold (cf. Proposition 1 in J. Brun [2]).

Next we consider the general case. We may assume that S is a closed submanifold of some $\mathbb{C}^N$ by the imbedding theorem. Moreover, there exist a Stein tubular neighbourhood U of S in $\mathbb{C}^N$ and a holomorphic retraction $\rho: U \rightarrow S$ (F. Docquier and H. Grauert [3]). We define a holomorphic map $\pi: U \times R_1 \times R_2 \times \cdots \times R_q \rightarrow S \times R_1 \times R_2 \times \cdots \times R_q$ by $\pi(u, \gamma) = (\rho(u), \gamma)$. Since Ω is K -complete, $\pi^{-1}(\Omega)$ is also K -complete. By the first case of the proof we conclude that $\pi^{-1}(\Omega)$ is a Stein manifold. Since Ω is a closed submanifold of $\pi^{-1}(\Omega)$, Ω is a

Stein manifold. Thus we proved the theorem.

The following proposition is due to J. Kajiwaru [7].

PROPOSITION 2. *Let R_1 and R_2 be Riemann surfaces and Ω be a domain in $R_1 \times R_2$. If there exists a positive dimensional complex Lie group L with $H^1(\Omega, \mathcal{A}_L)=0$, then Ω is pseudoconvex.*

By Proposition 2 and Theorem 1 we obtain the following theorem.

THEOREM 2. *Let Ω be a domain in $\mathbf{P} \times \mathbf{T}$. If there exists a complex linear group L whose Lie algebra contains a non-zero purely integral element such that $H^1(\Omega, \mathcal{A}_L)=0$, then Ω is a Stein manifold.*

J. Kajiwaru and K. Watanabe [10] showed that if there exists a positive dimensional complex Lie group L with $H^1(\Omega, \mathcal{A}_L)=0$ for a domain Ω in $\mathbf{P} \times \mathbf{P}$, then either $\Omega = \Delta \times \mathbf{P}$ for a domain Δ of $\mathbf{P}$ or Ω is a Stein manifold. J. Kajiwaru [7] showed that for a domain Ω in a product manifold $R \times \mathbf{P}$ of an open Riemann surface R and the one-dimensional complex projective space $\mathbf{P}$ there exists a positive dimensional complex Lie group L with $H^1(\Omega, \mathcal{A}_L)=0$, then either Ω is a Stein manifold or there exists a domain Δ in R with $\Omega = \Delta \times \mathbf{P}$. If L is a complex linear group satisfying the condition of Theorem 2, then the following theorem holds.

THEOREM 3. *Let R be an open Riemann surface. Let Ω be a domain in $R \times \mathbf{P}$ or in $\mathbf{P} \times \mathbf{P}$. If there exists a complex linear group L whose Lie algebra contains a non-zero purely integral element such that $H^1(\Omega, \mathcal{A}_L)=0$, then Ω is a Stein manifold.*

Especially take $\mathbf{C}^*$ as L , then we obtain the following corollaries.

COROLLARY 1. *Let Ω be a domain in $\mathbf{P} \times \mathbf{T}$ or in $\mathbf{P} \times \mathbf{P}$. If $H^1(\Omega, \mathcal{O}^*)=0$, then Ω is a Stein manifold.*

COROLLARY 2. *Let R be an open Riemann surface. If Ω is a domain in $R \times \mathbf{P}$ with $H^1(\Omega, \mathcal{O}^*)=0$, then Ω is a Stein manifold.*

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